Kober

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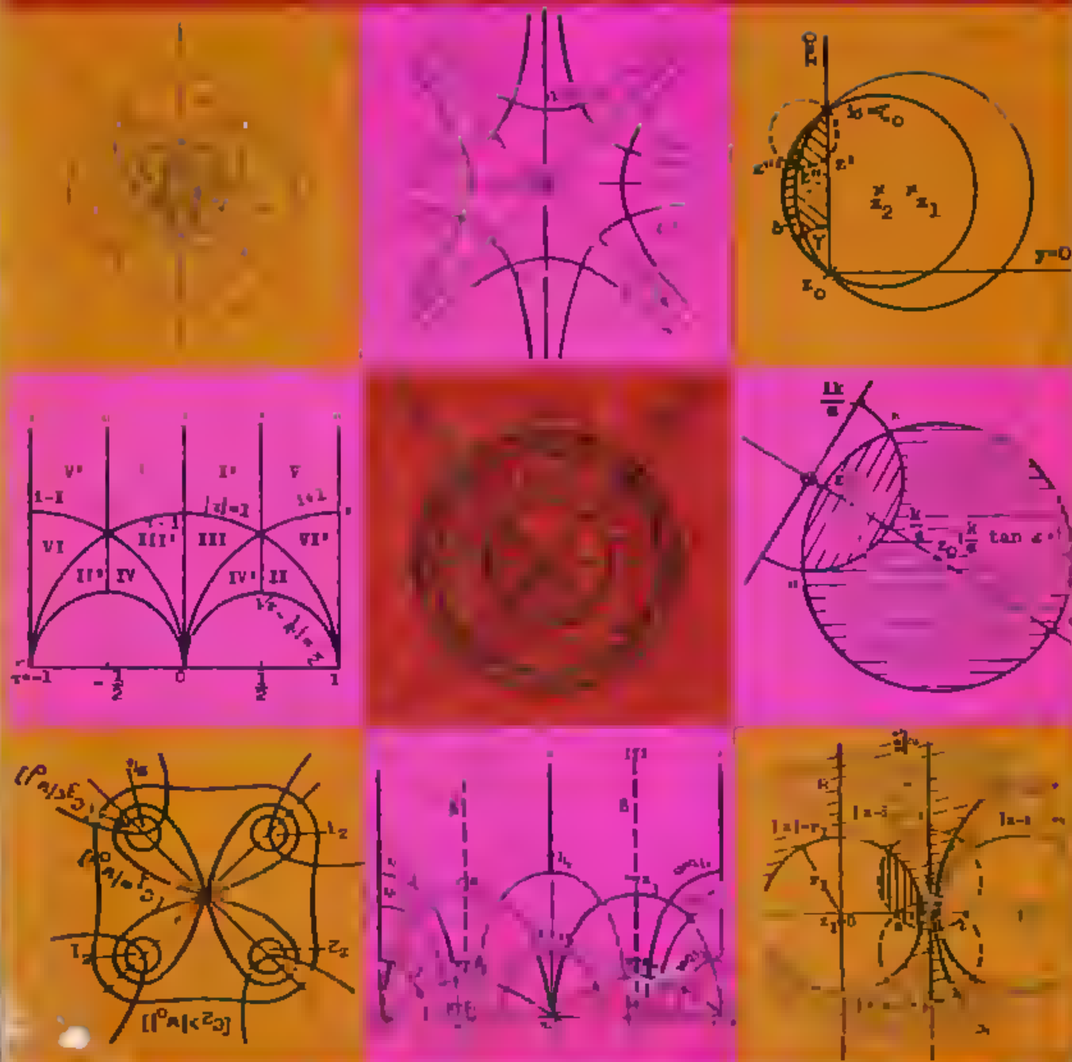
DICTIONARY OF CONFORMAL REPRESENTATIONS

H. Kober



DICTIONARY OF CONFORMAL REPRESENTATIONS

H. Kober

Written for the British Admiralty during the years 1944-1948, this unique book enables its users to solve Laplace's Equation in Two Dimensions for many boundary conditions. It contains scores of geometrical forms and their transformations for use in checking against specific problems.

Kober's *Dictionary of Conformal Representations* is especially helpful in hydrodynamics, gas dynamics, free streamline theory, heat flow, magnetic field problems, soil mechanics, electrostatics, map projection, etc. It contains the following specific features for workers in different fields: Linear and bilinear transformations for electrical engineers—Joukowski aerofoil for aerodynamicists—Transcendental functions for physicists and engineers—Schwartz-Christoffel transformations for hydrodynamicists and electrical engineers—Borda's mouthpiece for researchers in hydraulics and jet propulsion. The book is classified according to the analytic functions describing the transformations; it gives instant access to time-saving set-ups and eliminates much of the work involved in deriving a particular configuration.

"Of great value to workers in such fields as electricity, hydrodynamics, aerodynamics and heat flow," *British Journal of Applied Physics*. "Useful to engineers and physicists as well as to mathematicians," *Journal of Royal Naval Scientific Service*. "May well remain the standard work for many years," *Mathematical Gazette*.

Bibliography. Glossary. Topological index. 447 diagrams. xvi + 208pp. 5⅜ x 8½. 60160-9 Paperbound

DICTIONARY OF CONFORMAL REPRESENTATIONS

BY H. KOBER

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PART ONE

LINEAR AND BILINEAR TRANSFORMATIONS

1.* $\boxed{\text{Fixed points of } w = \frac{az+b}{cz+d};\quad ad - bc \neq 0.}$

F_1 and F_2 are the roots of the equation

$$cF^2 + (d-a)F - b = 0$$

The identical transformation $w = z$ is excluded.

$\boxed{F_1 = F_2 = \infty}$ $\quad w = z + b \quad (b \neq 0),\qquad p,q$ real.

Any line $\Im(\bar{b}z) = p$ is mapped on itself, but no other "circle".
The line $\Re(\bar{b}z) = q$ is mapped on $\Re(\bar{b}w) = q - |b|^2$.

$\boxed{F_1 = F_2 \neq \infty}$ $\quad \frac{1}{w-F} = \frac{1}{z-F} + k;\quad$ where $c \neq 0,\ F = \frac{a-d}{2c} = \frac{2b}{d-a},\ k = \frac{2c}{a+d}$

The transformation is "parabolic".
The straight line $\Im\left\{k(z-F)\right\} = 0$ and every circle touching this straight line at F is mapped on itself: no other "circle" has this property. The set of "circles" orthogonal to the above set is, as a whole, mapped on itself.

$\boxed{F_1 \neq F_2 = \infty}$ $\quad w - F_1 = \alpha(z-F_1);\ \alpha = \frac{a}{d}(\alpha \neq 0,1),\quad c = 0;\ F_1 = \frac{b}{d-a}$.

D = set of concentric circles, centre F_1.

E = set of straight lines through F_1.

The set D as a whole is mapped on itself; so is E. No "circle" other than those stated below is mapped on itself.

(i) α real, but $\alpha \neq -1$. Every line of E is mapped on itself.

(ii) $\alpha = -1$. Every circle of D, every line of E is mapped on itself. The transformation is involutory.

(iii) $|\alpha| = 1$, but $\alpha \neq -1$. Every circle of D is mapped on itself.

(iv) α not real, $|\alpha| \neq 1$. No circle nor line is mapped on itself: "loxodromic" transformation.

FOREWORD

The present book contains a collection of formulae and properties of a number of conformal representations. No proofs are given; in Part IV, however, the method of the Schwarz-Christoffel transformation is explained.

Conformal mappings may be classified either according to the analytic functions $w = f(z)$ describing them; or according to the curves and regions which they map, respectively, on certain curves and regions in the other plane, for instance on the unit circle and its interior, or on the real axis and the upper half-plane. The first classification can be made more systematic. It has, therefore, on the whole been adopted in the present dictionary. The other method, however, is used throughout Part IV, and also in some sections of the other parts; for instance in I, §5 and II, §7.

Completeness in the sense of including all relevant conformal mappings could not be aimed at in this short book, apparently the first of its kind; in which the author tries to deal with the subject in a systematic way and, incidentally, to fill the worst gaps. Also, in most cases transformations are omitted, if they can be obtained by combination of transformations included in this dictionary with linear or bilinear transformations. A typical example at the end of Part I will illustrate the use of the formulae of Part I in combinations.

A number of special notations are used in this dictionary. In the Theory of Functions nobody would use a notation other than $|z-z_1| = r$ for the circle with centre z_1 and radius r; though it is different from the formula familiar in Analytic Geometry. For the straight line, ellipse, etc., however, the forms of Analytic Geometry are usually employed in conformal transformations; though, in most cases, they are not suitable for this purpose. By means of the other notations, some of which were used by G. Pólya and G. Szegö, and which are adapted to the present problems, general results, apparently not yet stated, can be obtained in a number of cases; e.g. in §§3.2; 3.3; 6.1. The notation used for the rectangular hyperbola is due to Professor A. Erdélyi. The present form of the dictionary, aiming at conciseness by a proper arrangement of the results, is due to him and to Dr. J. Todd. I am greatly indebted to them for this and for other valuable advice. Again I wish to thank Professor L. Rosenhead, Liverpool.

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NOTATIONS AND NOMENCLATURE

Complex variables: z; w; ζ; ξ.

$z = x+iy$; $\Re(z)$ = real part of $z = x$; $\Im(z) = y$; $\bar{z} = x-iy$.

$z = re^{i\varphi}$; $r = |z| > 0$; $\varphi = \arg z$ real.

$w = u+iv$; $\Re(w) = u$; $\Im(w) = v$.

Critical points of the transformation $w = f(z)$:

The singular points of the analytic function $w = f(z)$, the points for which $\frac{dw}{dz} = 0$, and $z = \infty$.

Fixed points F_1, F_2, ..., of the transformation $w = f(z)$:

The solutions of the equation $z = f(z)$.

Asterisk *: It indicates that at the results concerned the w-plane is supposed to lie on the z-plane, the positive parts of the real axes coinciding.

"Circle": A circle or a straight line.

Line: Straight line.

Equation of the line $\ell x+my+p = 0$, where ℓ, m,p real, $\ell^2+m^2 > 0$:

$$\Re(\bar{\lambda}z) = p, \quad \text{or} \quad \Im(\mu z) = p,$$

where $\lambda = \ell+im \neq 0$, $\mu = m+i\ell = i\bar{\lambda}$.

Distance of a point z_0 from the line: $\left|\frac{\Re(\bar{\lambda}z_0)-p}{\lambda}\right|$.

Angle between x-axis and line: $\arg\lambda + \frac{\pi}{2} + n\pi$ $(n = 0,\pm1,\pm2, \ldots)$.

Ellipse:

Equation: $|z-z_1| + |z-z_2| = k$ $(k > |z_1-z_2|)$.

Its foci are z_1 and z_2.

Major axis: k

NOTATIONS AND NOMENCLATURE

Ellipse.

Eccentricity: $|z_1-z_2|/k$.

Exterior of ellipse: $|z-z_1| + |z-z_2| > k$.

Interior of ellipse: $|z-z_1| + |z-z_2| < k$.

Hyperbola.

Equation: $|z-z_1| - |z-z_2| = \pm k \qquad (0 < k < |z_1-z_2|)$,

the positive or negative sign, respectively, holding for the branch nearer the focus z_2 or for the branch nearer the focus z_1.

Real axis: k; eccentricity: $|z_1-z_2|/k$.

Equations of asymptotes:

$$\Re\left\{(\bar{z}_1-\bar{z}_2)e^{\pm i\varphi}\left(z - \frac{z_1+z_2}{2}\right)\right\} = 0 \qquad \left(\cos\varphi = \frac{k}{|z_1-z_2|}\right).$$

Exterior, not containing the foci: $\big||z-z_1| - |z-z_2|\big| < k$.

Rectangular hyperbola.

The above hyperbola is rectangular if $k\sqrt{2} = |z_1-z_2|$.

If, in addition, $z_2 = -z_1$ (i.e. origin is centre of curve), then the equation of the rectangular hyperbola is also

$$\Re(z^2/z_1^2) = \tfrac{1}{2} \qquad (\text{foci } \pm z_1)$$

Exterior: $-\infty \leq \Re(z^2/z_1^2) < \frac{1}{2}$.

Equations of asymptotes (three forms):

$$\Re(z^2/z_1^2) = 0; \text{ or } \Re(z/z_1) = \pm\,\Im(z/z_1); \text{ or } \Re\left\{z(1\pm i)/z_1\right\} = 0$$

Parabola.

Equation: $|z-z_0| = |k - \Re(\lambda z)|\,|\lambda|^{-1}; \quad \lambda \neq 0; \quad k \gtrless \Re(\lambda z_0)$.

Focus: $z = z_0$; directrix: $\Re(\lambda z) = k$.

Latus rectum: $2|k - \Re(\lambda z_0)|\,|\lambda|^{-1}$; vertex $z = z_0 + \dfrac{k-\Re(\lambda z_0)}{2\lambda}$

NOTATIONS AND NOMENCLATURE

<u>Cassinians and lemniscates:</u> $|z-z_1||z-z_2| = k; \quad k > 0.$

Foci: z_1, z_2.

If $k = \frac{1}{4}|z_1-z_2|^2$, the curve is a lemniscate.

<u>Notations for elliptic functions:</u> See part five, in front of §13, p. 169

<u>Single curled bracket:</u> See footnote to parabola, §6.1, part II, p. 35.

<u>Involutory transformation:</u> If $w = f(z)$, then $z = f(w)$.

<u>Region:</u> An open connected set of points, i.e.: any point of the region R is the centre of circular discs consisting entirely of points of R, and any two points of R can be joined by a Jordan arc consisting only of points of R.

<u>Domain:</u> It consists of a region and of its boundary points; e.g. the set of points $|z| \leq 1$, consisting of the region $|z| < 1$ [i.e. the interior of $|z| = 1$] and of the circle $|z| = 1$.

<u>A region R of the z-plane is mapped on a region S of the w-plane:</u>
This statement implies that

1) the transformation concerned represents R on S in a one-one correspondence.
2) the boundary of R is mapped on that of S, but, if the transformation is not linear or bilinear, not necessarily in a one-one correspondence.

The above connection between R and S is often indicated by similar shading or by like Roman numbers, and by denoting corresponding points by like letters in Script.

NOTATIONS AND NOMENCLATURE

If a simple closed contour C of the z-plane is mapped on a simple closed contour Γ of the w-plane in a one-one correspondence, and if directions on C and Γ are fixed by taking corresponding points in order, then the region to the left of C is mapped on the region to the left of Γ. This rule is often used to decide whether the interior of a closed curve in the z-plane is mapped on the interior or exterior of the corresponding curve in the w-plane.

$\boxed{F_1 \neq F_2, \quad F_1 \neq \infty, \quad F_2 \neq \infty}$

$$\frac{w-F_1}{w-F_2} = \alpha \frac{z-F_1}{z-F_2}.$$

$$\alpha = \frac{cF_2+d}{cF_1+d} = \frac{(a+d-\sqrt{(a-d)^2+4bc})^2}{4(ad-bc)}$$

E = pencil of "circles" through F_1 and F_2

D = set of "circles" orthogonal to the circles of E (co-axal system).

The results are the same as in the case $F_1 \neq F_2 = \infty$, but read "circle" for circle or straight line there.

2. THE LINEAR TRANSFORMATION $w = az+b$, $a \neq 0$.

2.1 * $\boxed{w = az}$

(i) $a > 0$. Magnification, centre O ratio a.

(ii) $a = e^{i\tau}$, τ real. Rotation about O through angle τ.

(iii) $a = ke^{i\tau}$, $k > 0$, τ real. Rotation about O through angle τ, with subsequent magnification, centre O ratio k.

2.2 * $\boxed{w = z+b}$ Translation by the vector Ob.

2.3 * $\boxed{w = az+b}$ $a \neq 1$. $F_0 = \frac{b}{1-a}$, $a = ke^{i\tau}$, $k > 0$, τ real.

Rotation about F_0 through angle τ, with subsequent magnification, centre F_0 ratio k.

ℓ, m, p, q, r, L, M, P, Q, R real

z - plane	w - plane
line $x = p$	line $\Re(\frac{w-b}{a}) = p$
line $y = q$	line $\Im(\frac{w-b}{a}) = q$

z - plane	w - plane
line $\ell x+my = p$	line $\Re\left\{(\ell-im)\frac{w-b}{a}\right\} = p$
line $\Re(az+b) = P$	line $u = P$
line $\Im(az+b) = Q$	line $v = Q$
line $\Re\left\{(L-iM)(az+b)\right\} = P$	line $Lu+Mv = P$
circle $\|z-z_o\| = r$	circle $\|w-(az_o+b)\| = \|a\|r$
circle $\left\|z - \frac{w_o-b}{a}\right\| = \frac{R}{\|a\|}$	circle $\|w-w_o\| = R$

3. THE GENERAL BILINEAR TRANSFORMATION.

$$w = \frac{az+b}{cz+d}, \qquad c \neq 0, \qquad ad-bc \neq 0.$$

$$z = \frac{dw-b}{-cw+a}.$$

3.1 $\boxed{w = \frac{az+b}{cz+d}}$ If $a = -d$, the transformation is involutory, i.e. w and z may be interchanged.

Magnification $\left|\frac{dw}{dz}\right| = \left|\frac{ad-bc}{(cz+d)^2}\right|$.

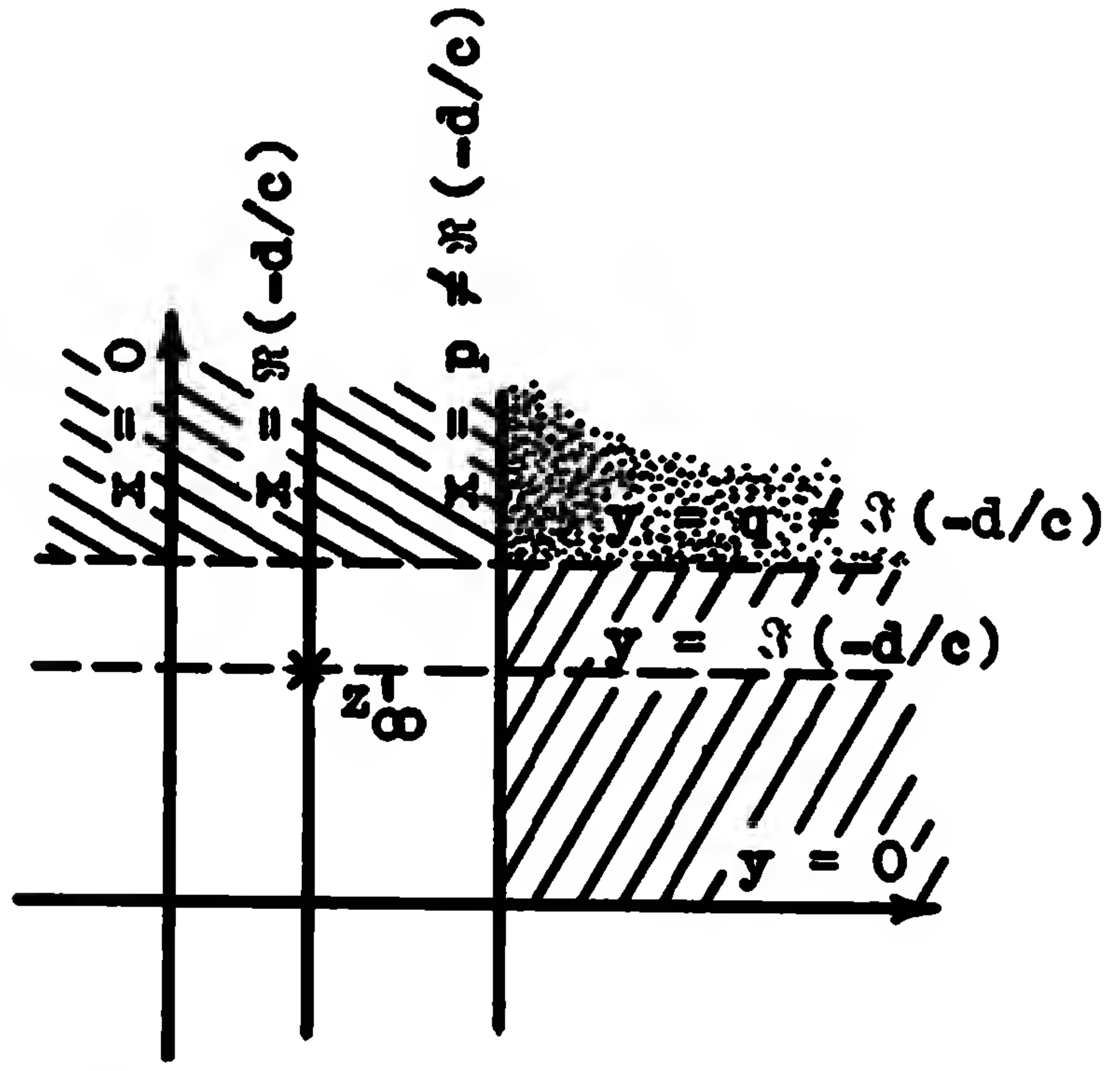
Critical points: $z = -d/c$ and $z = \infty$. At these points the transformation is not conformal, in the familiar sense.

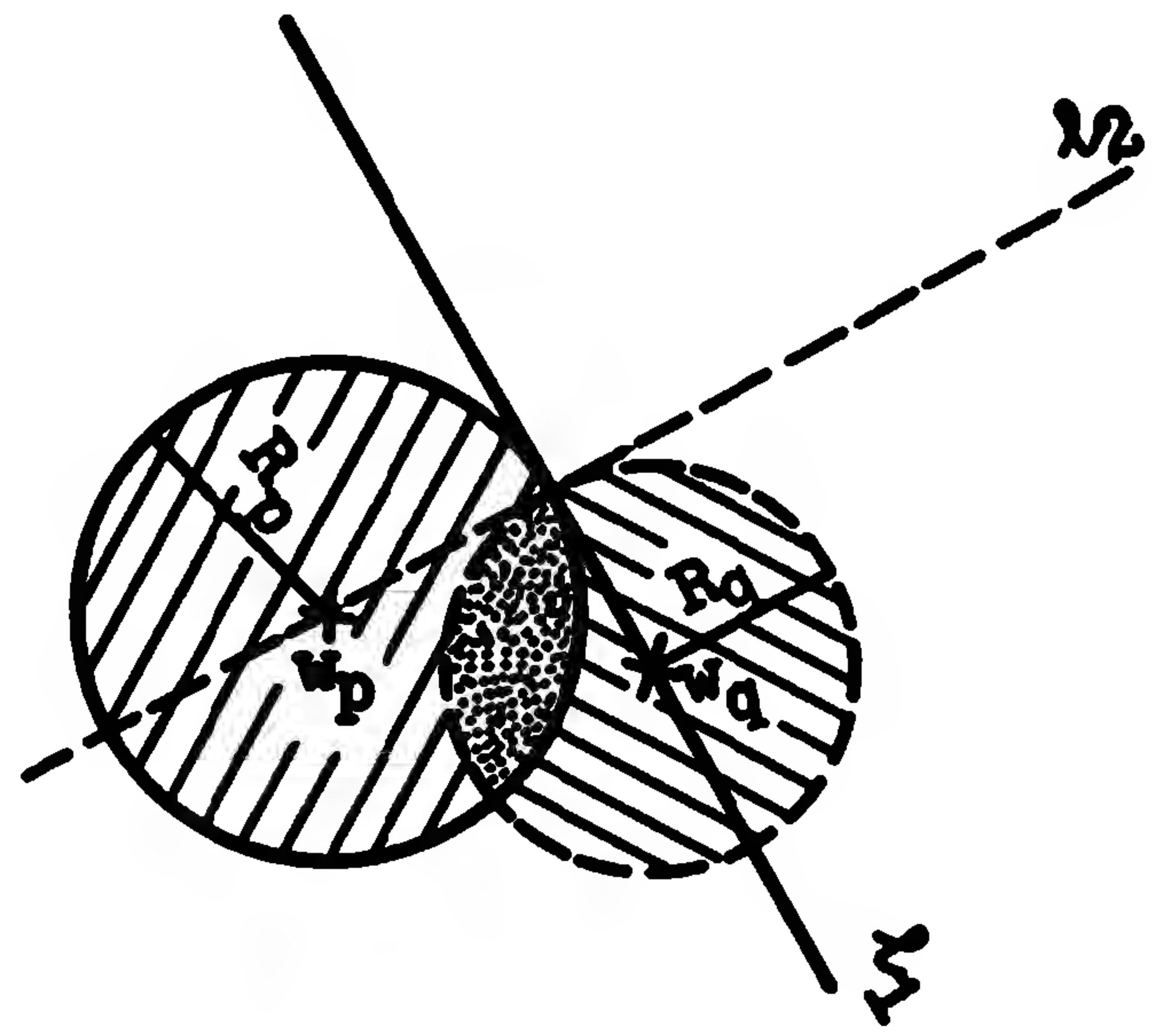
z - plane	w - plane
$z = 0;\quad -b/a;\quad z_\infty^1 = -d/c$	$w = b/d;\quad 0;\quad \infty$
∞	$a/c = w_\infty$

3.2 **Lines parallel to axes.** $z'_\infty = -d/c$, $w_\infty = a/c$.

$p, q, r, r_p, r_Q,$ P, Q, R, R_p, R_q real.

z - plane	w - plane





line $x = -\Re(d/c)$ — line ℓ_1: $\Re\left\{c(\bar a\bar d-\bar b\bar c)(cw-a)\right\} = 0$

line $x = p$, $p \neq -\Re(d/c)$ — circle $|w-w_p| = R_p$;

$$w_p = \frac{a\bar c p + \frac{1}{2}(a\bar d+b\bar c)}{p|c|^2 + \Re(c\bar d)},$$

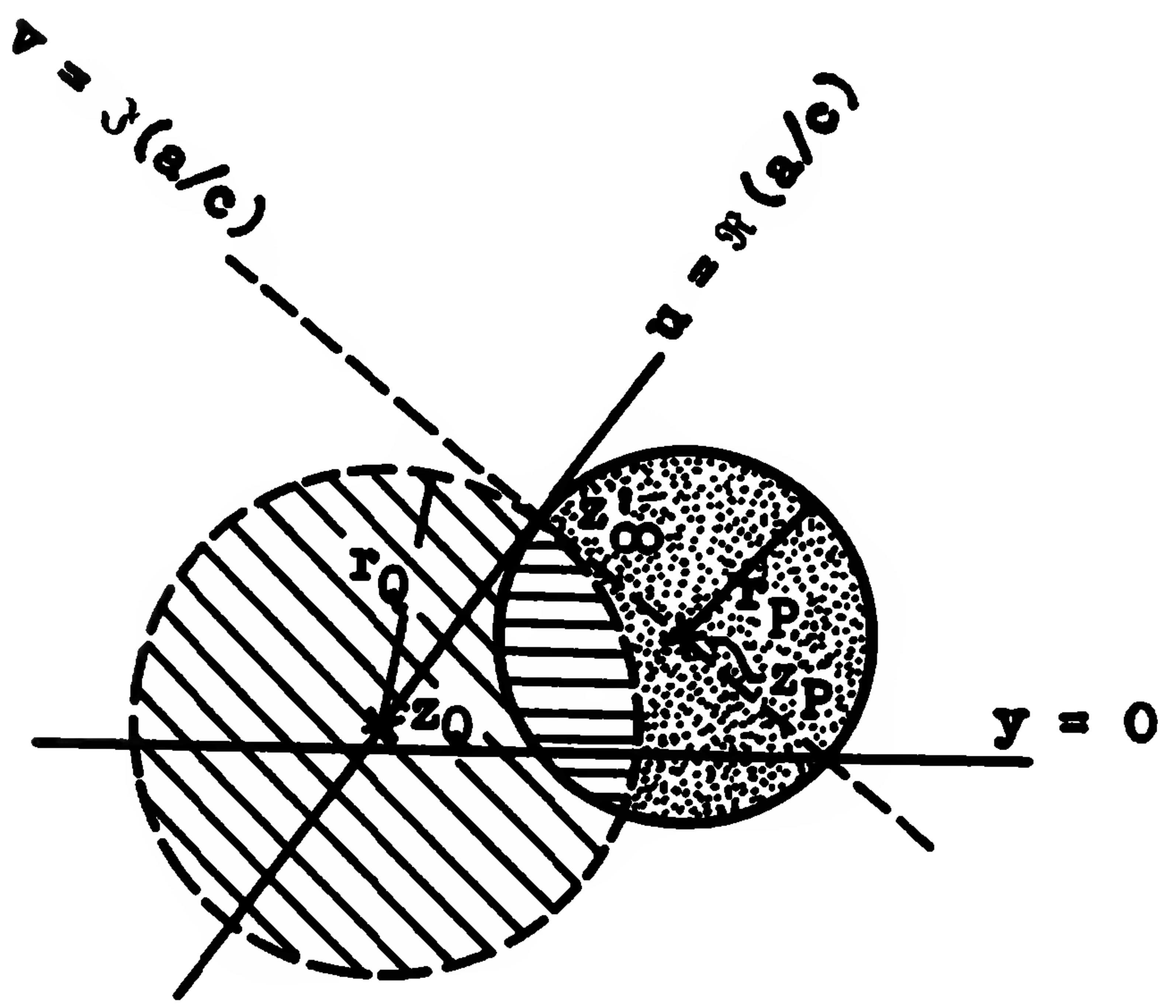
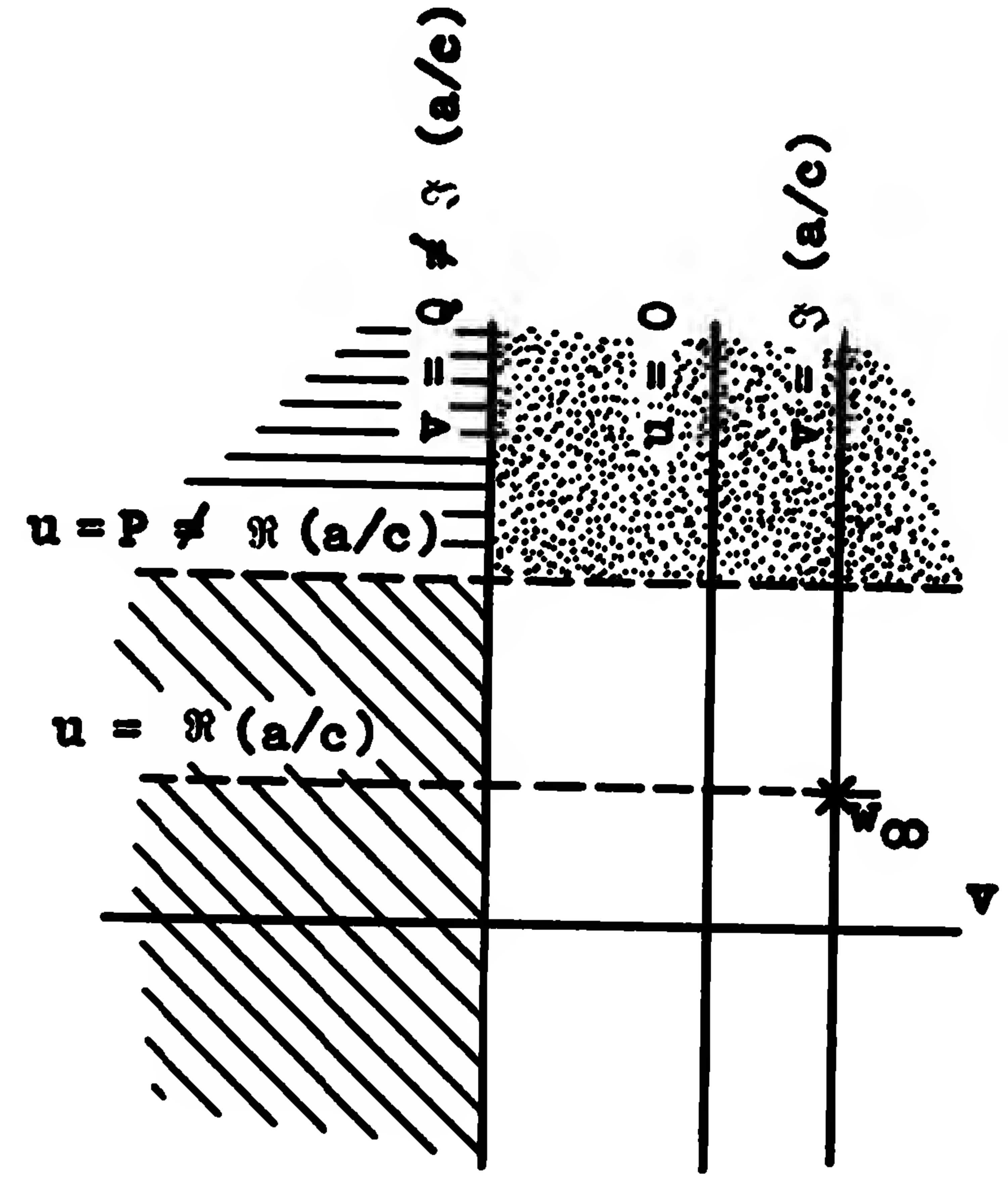
$$R_p = \left|\frac{ad - bc}{2p|c|^2 + 2\Re(\bar c d)}\right|$$

line $y = -\Im(d/c)$ — line ℓ_2: $\Im\left\{c(\bar a\bar d-\bar b\bar c)(cw-a)\right\} = 0$

line $y = q$, $q \neq -\Im(d/c)$ — circle $|w-w_q| = R_q$;

$$w_q = \frac{a\bar c q + \frac{1}{2}i(a\bar d-b\bar c)}{q|c|^2 + \Im(\bar c d)},$$

$$R_q = \left|\frac{ad - bc}{2q|c|^2 + 2\Im(\bar c d)}\right|$$

z - plane	w - plane
	
line $\Re\left\{c(\bar{a}\bar{d}-\bar{b}\bar{c})(cz+d)\right\} = 0$	line $u = \Re\,(a/c)$
circle with centre $z_P = \dfrac{\frac{1}{2}(\bar{a}d+b\bar{c}) - \bar{c}dP}{P\lvert c\rvert^2 - \Re\,(a\bar{c})}$,	line $u = P,\quad P \neq \Re\,(a/c)$
radius $r_P = \left\lvert\dfrac{ad - bc}{2P\lvert c\rvert^2 - 2\Re\,(a\bar{c})}\right\rvert$	
line $\Im\left\{c(\bar{a}\bar{d}-\bar{b}\bar{c})(cz+d)\right\} = 0$	line $v = \Im\,(a/c)$
circle with centre $z_Q = \dfrac{\frac{1}{2}i(\bar{a}d-b\bar{c}) - \bar{c}dQ}{Q\lvert c\rvert^2 - \Im\,(a\bar{c})}$,	line $v = Q,\quad Q \neq \Im\,(a/c)$
radius $r_Q = \left\lvert\dfrac{ad - bc}{2Q\lvert c\rvert^2 - 2\Im\,(a\bar{c})}\right\rvert$	

3.3 **Other lines, and circles.** $z'_\infty = -d/c$, $w_\infty = a/c$.

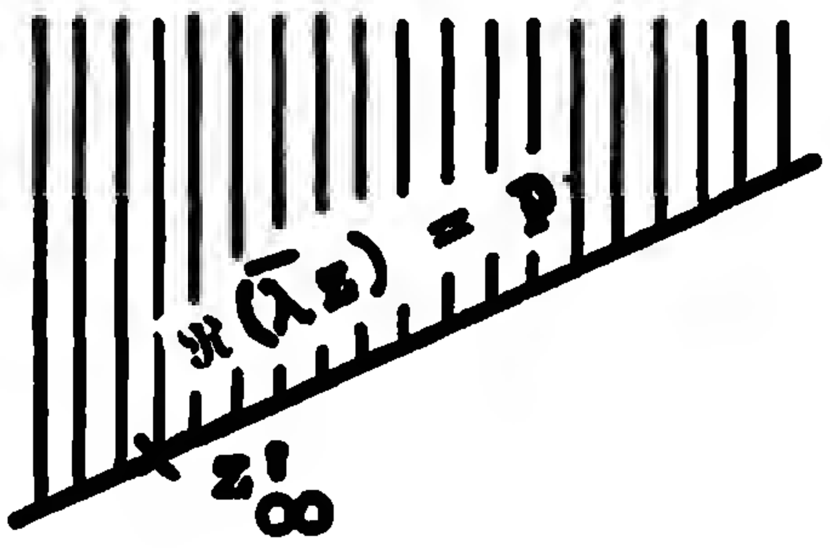
p, r, P, R real.

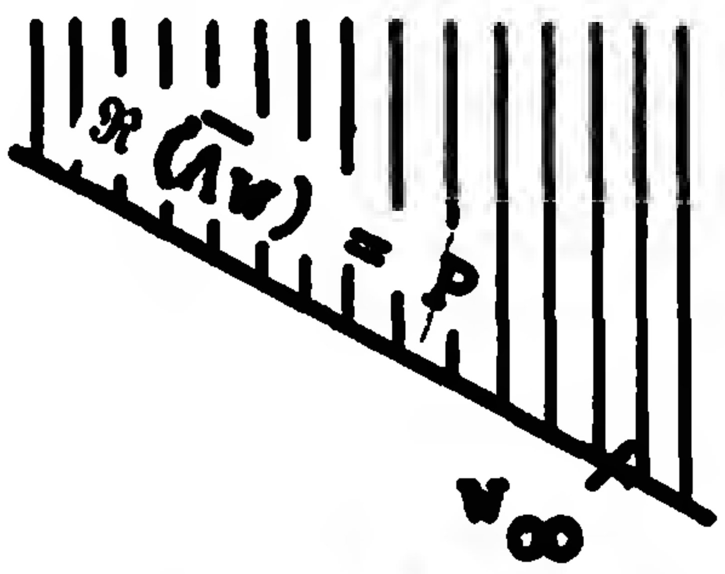
z - plane	w - plane
(i) & (i)' <u>line</u> $\Re(\bar{\lambda}z) = p$ passing through z'_∞	line $\Re(\bar{\Lambda}w) = P$ passing through w_∞
(ii) <u>line</u> $\Re(\bar{\lambda}z) = p$ <u>not</u> passing through z'_∞	circle $\lvert w-w_o\rvert = R$ passing through w_∞

(i) & (i)' $p = \Re(\bar{\lambda}z'_\infty) = -\Re(\bar{\lambda}d/c)$

$\Lambda = \dfrac{ad-bc}{c^2}\bar{\lambda}$; $P = \Re(\bar{\Lambda}w_\infty)$
$= \Re(\bar{\Lambda}\,a/c)$

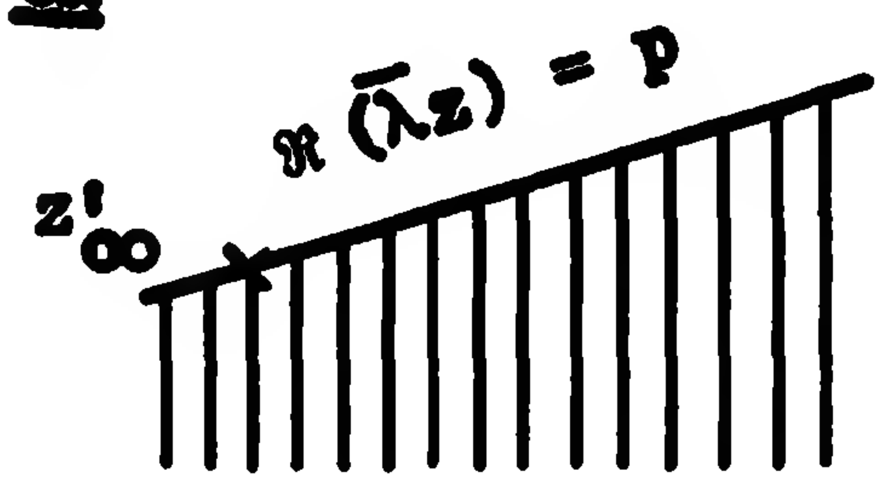
(i)

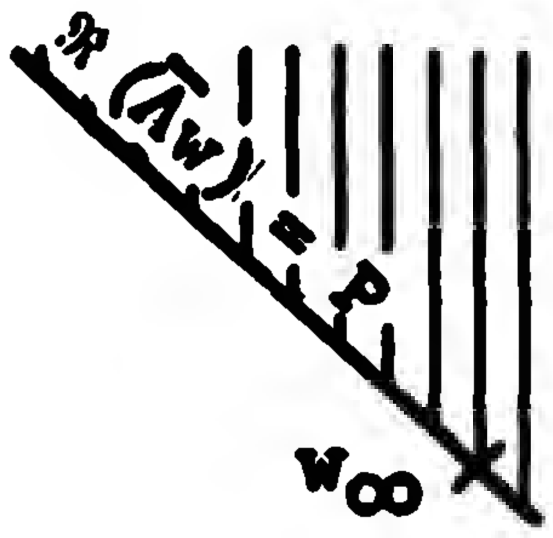




OR

(i)'



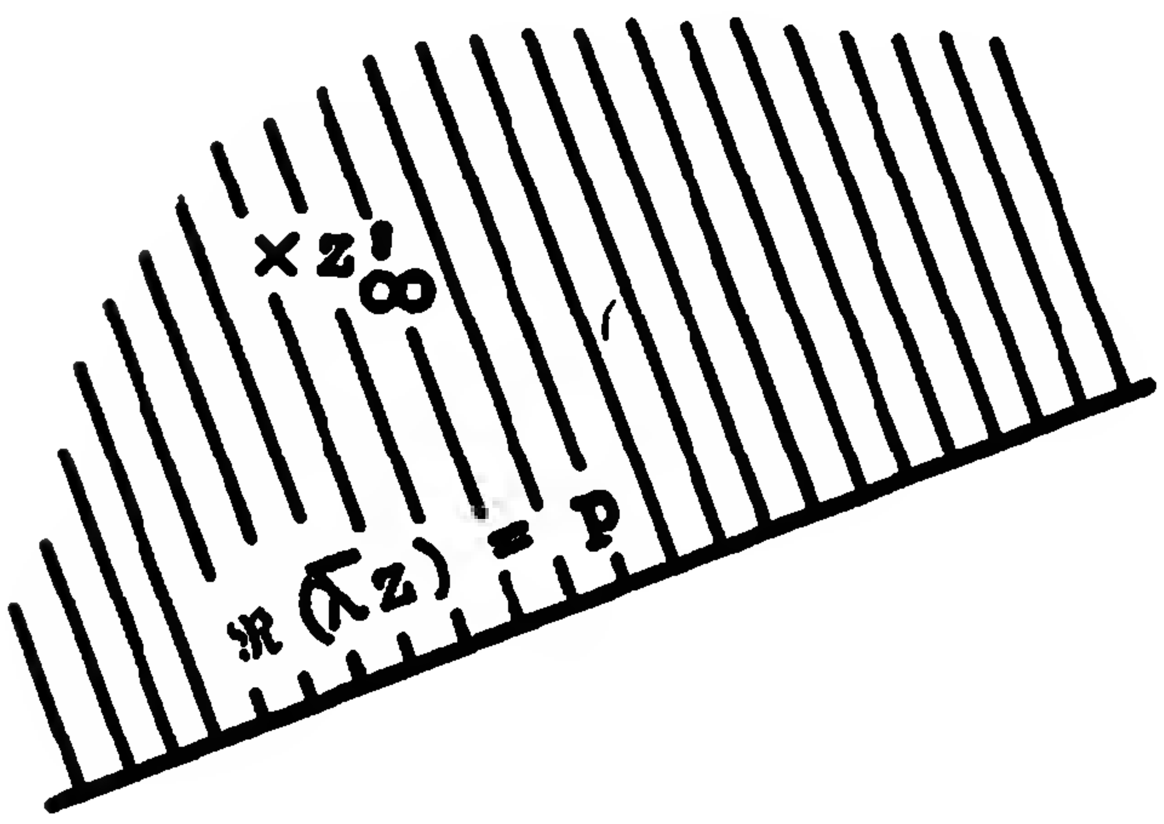


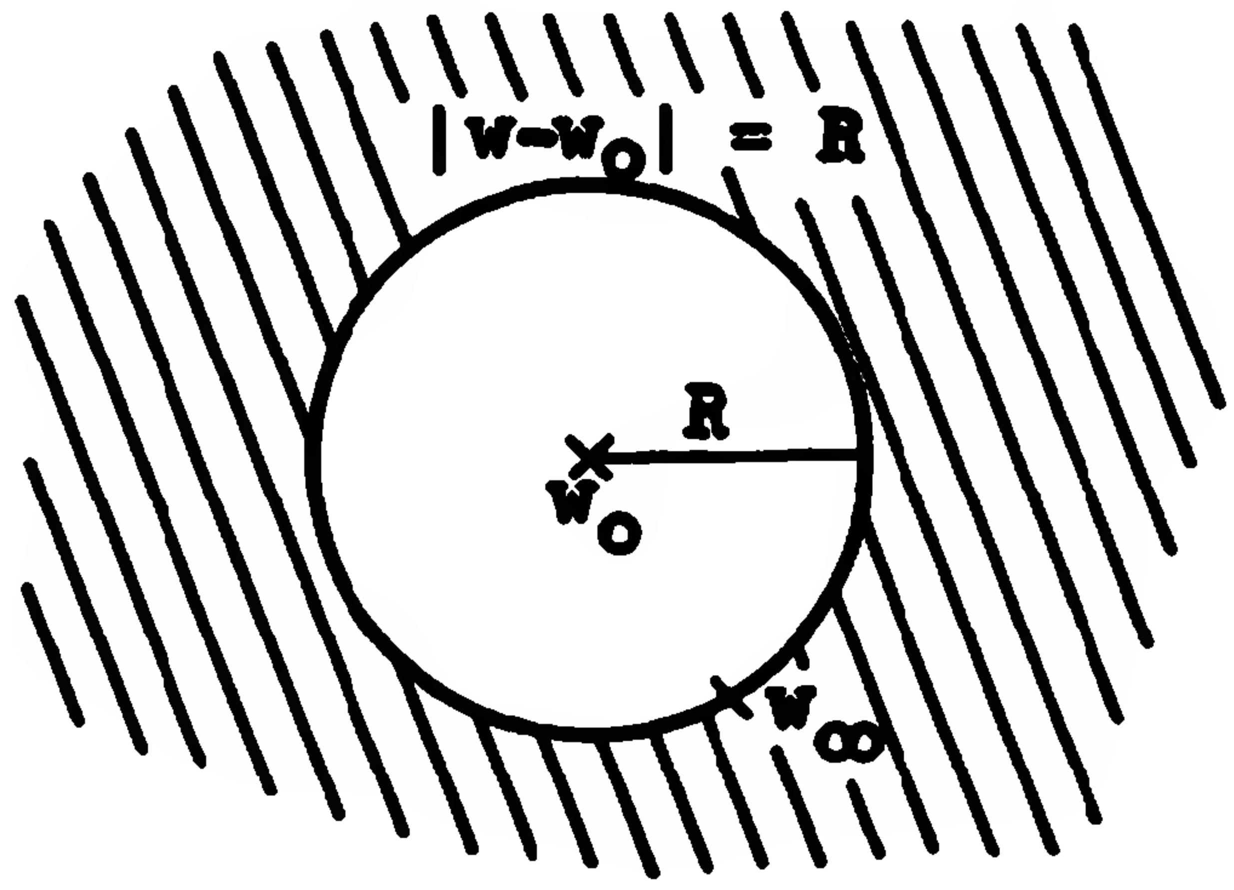
(ii) $p \neq \Re(\bar{\lambda}z'_\infty)$

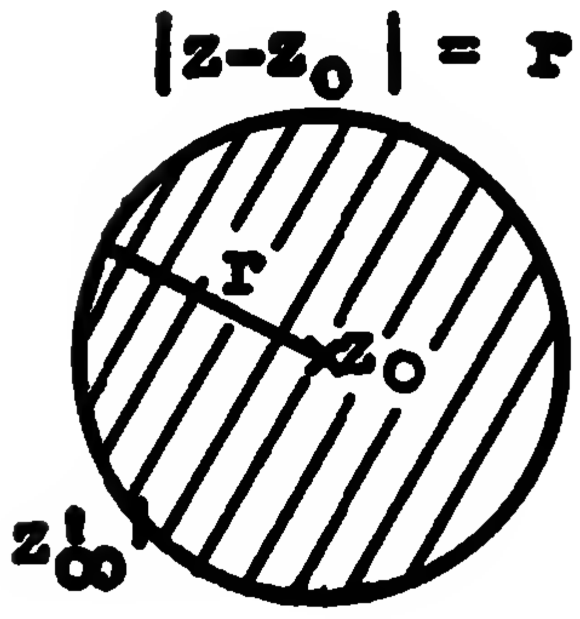
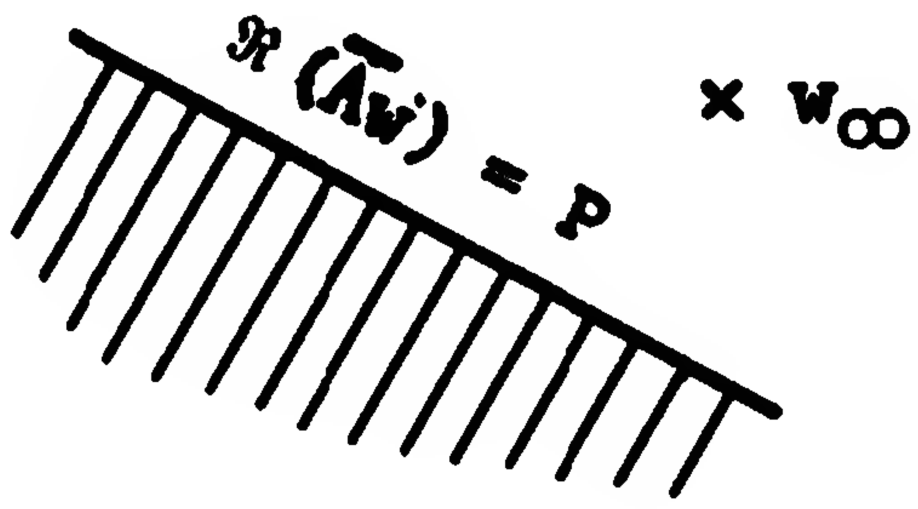
centre $w_o = \dfrac{2ap\bar{c} + a\bar{d}\lambda + b\bar{c}\bar{\lambda}}{2p\lvert c\rvert^2 + c\bar{d}\lambda + \bar{c}d\bar{\lambda}}$

radius $R = \left\lvert \dfrac{(ad-bc)\lambda}{2p\lvert c\rvert^2 + 2\Re(c\bar{d}\lambda)} \right\rvert$

$= \lvert w_\infty - w_o \rvert$





z - plane	w - plane
(iii) circle $\lvert z-z_o\rvert = r$ passing through z'_∞ (iv) circle $\lvert z-z_o\rvert = r$ not passing through z'_∞	line $\Re(\bar{\Lambda}w) = P$ not passing through w_∞ circle $\lvert w-w'_o\rvert = R$ not passing through w_∞
(iii) $r = \lvert z'_\infty - z_o\rvert = \lvert z_o + \frac{d}{c}\rvert$ 	$\Lambda = -\dfrac{h}{c(cz_o+d)}$, where $h = ad-bc$; $P = \dfrac{-\lvert h\rvert^2 - 2\Re\left\{c(az_o+b)\bar{h}\right\}}{2\lvert c(cz_o+d)\rvert^2}$ $= -\frac{1}{2}\lvert\Lambda\rvert^2 + \Re\left(\bar{\Lambda}\,\dfrac{az_o+b}{cz_o+d}\right)$ Also $\Lambda = w_{oo} - w_\infty$, $P = \frac{1}{2}\Re\left\{\bar{\Lambda}[w_{oo} + w_\infty]\right\}$, where $w_{oo} = (az_o+b)(cz_o+d)^{-1}$ 
(iv) $\lvert z_o + \frac{d}{c}\rvert \neq r$	$w'_o = \dfrac{(az_o+b)(\bar{c}\bar{z}_o+\bar{d}) - a\bar{c}r^2}{\lvert cz_o+d\rvert^2 - \lvert c\rvert^2 r^2}$ $R = \dfrac{r\lvert ad-bc\rvert}{\left\lvert\lvert cz_o+d\rvert^2 - \lvert c\rvert^2 r^2\right\rvert}$; also:

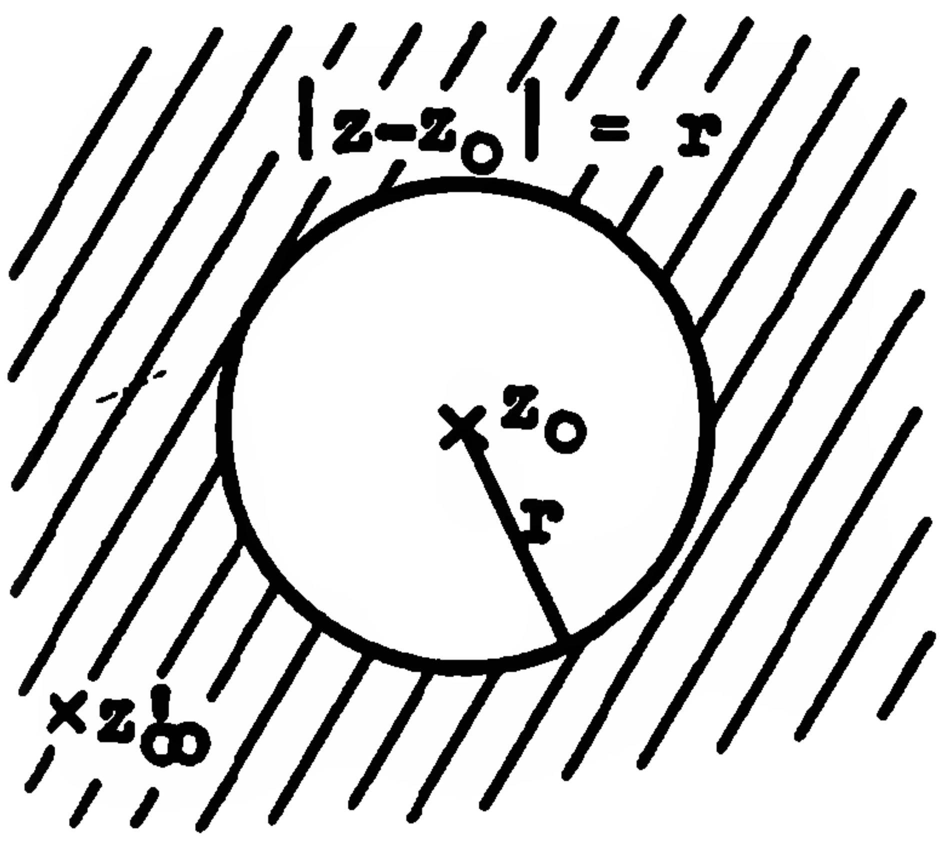
Special cases

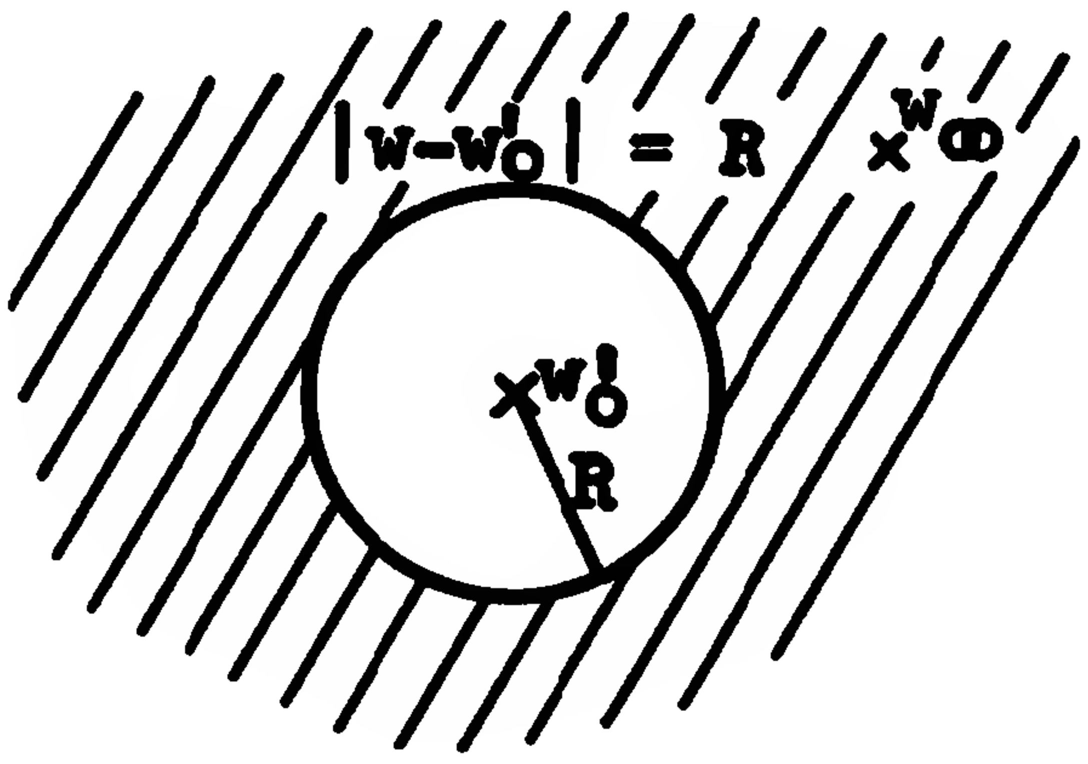
(iva)

$|cz_o+d| > |c|r$

$w_o' = \dfrac{az_o + b - \bar{c}r^2s}{cz_o + d}$; $R = r|s|$,

where $s = \dfrac{ad - bc}{|cz_o+d|^2 - |cr|^2}$

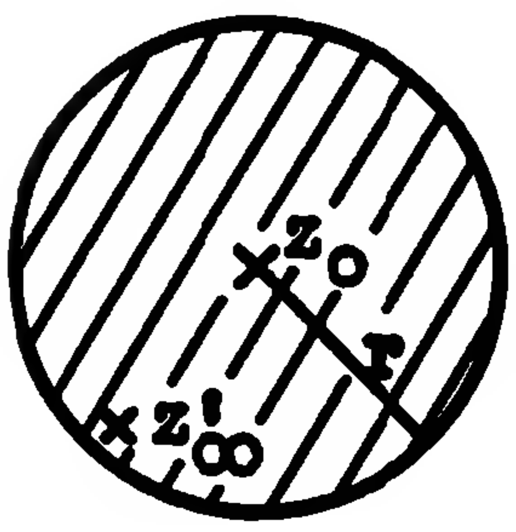


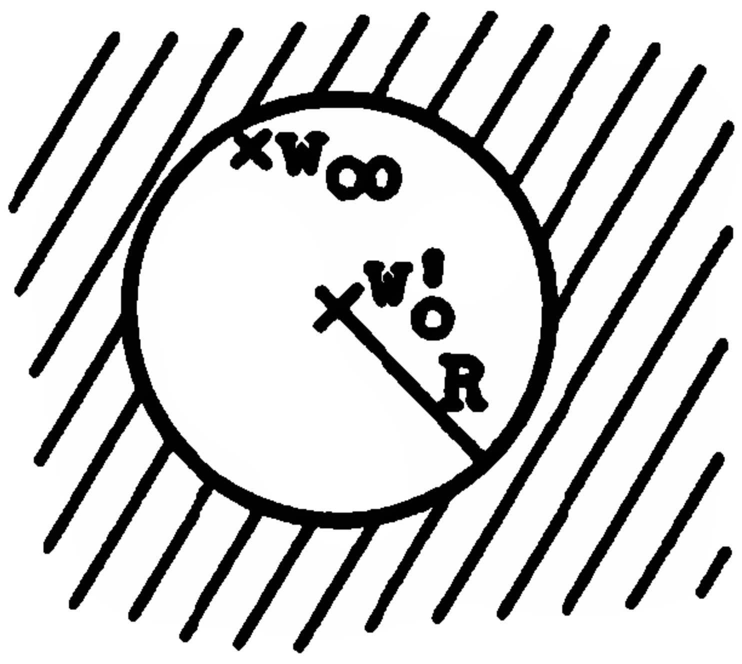


(ivb)

$0 < |cz_o + d| < |c|r$

w_o', R as above

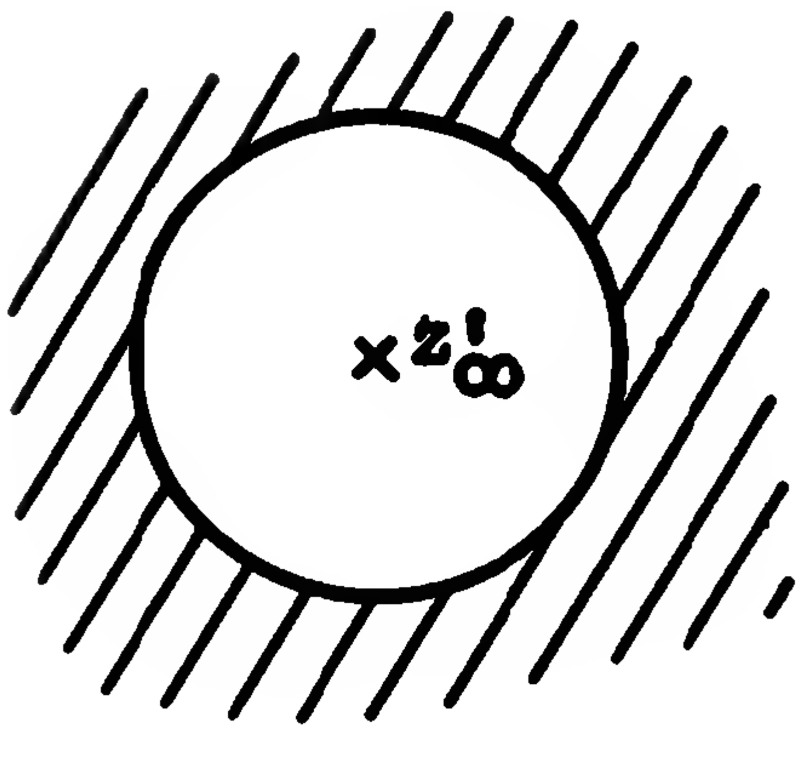


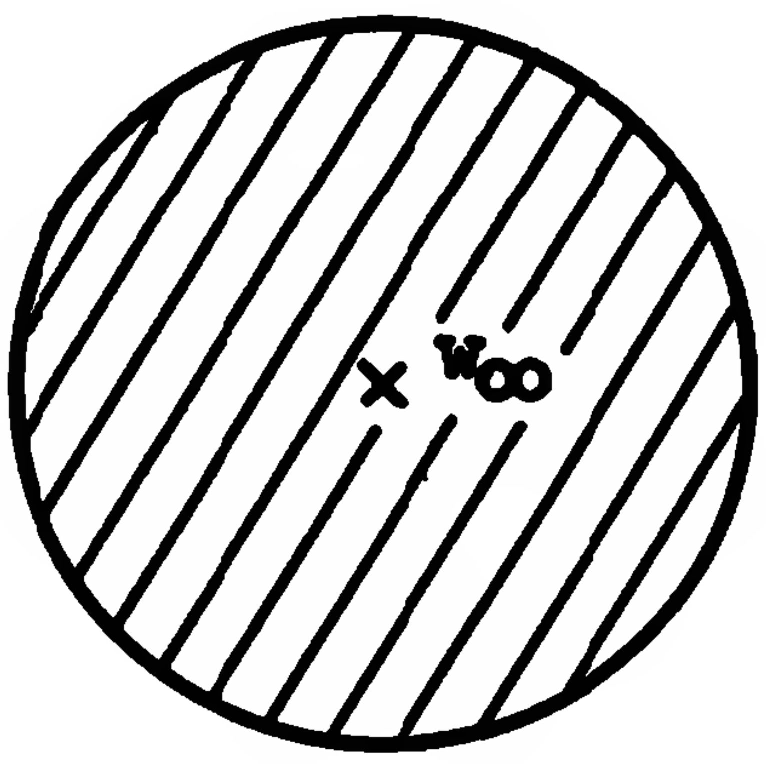


(iv')

$z_o = -d/c = z_\infty'$

$w_o' = a/c = w_\infty$; $R = \left|\dfrac{ad-bc}{c^2r}\right|$

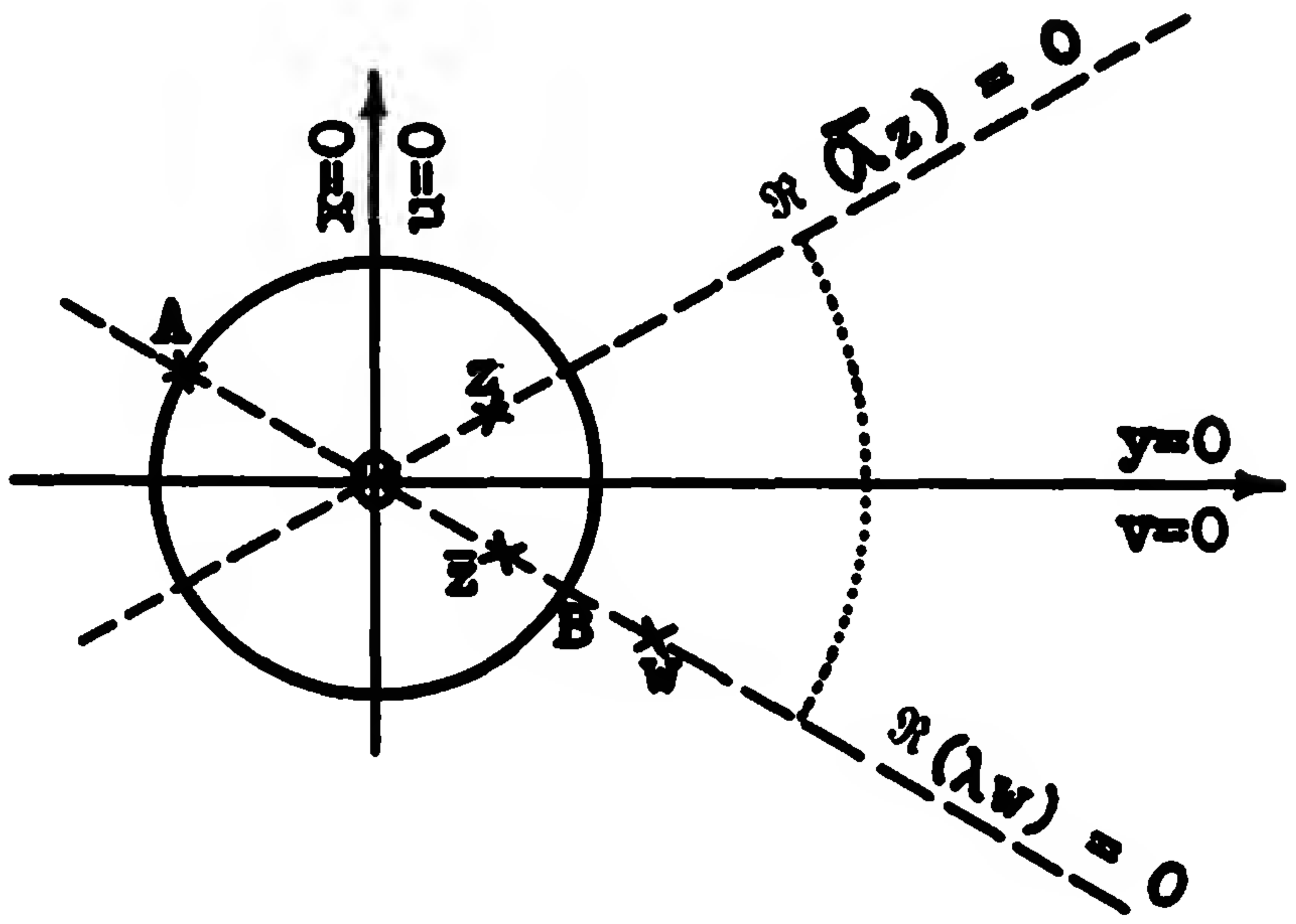




4. SPECIAL BILINEAR TRANSFORMATIONS.

4.1 * $\boxed{w = \frac{1}{z}}$ The transformation is involutory (See §3.1).

$F_1 = 1, \quad F_2 = -1.$



z-plane
w-plane

Radius $|OA| = 1$.

$\angle\, z0w$ bisected by x-axis.

$A\bar{z}:\bar{z}B = Aw:Bw$

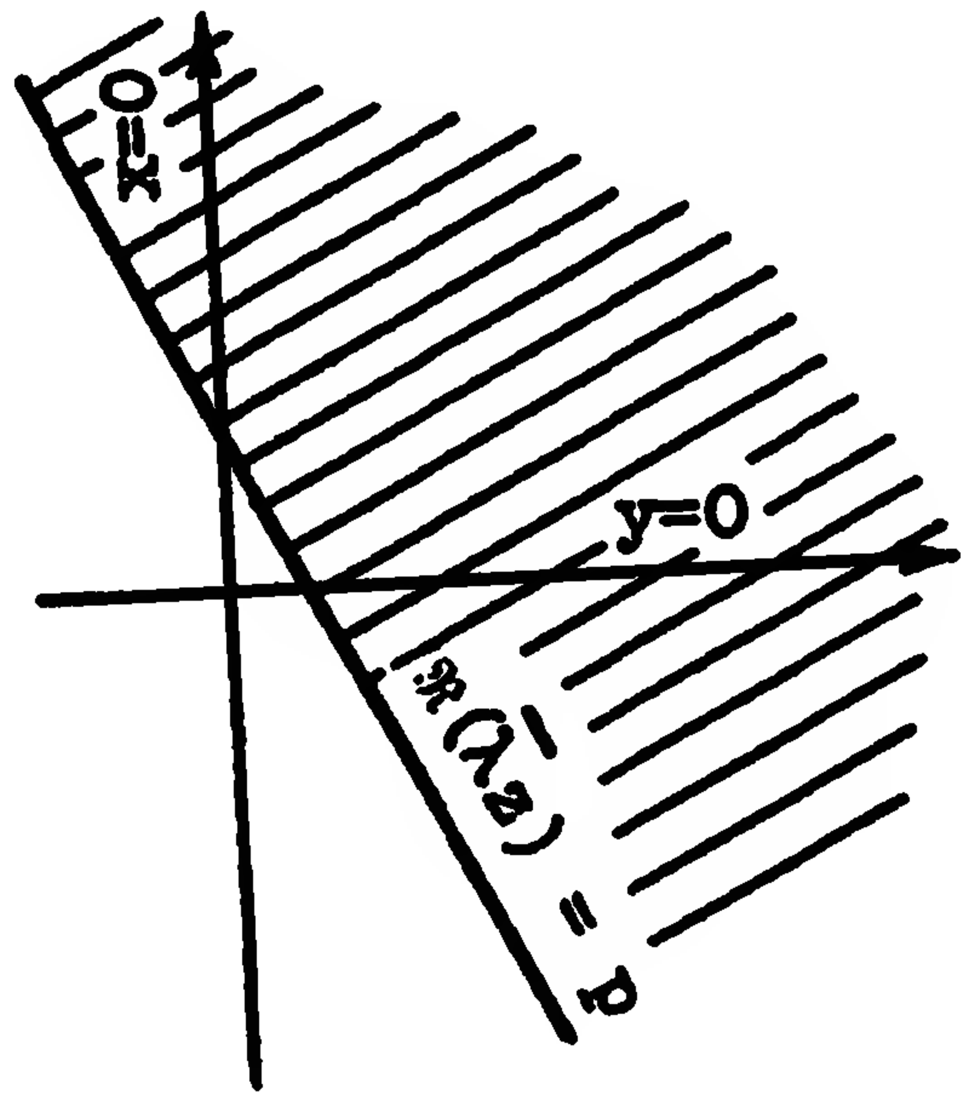
The points $\bar{z}$ and w are "inverse" with respect to the unit circle.

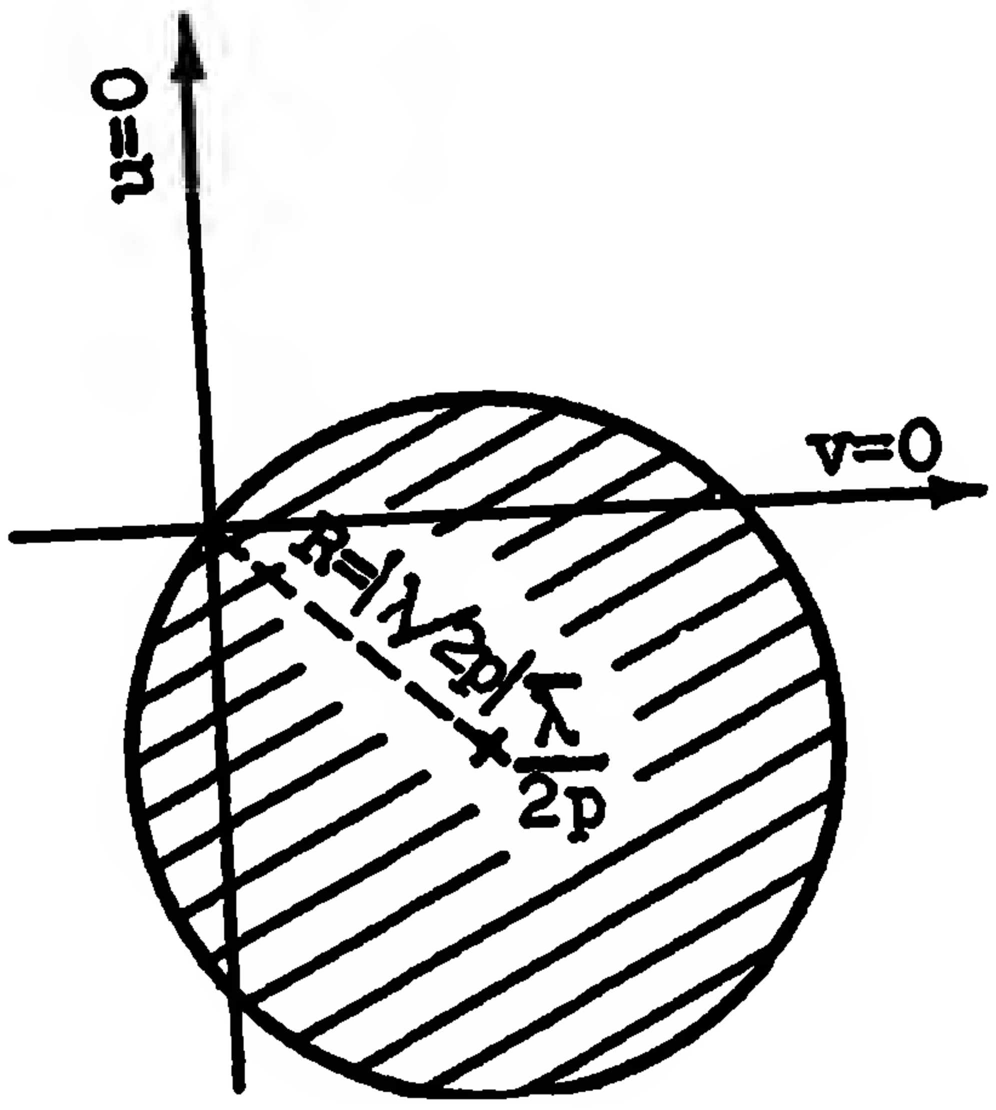
p, q, r real.

z - plane	w - plane
points $z'_\infty = 0$; ∞; i; - i	points ∞; $w_\infty = 0$; -i; +i
* interior of "circle" of D (see §1)	interior of the same "circle"
* exterior of "circle" of E (see §1)	interior of the same "circle"
(i) line $\Re(\bar{\lambda}z) = 0$	line $\Re(\lambda w) = 0$
(ii) line $\Re(\bar{\lambda}z) = p \neq 0$	circle $\left\|w - \frac{\bar{\lambda}}{2p}\right\| = \left\|\frac{\lambda}{2p}\right\|$
(iii) circle $\|z-z_o\| = \|z_o\| \neq 0$	line $\Re(wz_o) = \frac{1}{2}$
(iv) circle $\|z-z_o\| = r \neq \|z_o\|$	circle $\left\|w - \frac{\bar{z}_o}{\|z_o\|^2 - r^2}\right\| = \frac{r}{\left\|\|z_o\|^2 - r^2\right\|}$
(v) circle $\|z\| = r$	circle $\|w\| = \frac{1}{r}$

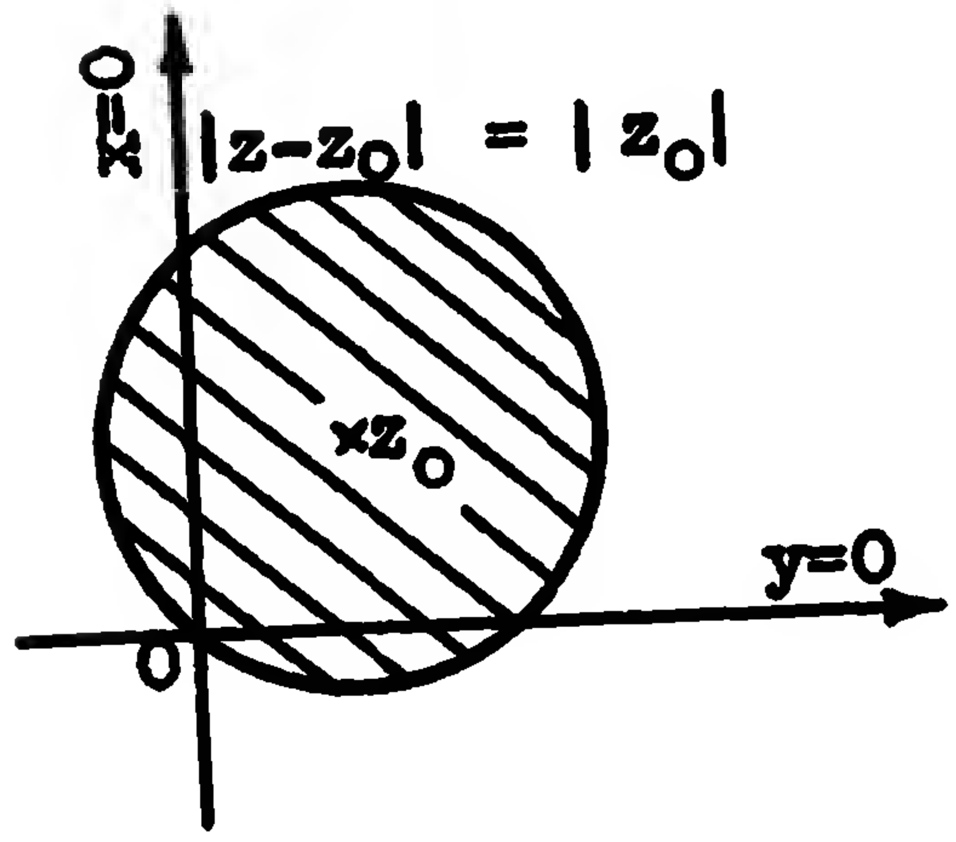
	z - plane	w - plane

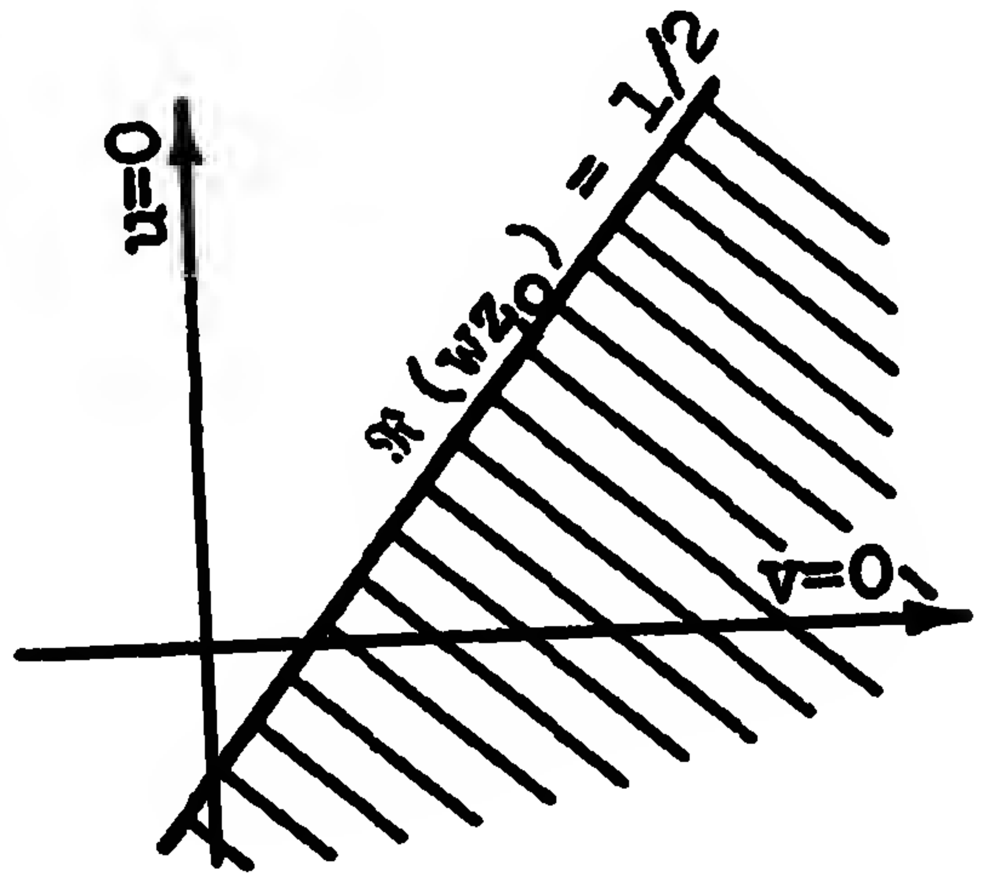
(ii)



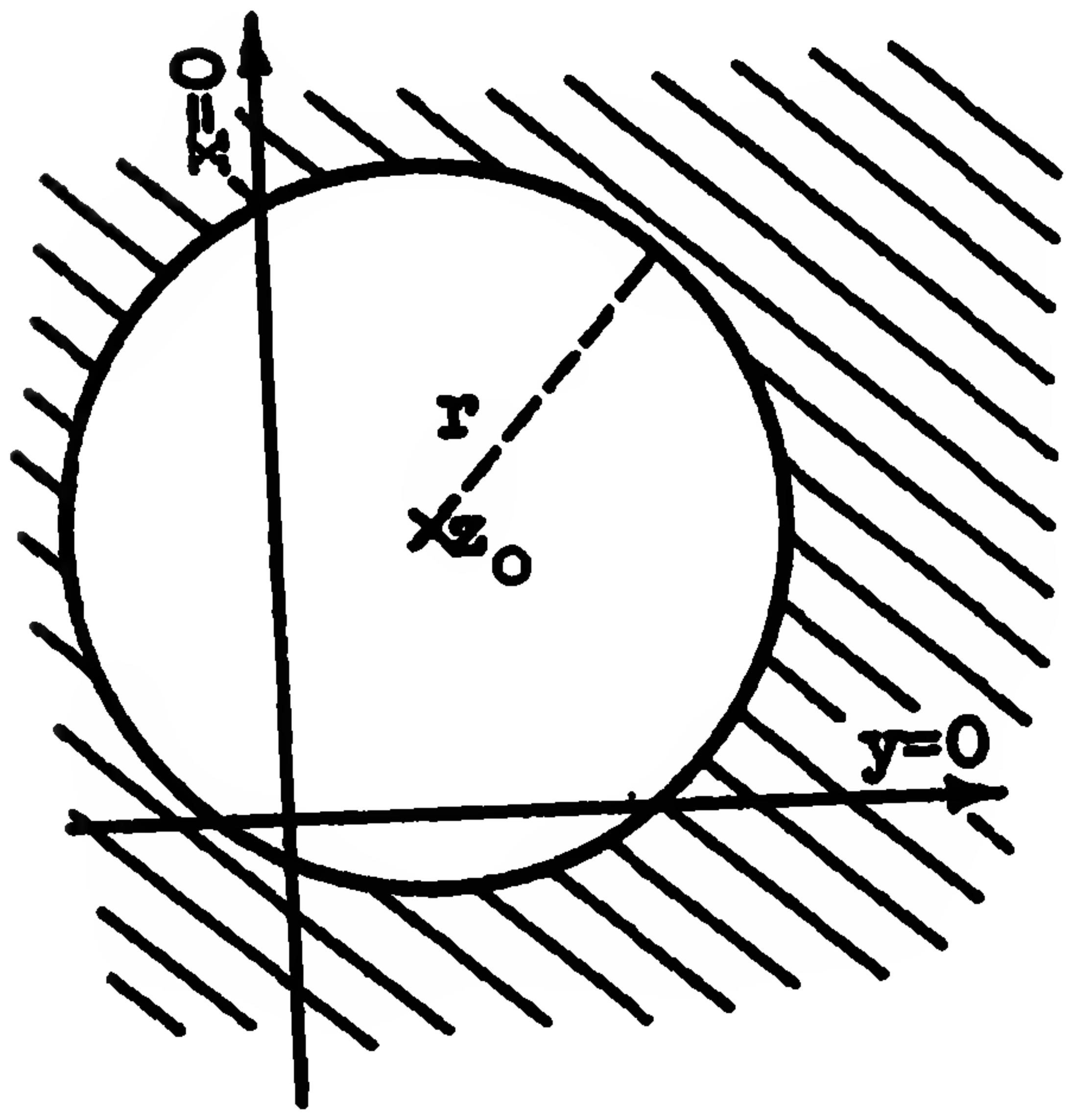


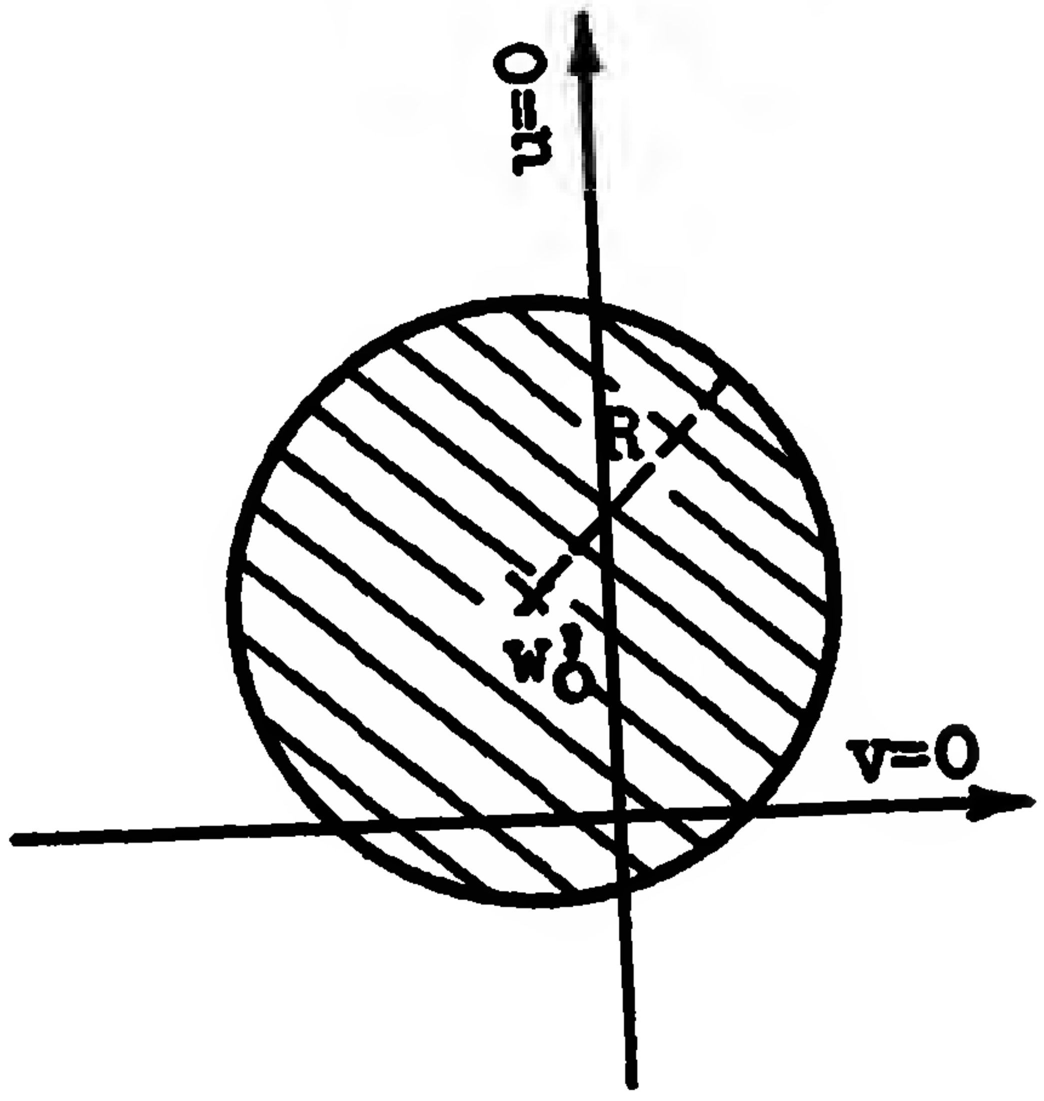
(iii)





(iv)





$$w'_o = \frac{\bar{z}_o}{|z_o|^2 - r^2}$$

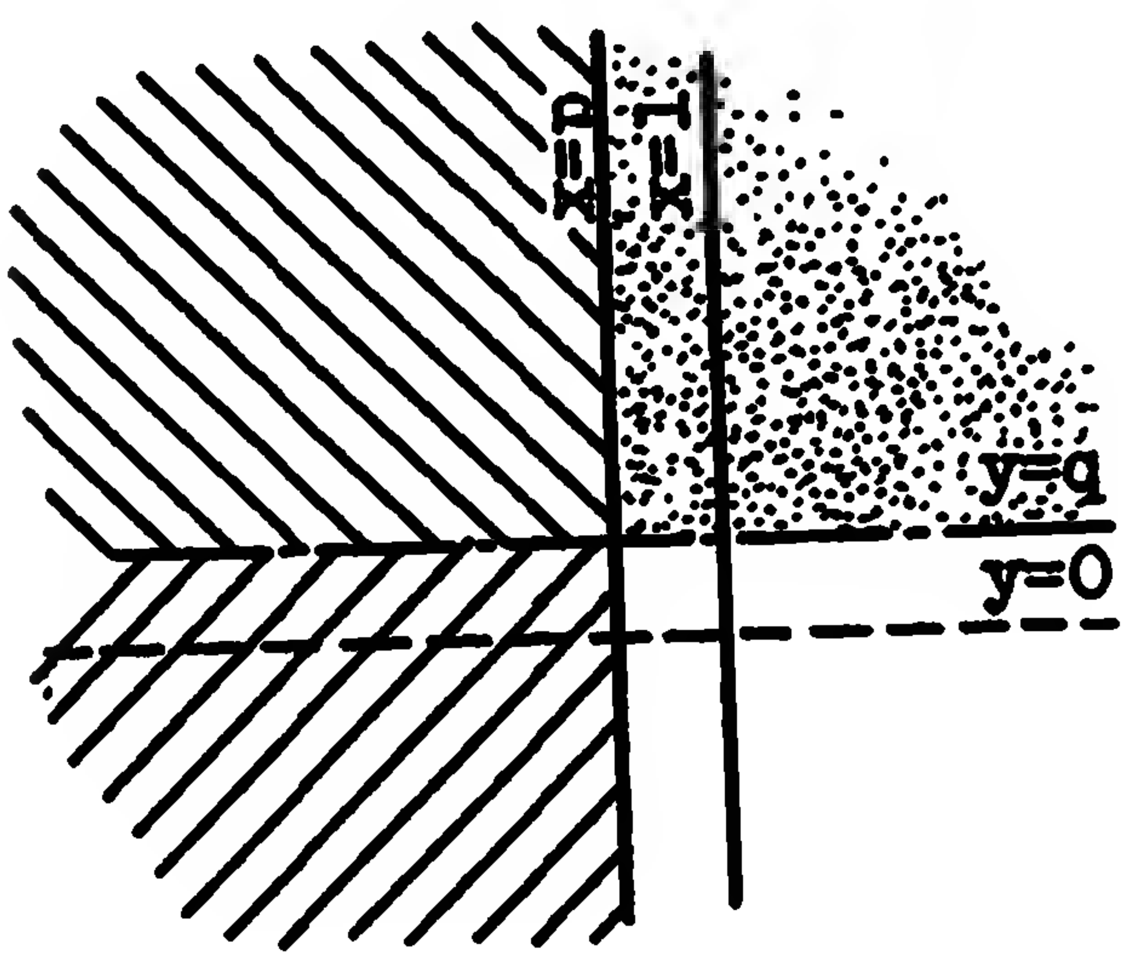
$$R = \frac{r}{\left||z_o|^2 - r^2\right|}$$

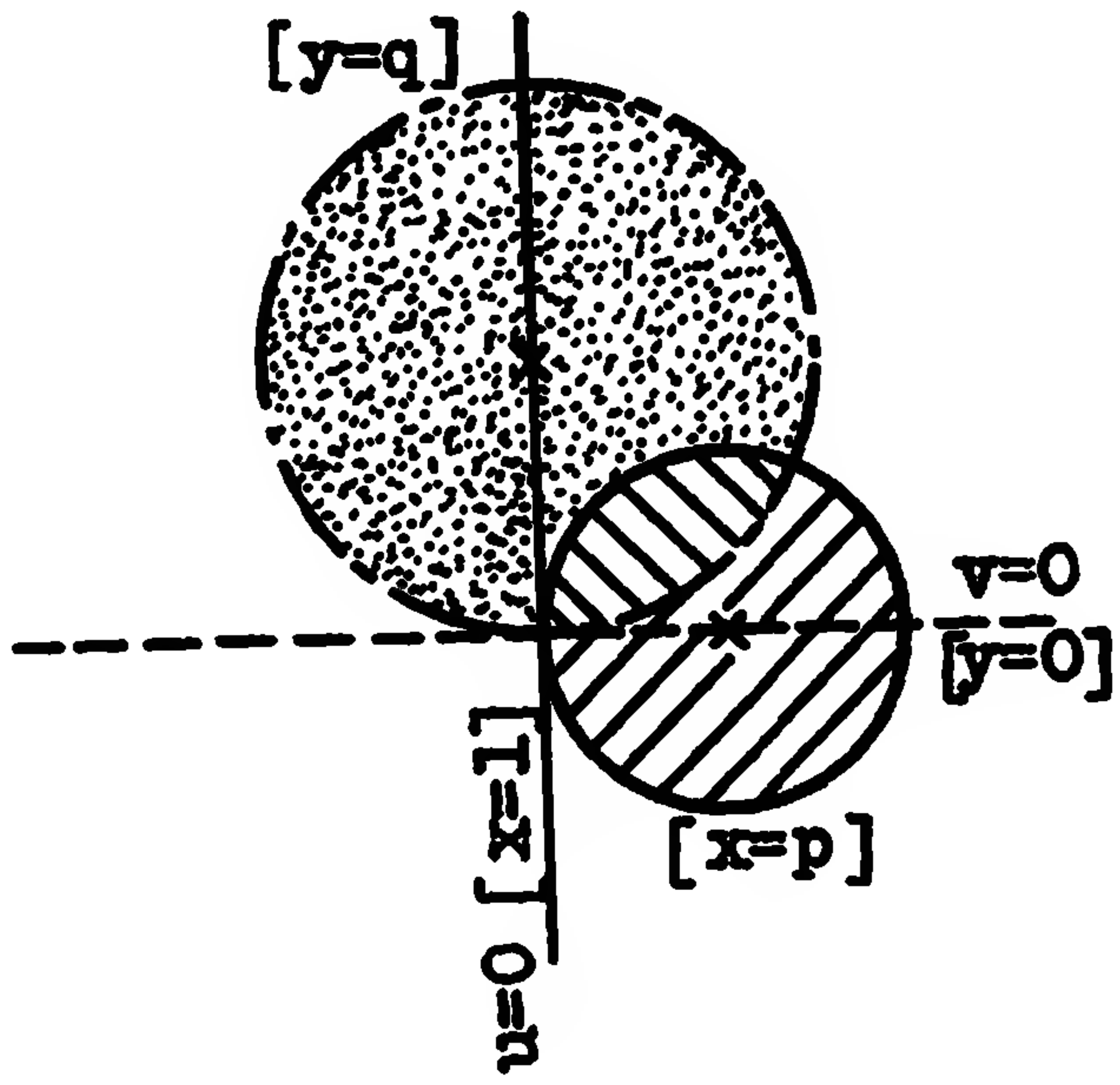
4.2 $\boxed{w = \frac{1}{1-z}}$ $z = \frac{w-1}{w}$

$z'_{\infty} = 1;$ $w_{\infty} = 0;$ $*F_1 = \frac{1}{2} + \frac{1}{2}\, i\sqrt{3}$, $*F_2 = \frac{1}{*F_1}$

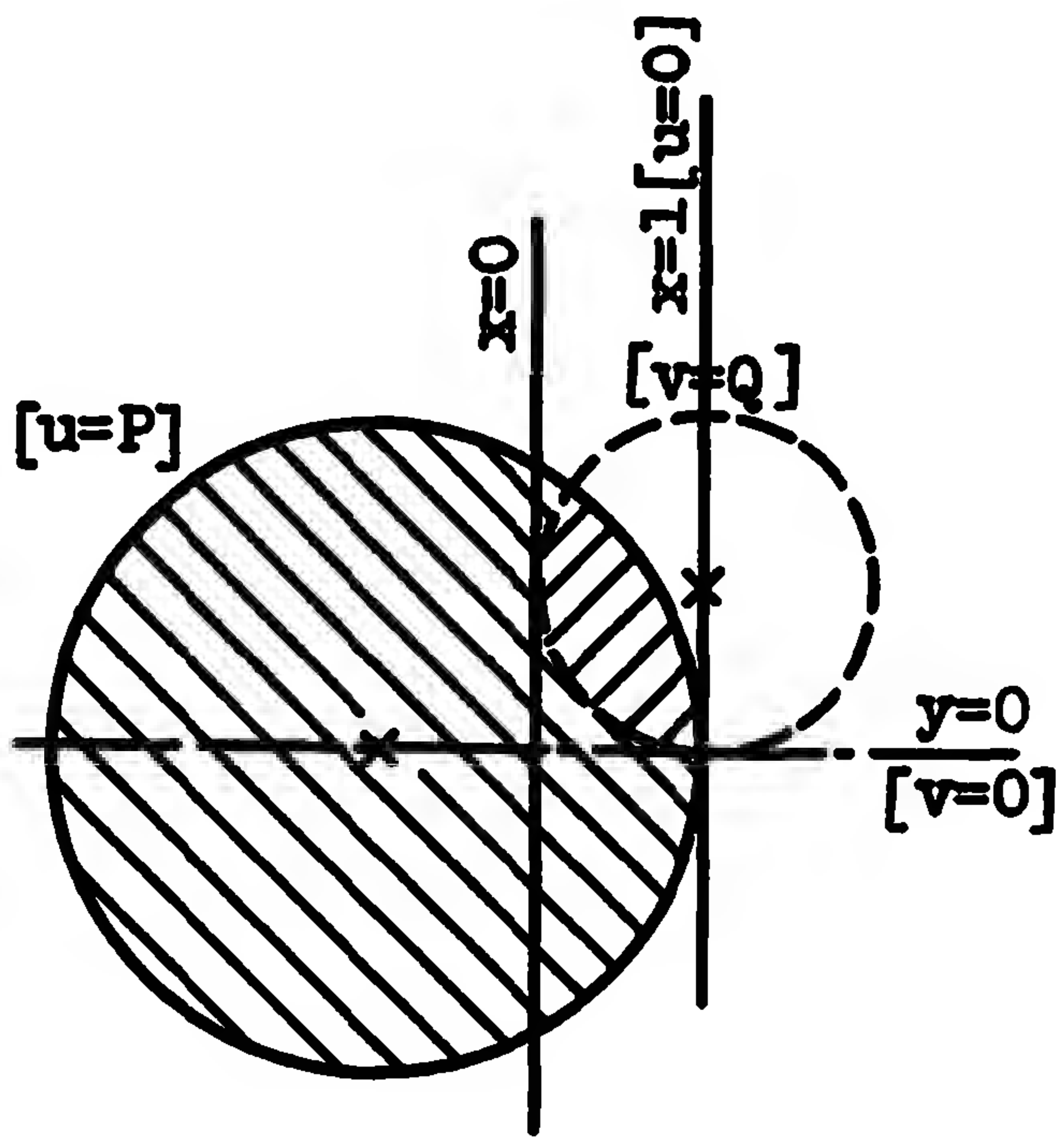
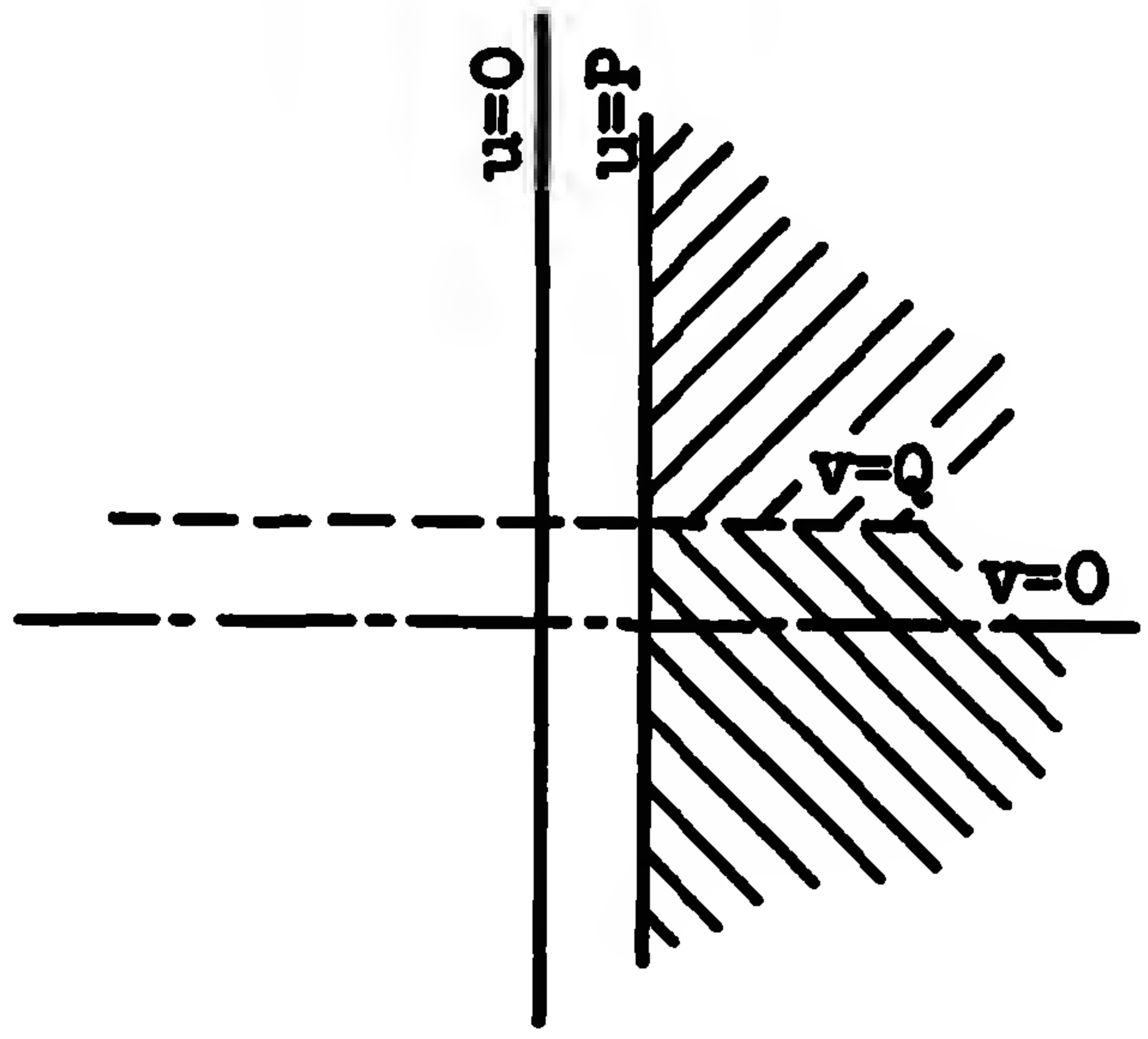
$p, q,$ P, Q real.

z - plane	w - plane
half-plane $y \geq 0$	half-plane $v \geq 0$
half-plane $x \geq 0$	domain $\lvert w - \frac{1}{2}\rvert \geq \frac{1}{2}$
domain $\lvert z\rvert \leq 1$	half-plane $u \geq \frac{1}{2}$
domain $\lvert z-1\rvert \leq 1$	domain $\lvert w\rvert \geq 1$
*interior of circle of D (see §1)	interior of circle of D





line $x = 1$	line $u = 0$
line $x = p \neq 1$	circle $\left\lvert w - \frac{1}{2(1-p)}\right\rvert = \frac{1}{2\lvert 1-p\rvert}$
line $y = q \neq 0$	circle $\left\lvert w - \frac{1}{2q}\right\rvert = \frac{1}{2\lvert q\rvert}$

z - plane	w - plane
	
circle $\lvert z - (1 - \frac{1}{2P}) \rvert = \frac{1}{2\lvert P \rvert}$	line $u = P \neq 0$
circle $\lvert z - (1 + \frac{i}{2Q}) \rvert = \frac{1}{2\lvert Q \rvert}$	line $v = Q \neq 0$

4.3 $\boxed{w = \frac{z}{z-1}}$ $z = \frac{w}{w-1}$. The transformation is involutory (see §3.1).

$z'_\infty = 1, \qquad w_\infty = 1; \qquad {}^*F_1 = 0, \quad {}^*F_2 = 2.$

p, q real.

z - plane	w - plane
half-plane $y \geq 0$	half-plane $v \leq 0$
half-plane $x \geq 0$	domain $\lvert w - \frac{1}{2} \rvert \geq \frac{1}{2}$
domain $\lvert x \rvert \leq 1$	half-plane $u \leq \frac{1}{2}$
half-plane $x \leq \frac{1}{2}$	domain $\lvert w \rvert \leq 1$
* exterior of "circle" of E (see §1)	interior of "circle" of E
* interior of "circle" of D (see §1)	interior of "circle" of D
line $x = 1$	line $u = 1$, passing through $w = 1$

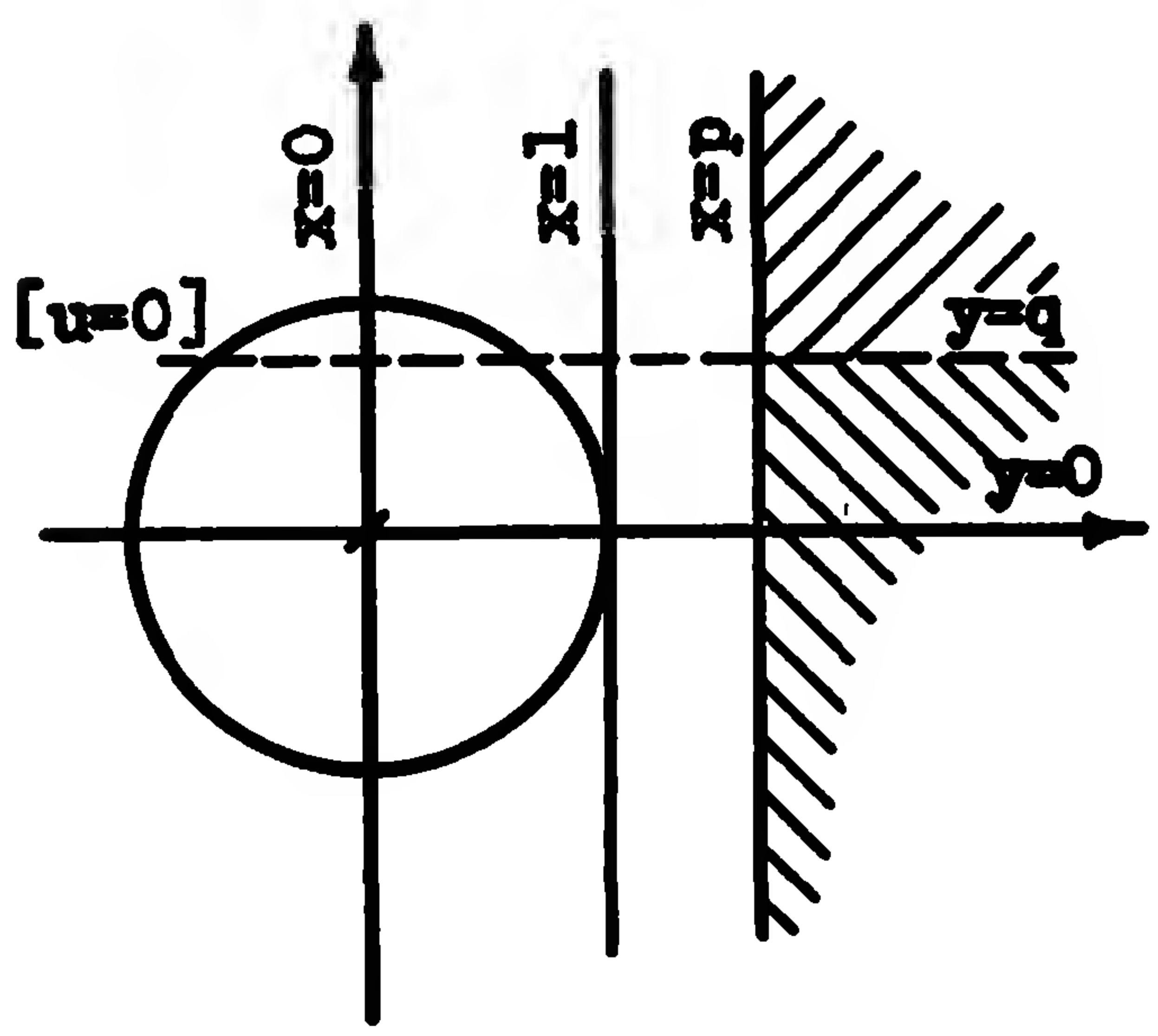
z - plane	w - plane
line $x = p$ $(p \neq 1)$	circle $\left\| w - \frac{2p-1}{2(p-1)} \right\| = \frac{1}{\|2p-2\|}$, passing through $w = 1$
line $y = q$ $(q \neq 0)$	circle $\left\| w - (1 - \frac{i}{2q}) \right\| = \frac{1}{2\|q\|}$, passing through $w = 1$.

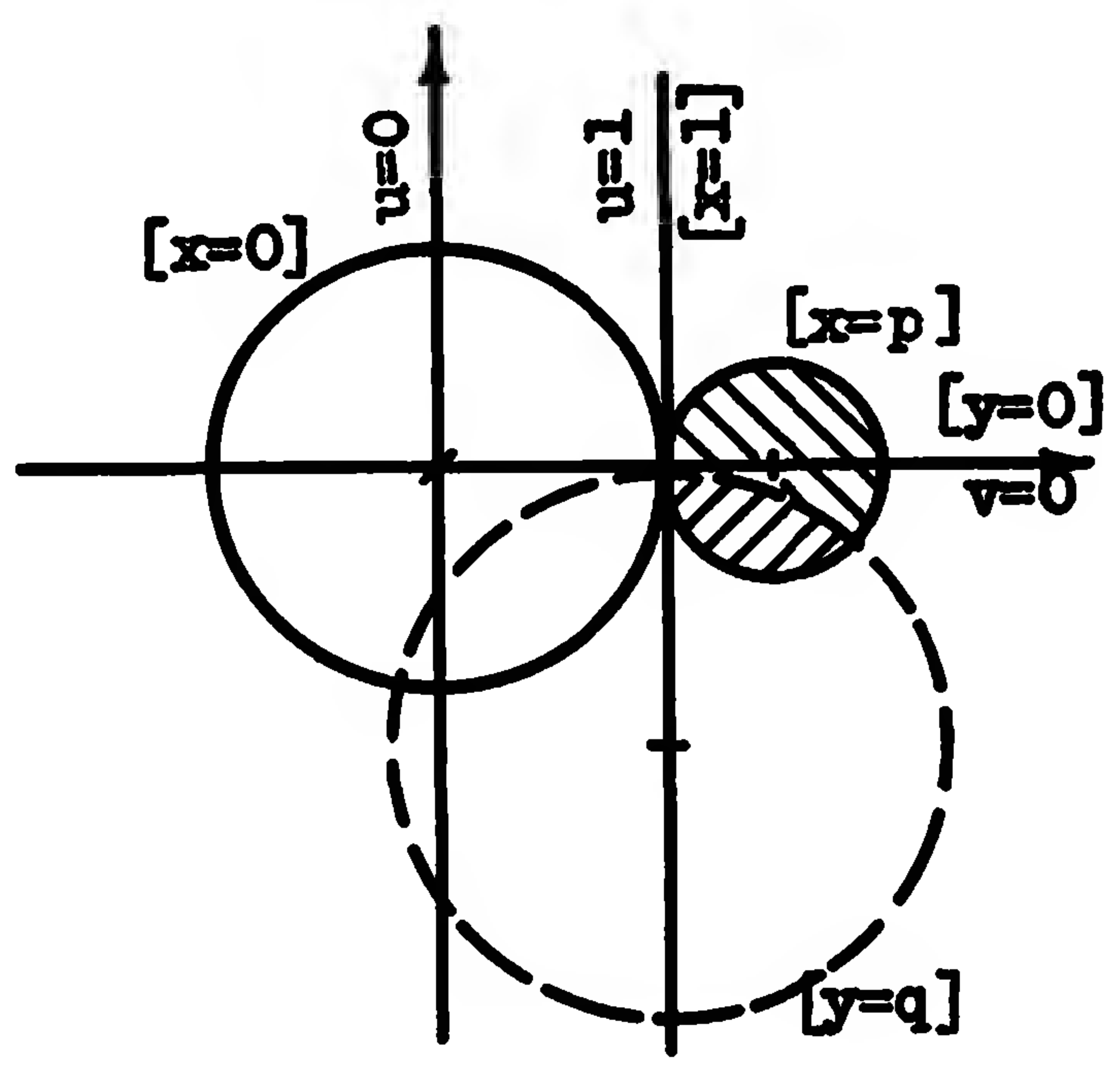
4.4 $\boxed{w = \frac{z+1}{z-1}}$ The transformation is involutory (see §3.1)

$$*F_{\substack{1\\2}} = 1 \pm \sqrt{2}$$

p, q real.

z - plane	w - plane
points $z = 0$; $z'_{\infty} = 1$; ∞	points $w = -1$; ∞; $w_{\infty} = 1$
half-plane $x \geq 0$	domain $\|w\| \geq 1$
half-plane $y \geq 0$	half-plane $v \leq 0$
domain $\|z\| \leq 1$	half-plane $u \leq 0$
*interior of "circle" of D (see §1)	interior of the same "circle"
*exterior of "circle" of E (see §1)	interior of the same "circle"
line $x = 1$, belonging to D	line $u = 1$





z - plane	w - plane
line $x = p$ $(p \neq 1)$	circle $\left\lvert w - \frac{p}{p-1} \right\rvert = \frac{1}{\lvert p-1 \rvert}$
line $y = q$ $(q \neq 0)$	circle $\left\lvert w - \frac{q-i}{q} \right\rvert = \frac{1}{\lvert q \rvert}$

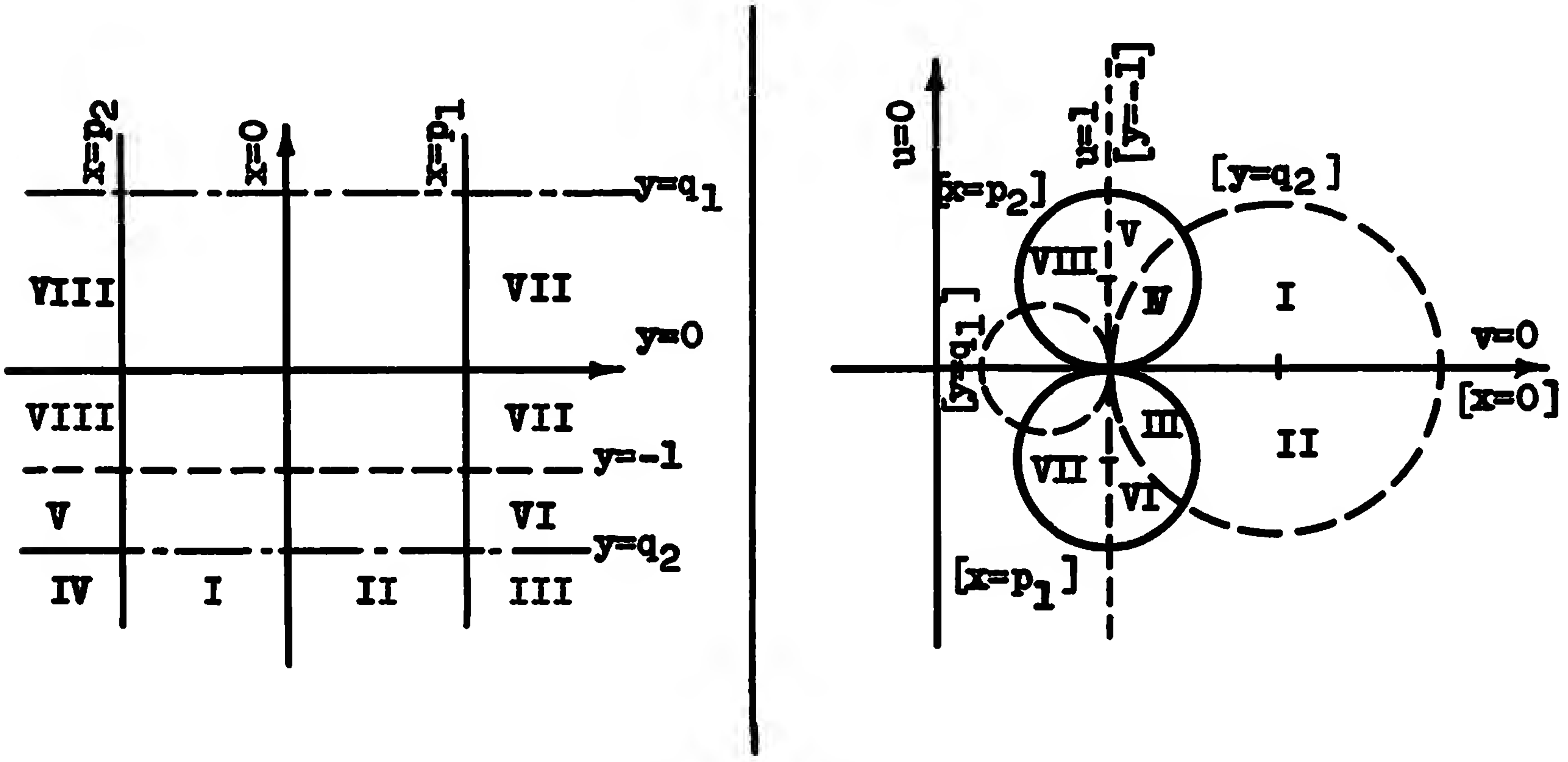
4.5 $\boxed{w = \frac{z-i}{z+i}}$ $z = i\,\frac{1+w}{1-w}$

$*F_{\substack{1\\2}} = \frac{1}{2}(1-i)(1 \pm \sqrt{3})$

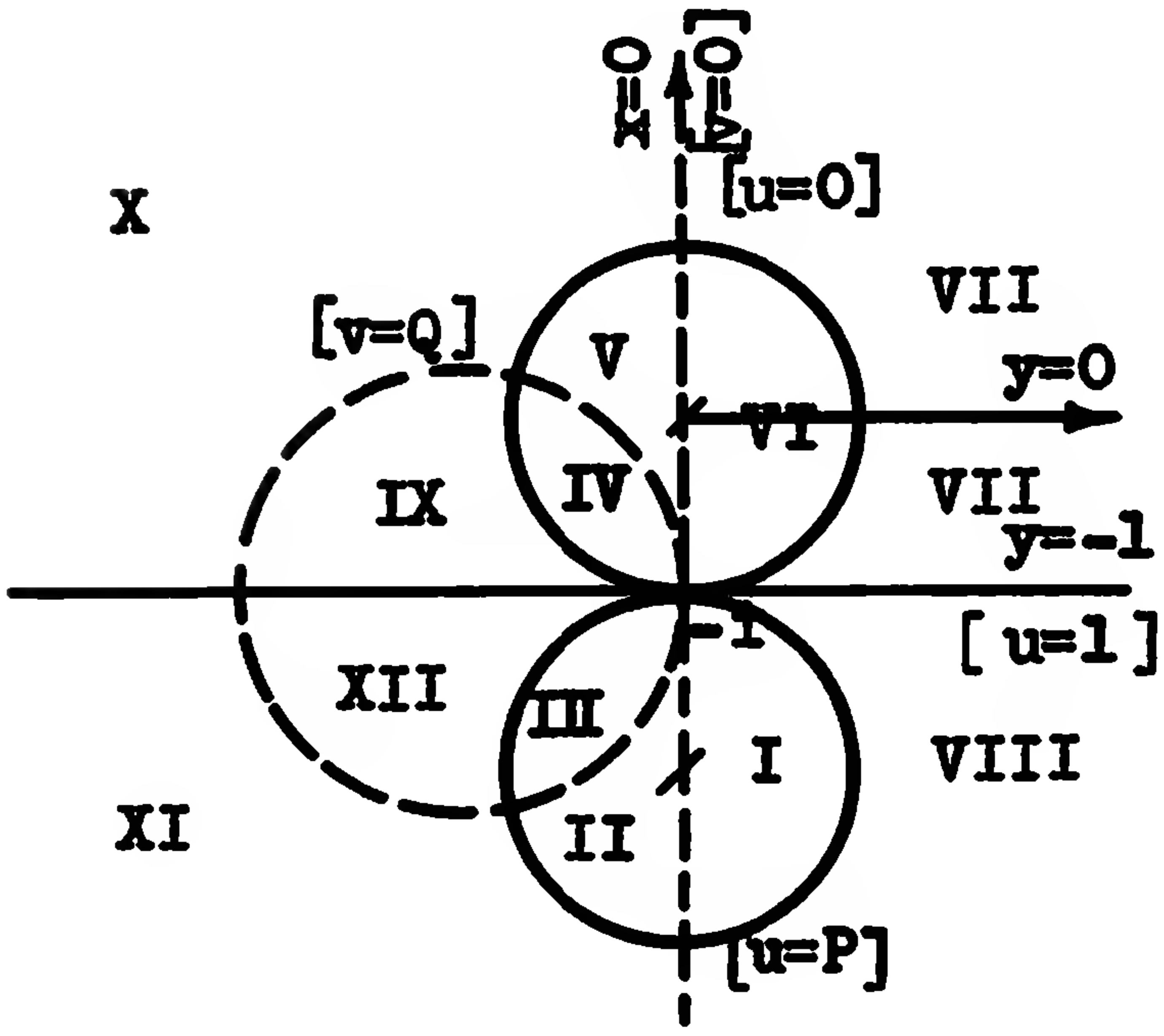
p, q, P, Q real.

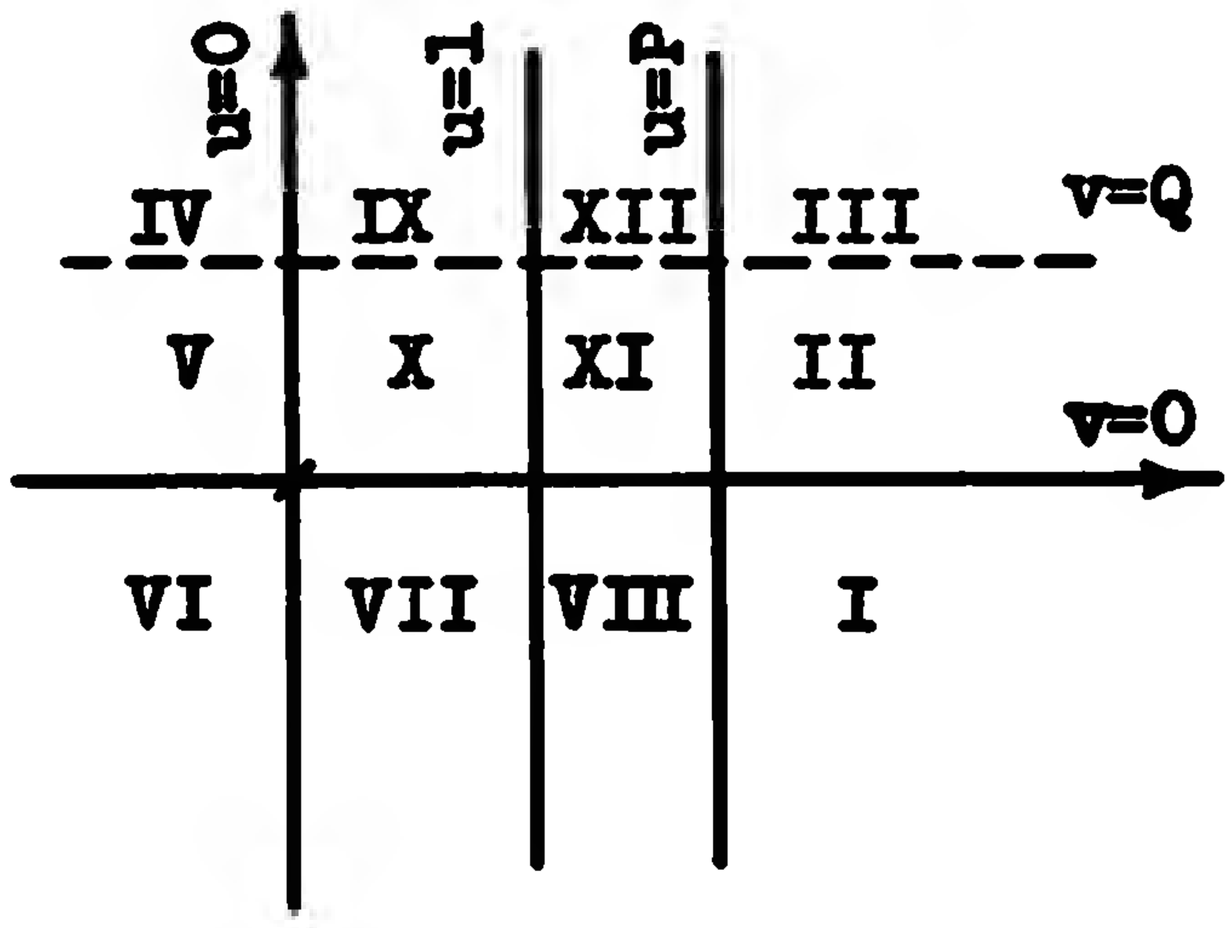
z - plane	w - plane
$z = 0;\ \ i;\ \ z'_{\infty} = -i;\ \ \infty$	$w = -1;\ \ 0;\ \ \infty;\ \ w_{\infty} = 1$
half-plane $y > 0$	region $\lvert w \rvert < 1$
half-plane $x > 0$	half-plane $v < 0$
region $\lvert z \rvert < 1$	half-plane $u < 0$
*interior of "circle" of D (see §1)	interior of the same "circle"
line $x-y = 1$, belonging to D	line $u-v = 1$

LINES PARALLEL TO THE AXES [p, q, P, Q real]



z - plane	w - plane
line $x = p \quad (p \neq 0)$	circle $\left\lvert w - (1 - \frac{i}{p})\right\rvert = \frac{1}{\lvert p\rvert}$
line $y = -1$	line $u = 1$
line $y = q \quad (q \neq -1)$	circle $\left\lvert w - \frac{q}{q+1}\right\rvert = \frac{1}{\lvert q+1\rvert}$





line $y = -1$	line $u = 1$
circle $\left\lvert z - \frac{iP}{1-P}\right\rvert = \frac{1}{\lvert 1-P\rvert}$	line $u = P \quad (P \neq 1)$
circle $\left\lvert z - (-i - \frac{1}{Q})\right\rvert = \frac{1}{\lvert Q\rvert}$	line $v = Q \quad (Q \neq 0)$

$$\boxed{w = \frac{i+z}{i-z}}$$

This is a combination of $-w = (\xi - i)/(\xi + i)$ and $\xi = -z$.

5. CONSTRUCTION OF LINEAR AND BILINEAR TRANSFORMATIONS

which map given elements (points, lines, etc.) on given elements.

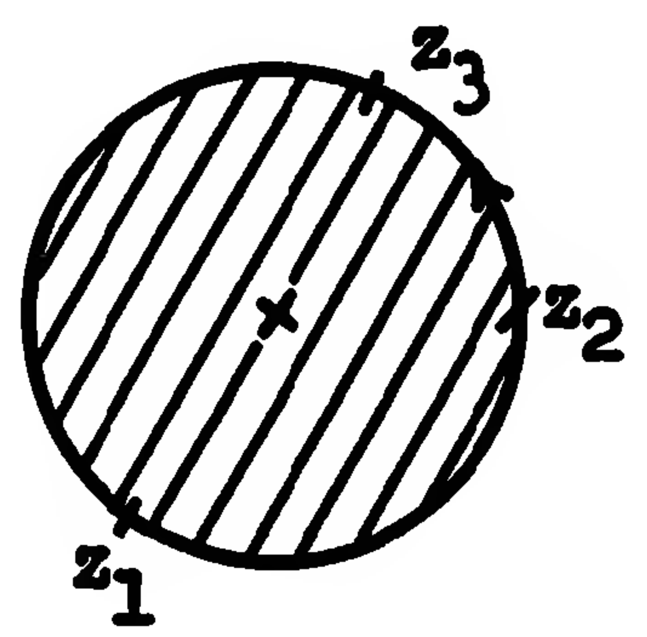
Constants, other than those labelled "given", are arbitrary except for the restrictions explicitly stated (such as "real", $\neq 0$, etc.).

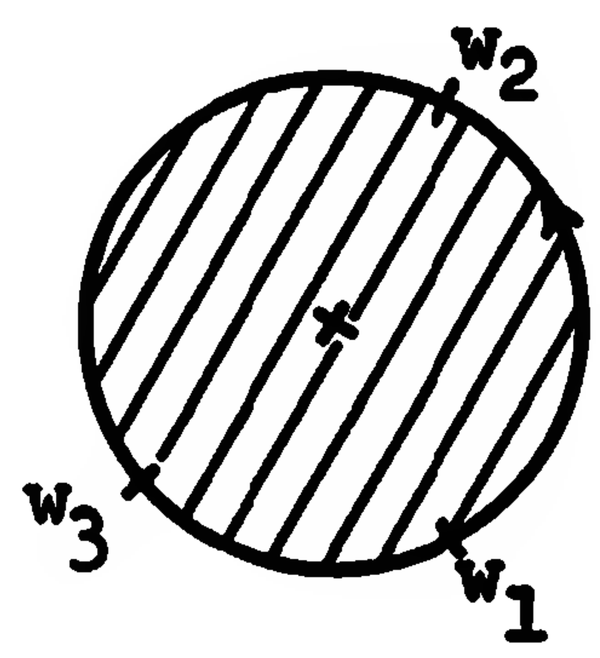
5.1 Three points on three points.

z - plane	w - plane

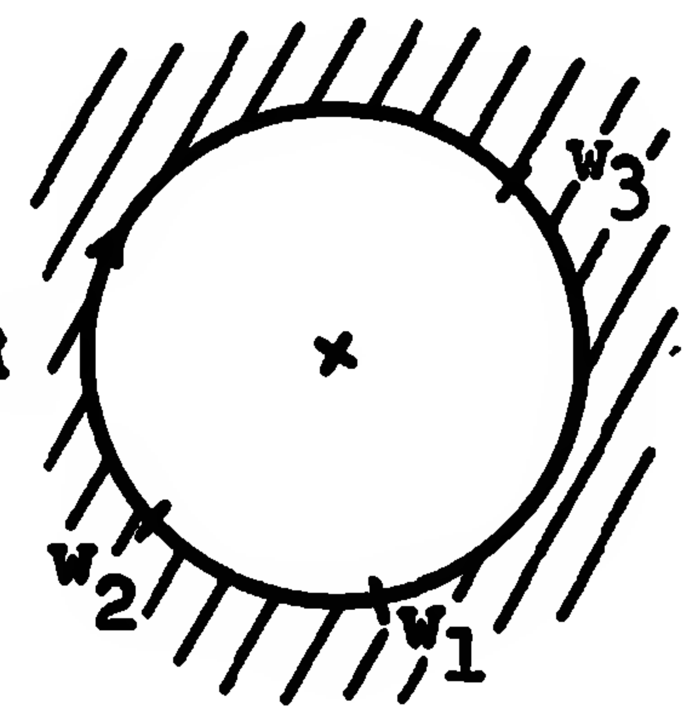
(i) Transformation: $\dfrac{w-w_1}{w-w_2}\,\dfrac{w_3-w_2}{w_3-w_1} = \dfrac{z-z_1}{z-z_2}\,\dfrac{z_3-z_2}{z_3-z_1}$;

given z_1, z_2, z_3, and w_1, w_2, w_3, all the points being finite.



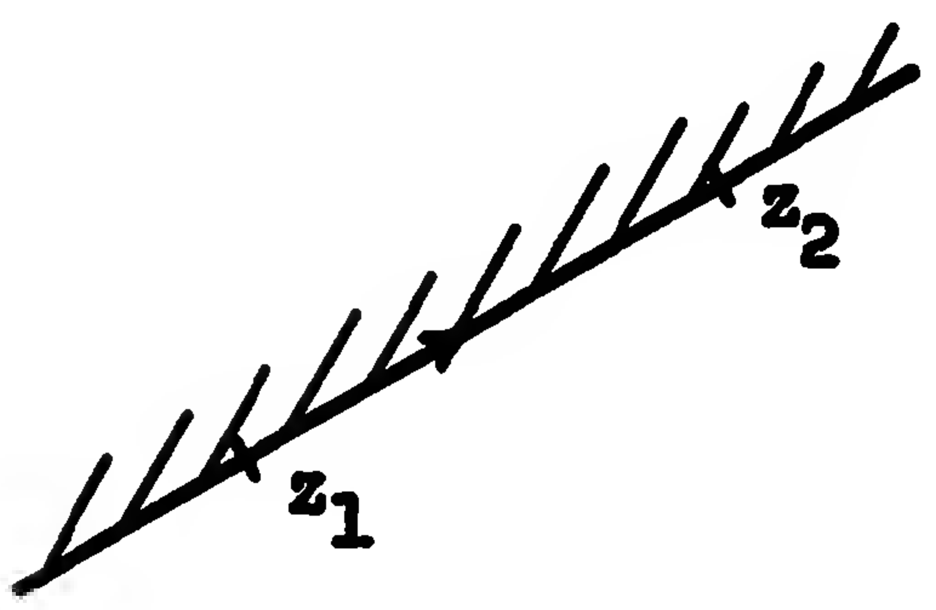


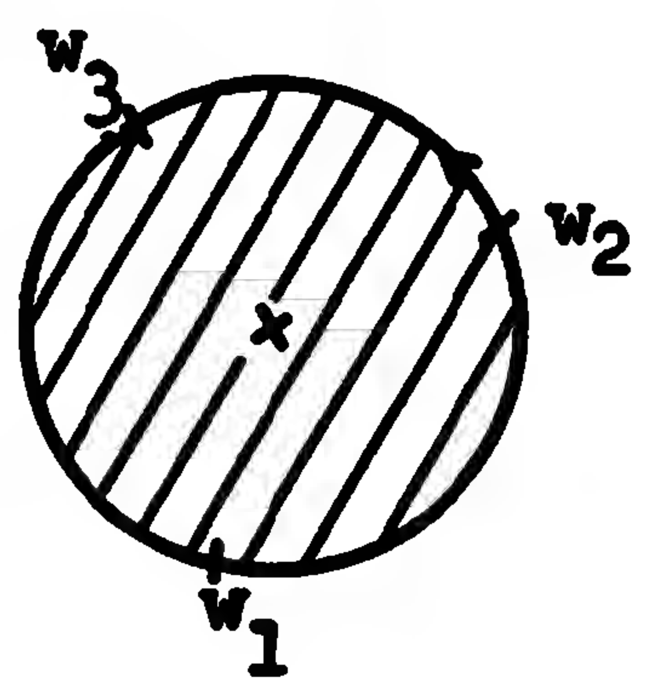
OR



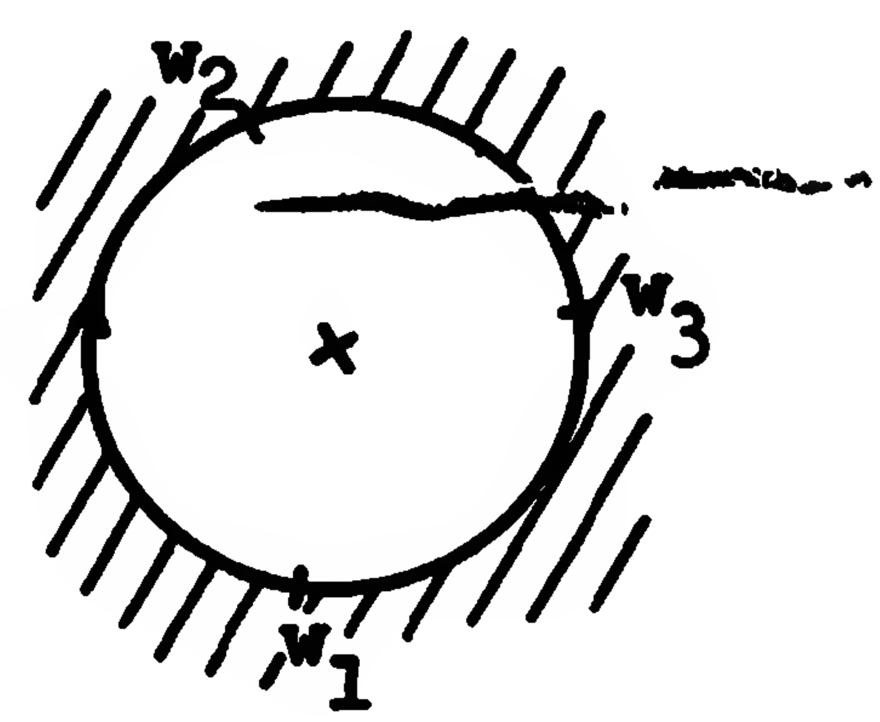
(ii) Transformation: $\dfrac{w-w_1}{w-w_2}\,\dfrac{w_3-w_2}{w_3-w_1} = \dfrac{z-z_1}{z-z_2}$;

given z_1, z_2, ∞, and w_1, w_2, w_3.





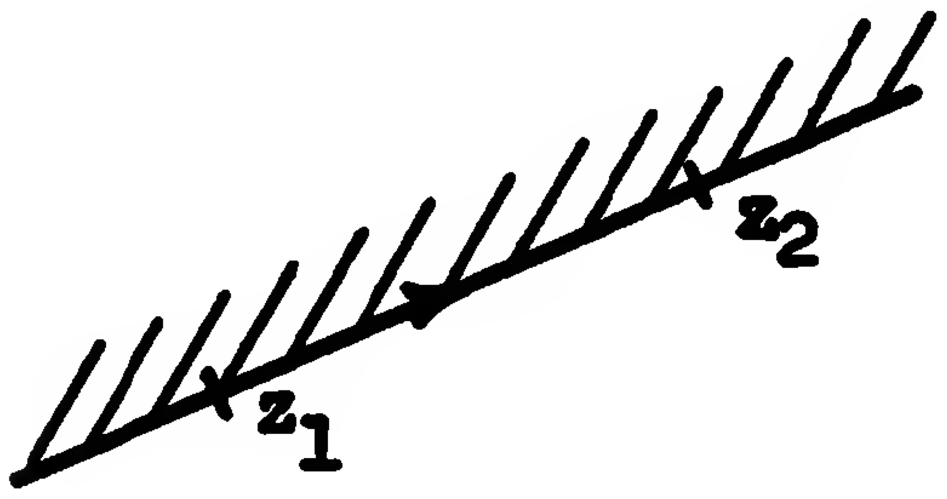
OR

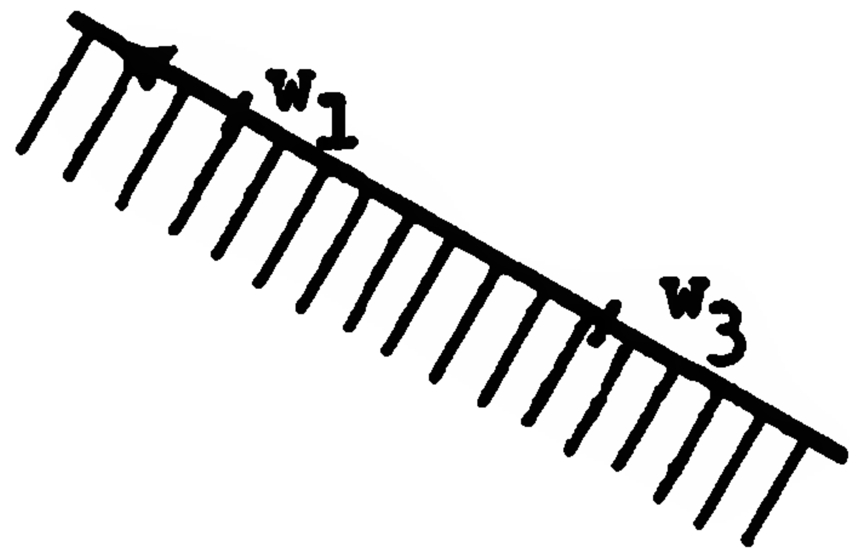


z - plane	w - plane

(iii) Transformation: $\dfrac{w-w_3}{w_3-w_1} = \dfrac{z_2-z_1}{z-z_2}$;

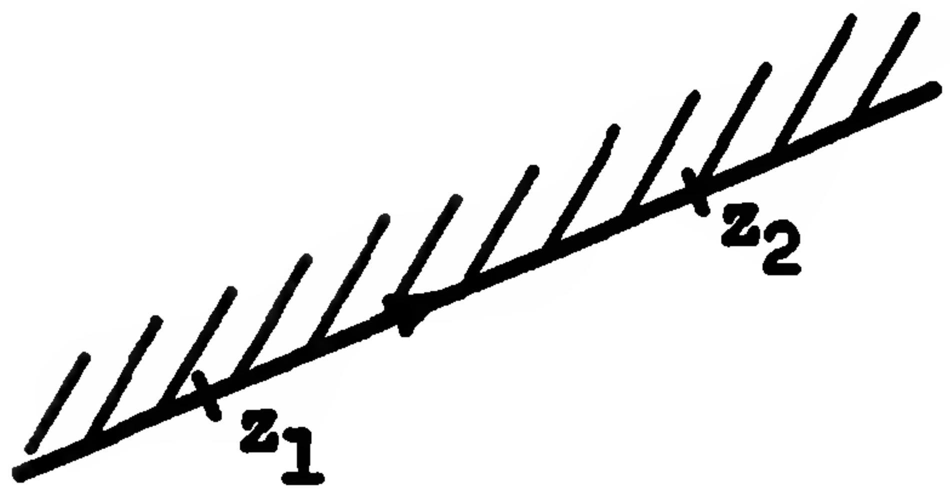
given z_1, z_2, ∞, and w_1, ∞, w_3.





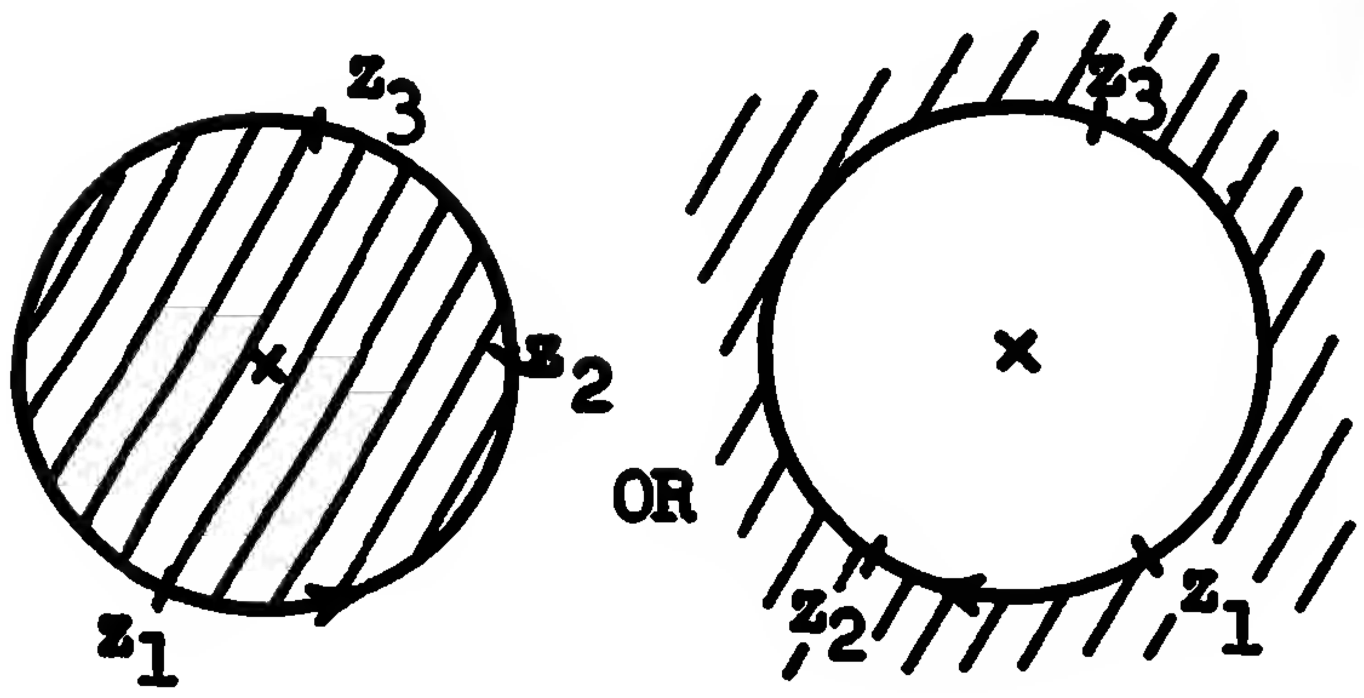
(iv) Transformation: $\dfrac{w-w_1}{w_1-w_2} = \dfrac{z-z_1}{z_1-z_2}$;

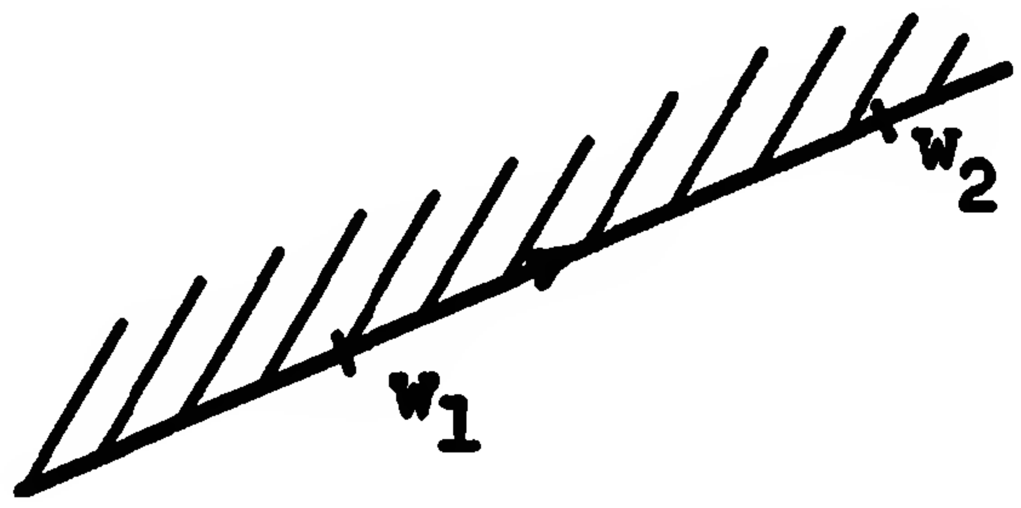
given z_1, z_2, ∞, and w_1, w_2, ∞



(v) Transformation: $\dfrac{w-w_1}{w-w_2} = \dfrac{z-z_1}{z-z_2}\,\dfrac{z_3-z_2}{z_3-z_1}$;

given z_1, z_2, z_3, and w_1, w_2, ∞. (conf. ii).





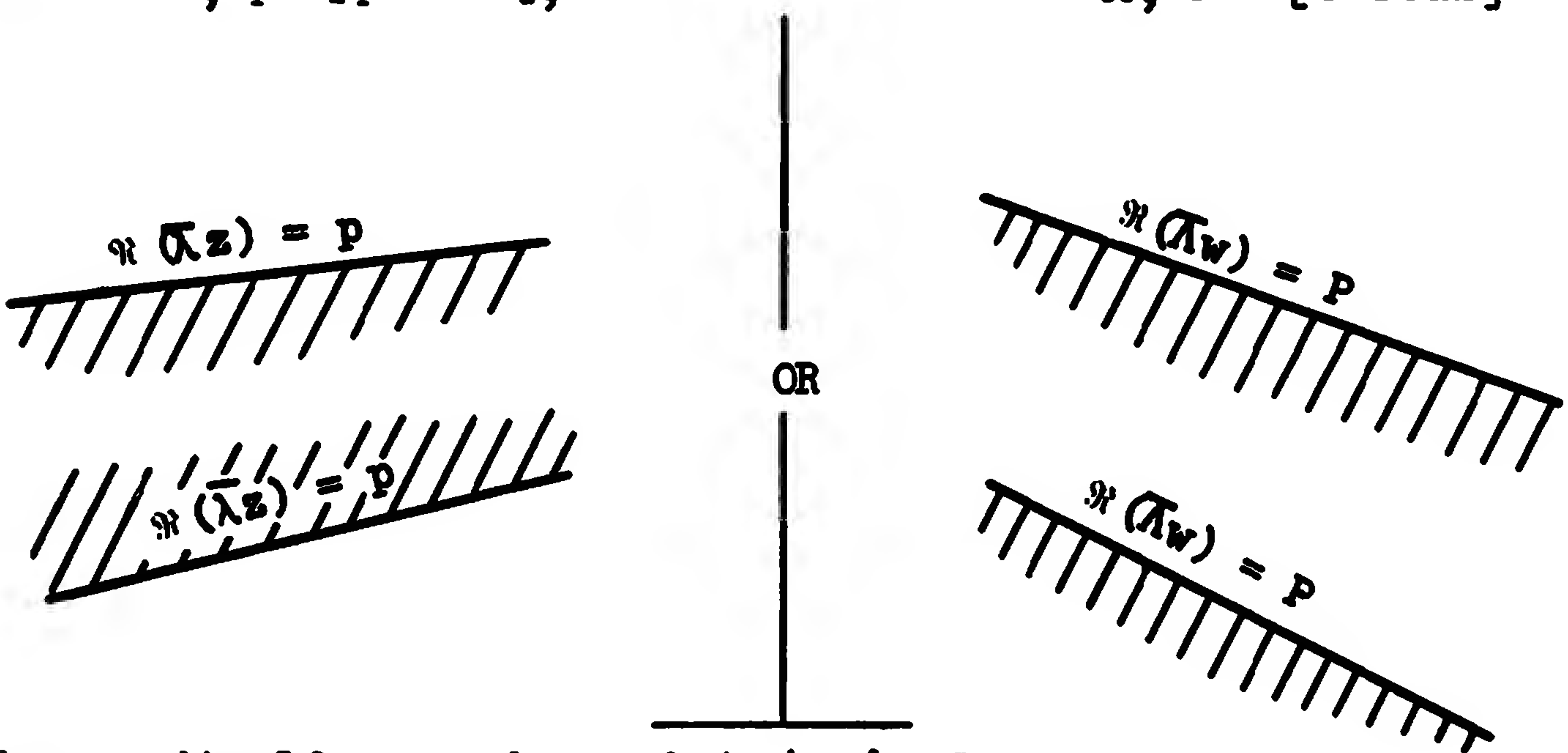
5.2 Straight line on straight line

z - plane	w - plane

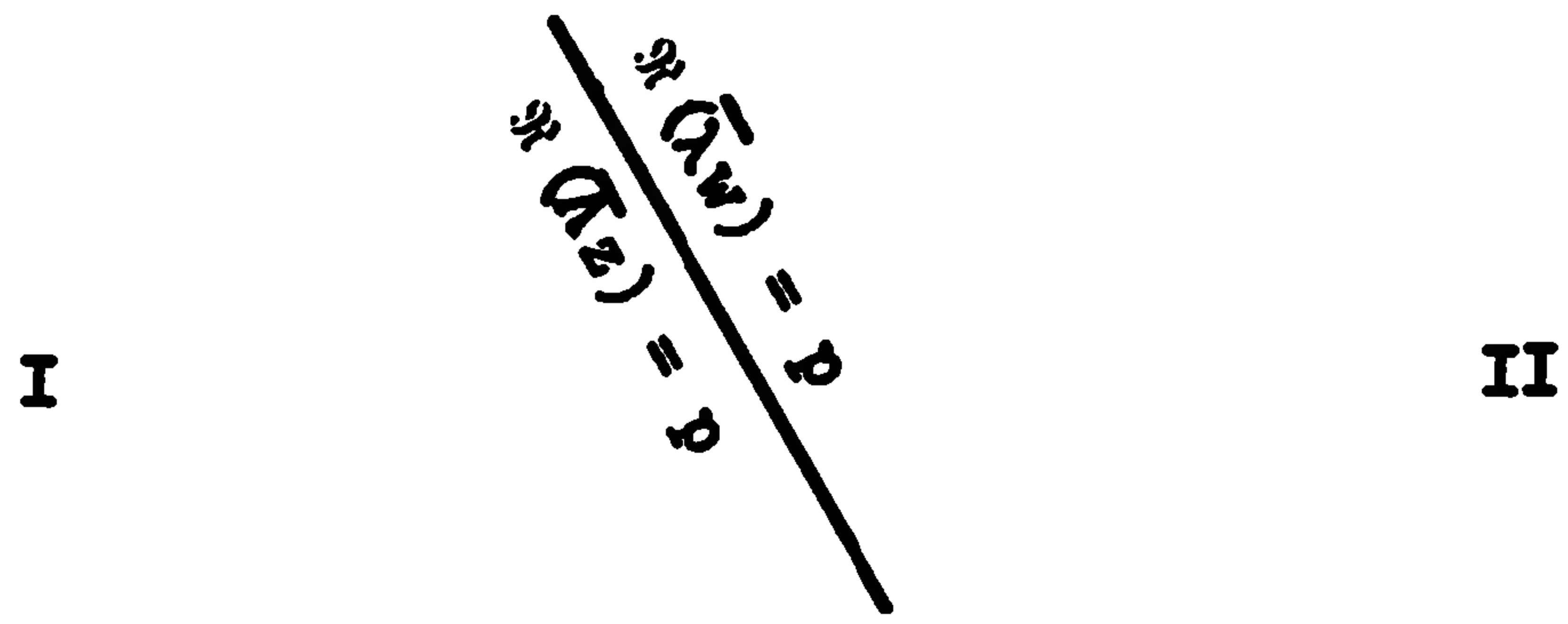
Transformation: $\bar{\Lambda} w = P + \dfrac{a(\bar{\lambda} z - p) - ib}{ic(\bar{\lambda} z - p) + d}$,

a, b, c, d real, $ad - bc \neq 0$ (See also 5.1);

given λ, p [p real], and Λ, P [P real]

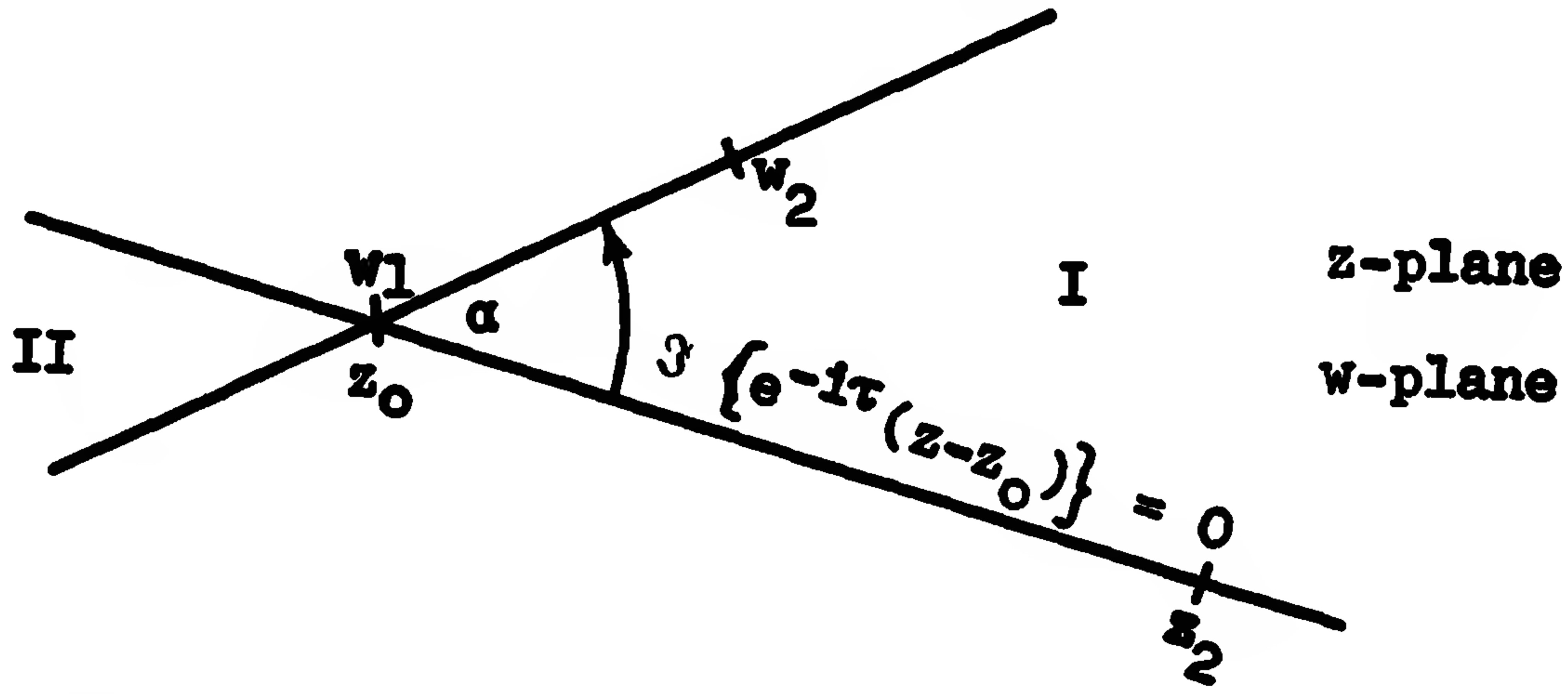


*<u>Line on itself</u>: as above, but $\Lambda = \lambda$, $P = p$.



z - plane	w - plane
(i) $ad-bc > 0$	
half-plane I	half-plane I
half-plane II	half-plane II
(ii) $ad-bc < 0$	
half-plane I	half-plane II

*Angle on itself, with arms interchanged.



Given z_o, $\alpha \neq 0$ $(-\pi < \alpha < \pi)$, τ [α,τ real].

Involutory transformation: $(w-z_o)(z-z_o) = be^{i(\alpha+2\tau)}$; $b > 0$.

z - plane	w - plane
points z_o; $z_1=\infty$; $z_2=z_o+ae^{i\tau}$ [a real; $a>0$ in diagram].	points $w_o=\infty$; $w_1=z_o$; $w_2=z_o+\frac{b}{a}e^{i(\tau+\alpha)}$
regions I, II, respectively.	regions I, II, respectively.

If $b<0$, I is mapped on II.

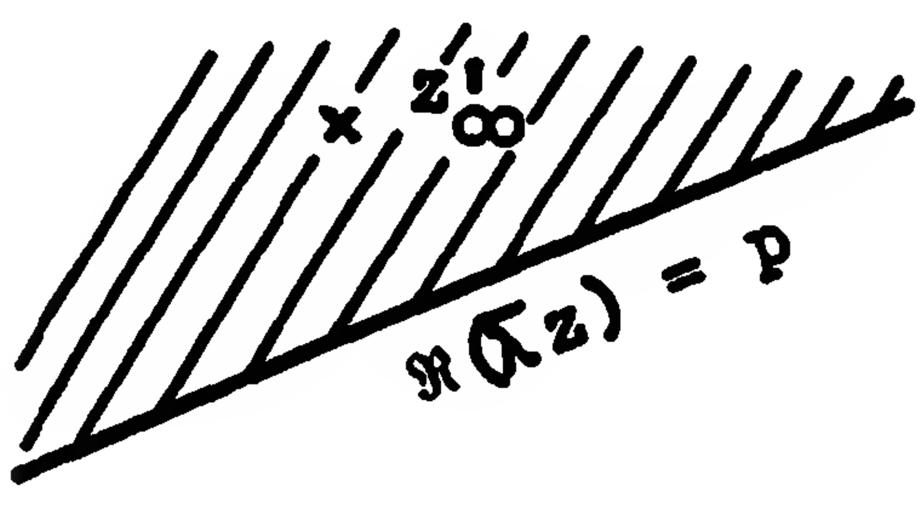
5.3 **Straight line on circle**

z - plane	w - plane

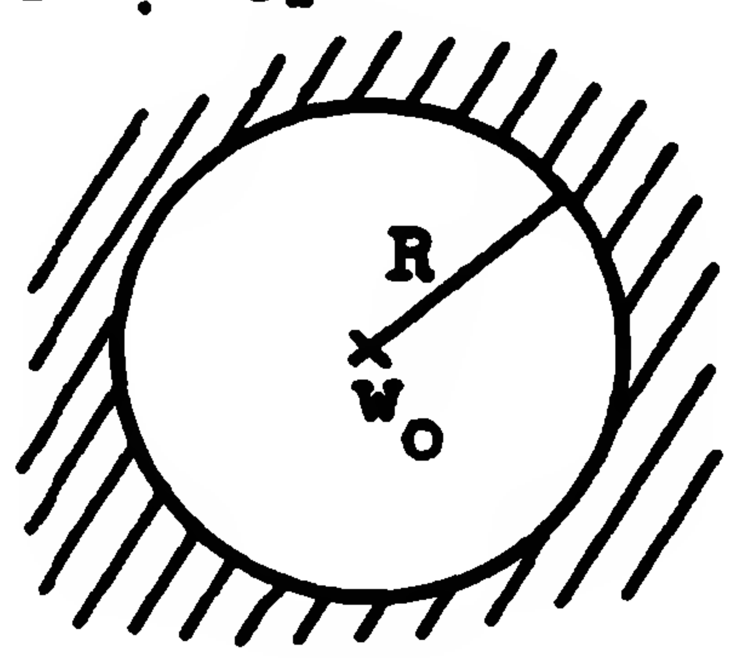
Transformation required: $w-w_o = Re^{i\tau}\dfrac{\bar\lambda z-p+\beta}{\bar\lambda z-p-\bar\beta}$,

τ real; $\Re(\beta) \neq 0$ (See also 5.1);

given λ, p [p real], and w_o, R [$R > 0$]



$z'_\infty = (p+\bar\beta)/\bar\lambda$



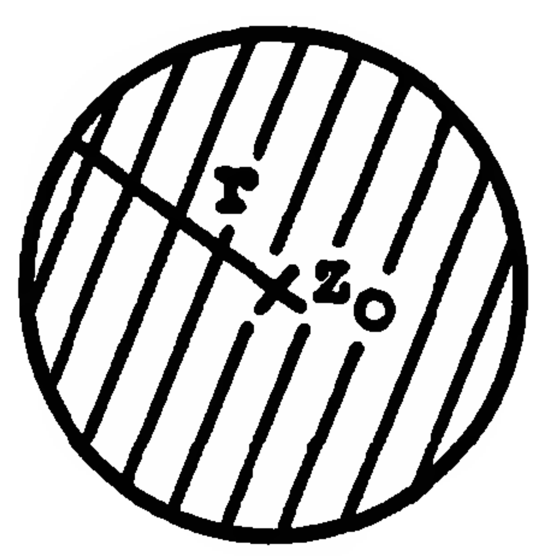
5.4 Circle on straight line

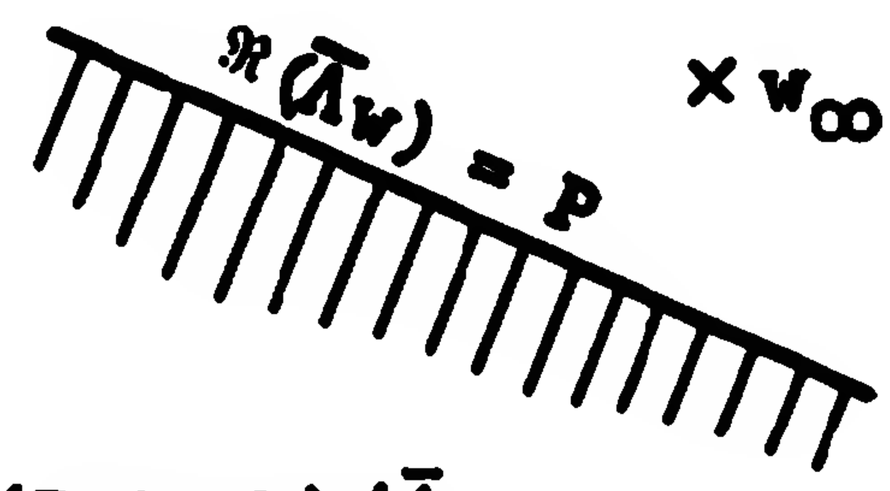
z - plane	w - plane

Transformation required: $\bar{\Lambda} w = P + ia + b\,\dfrac{z-z_0+re^{i\tau}}{z-z_0-re^{i\tau}}$,

a, b, τ real, $b \neq 0$. (See also 5.1);

given z_0, $r > 0$, and Λ, P [P real]





$w_\infty = (P+ia+b)/\bar{\Lambda}$

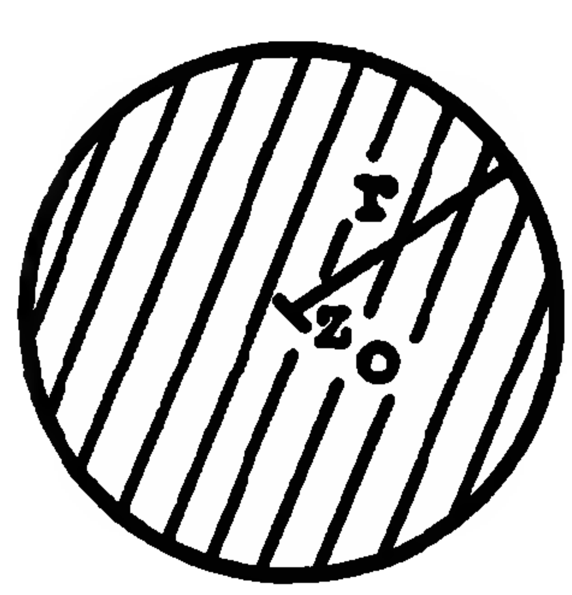
5.5 Circle on circle

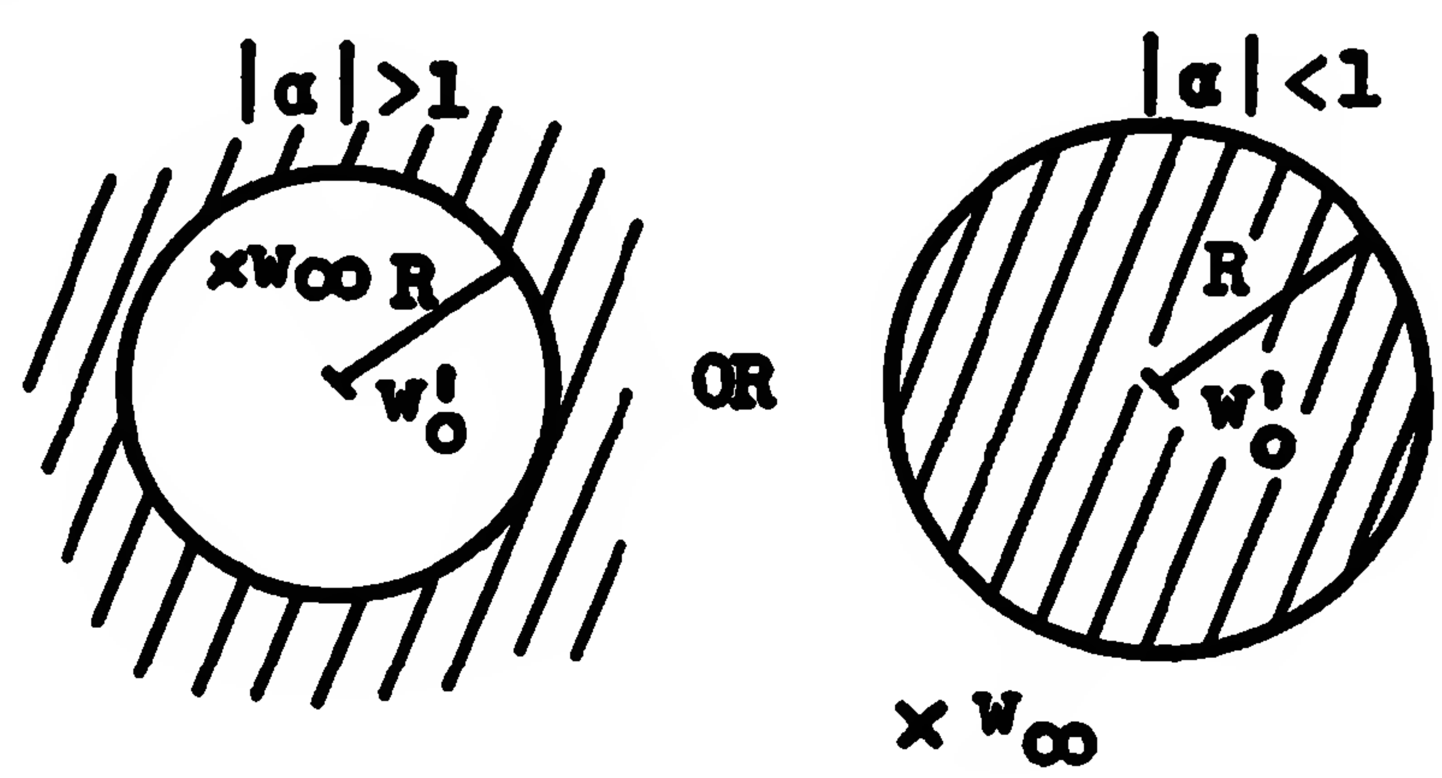
z - plane	w - plane

Transformation required: $w-w_0' = Re^{i\tau}\,\dfrac{z-z_0-r\alpha}{\bar{\alpha}z-\bar{\alpha}z_0-r}$;

τ real; $|\alpha| \neq 1$ (See also 5.1);

given z_0, $r > 0$, and w_0', $R > 0$





$w_\infty = w_0' + Re^{i\tau}/\bar{\alpha}$

Mapping <u>unit circle on itself</u>: $w = e^{i\tau}\,\dfrac{z-\alpha}{\bar{\alpha}z-1}$ [α, τ as above].

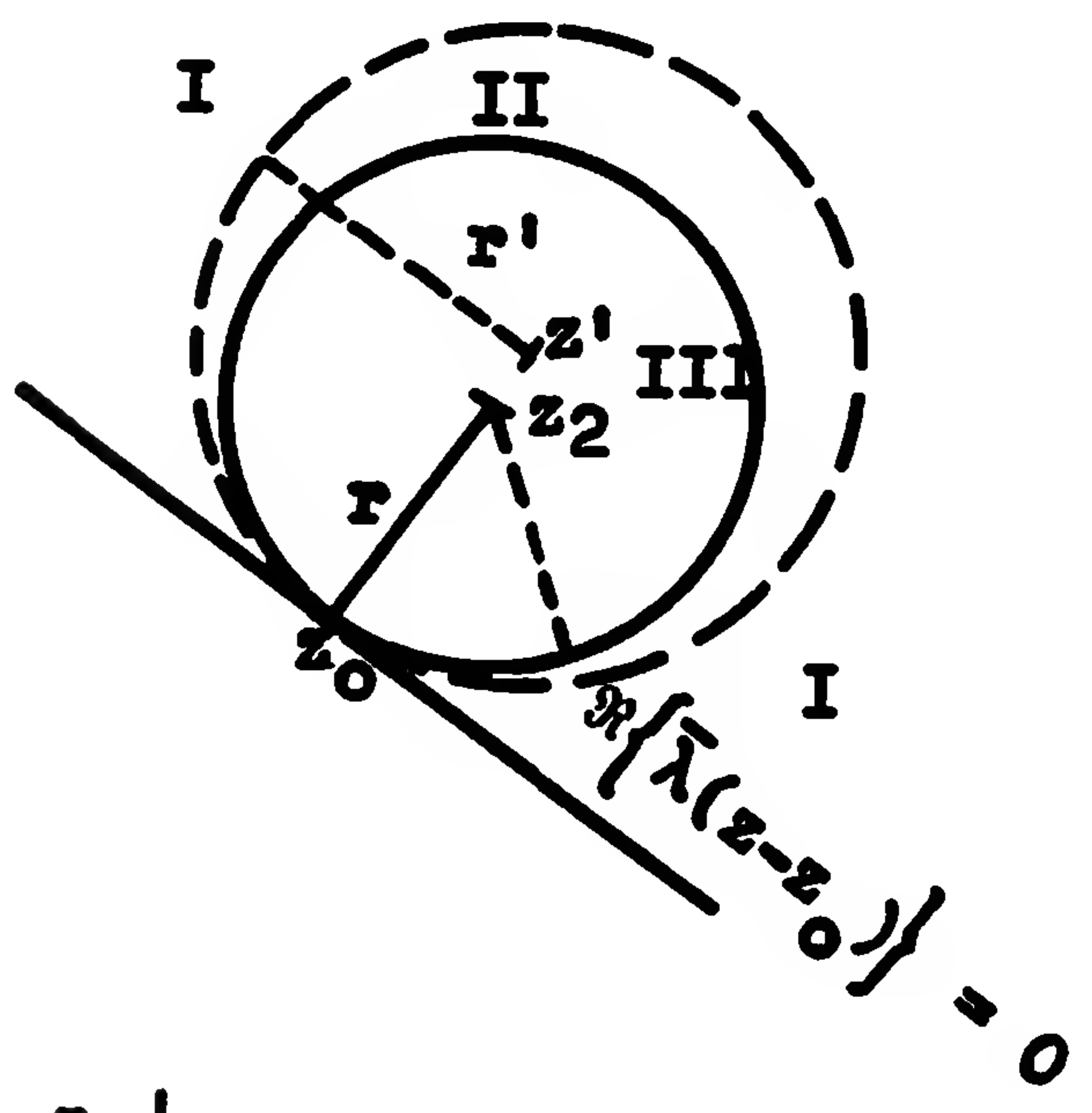
5.6 Circle and line in contact on two parallel lines

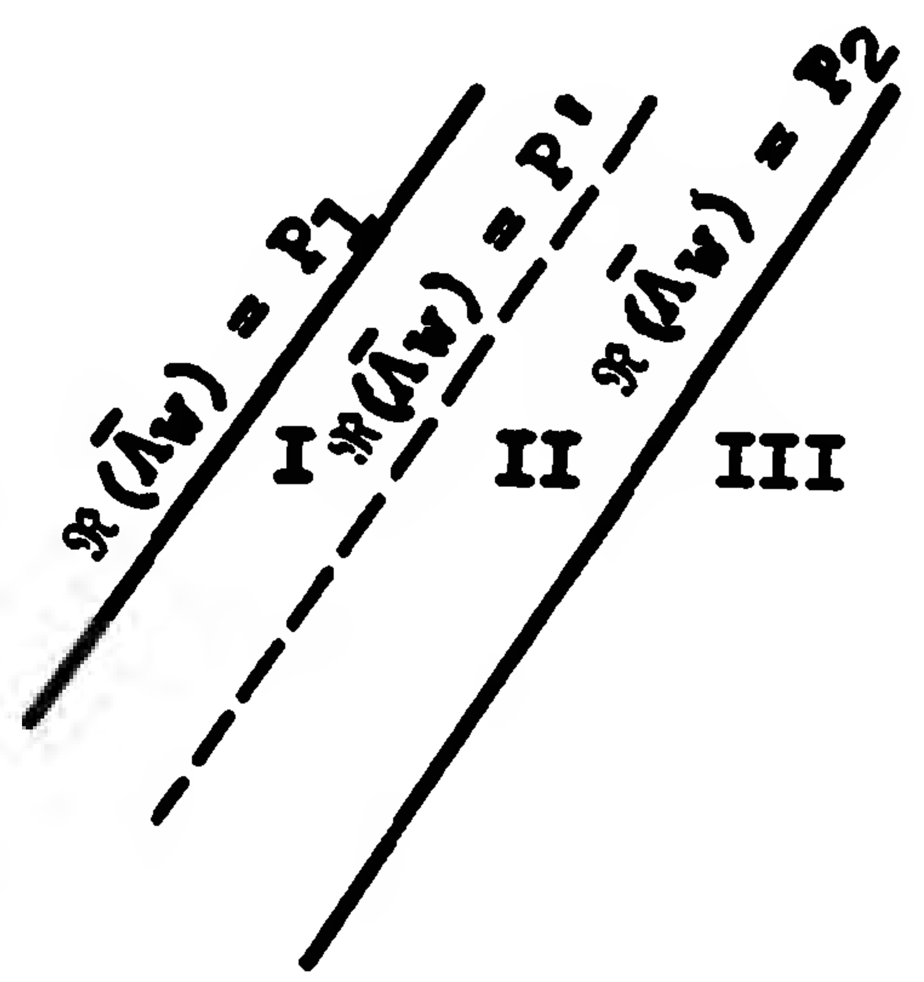
z - plane	w - plane

Transformation: $$\bar{\Lambda} w = ai + P_1 + \frac{2(z_2-z_o)(P_2-P_1)}{z-z_o};$$

a real;

given z_o, z_2 and Λ; P_1, P_2 (real).





$r = |z_2-z_o|$

$\lambda = z_2-z_o$

z - plane	w - plane
line $\Re(\bar{\Lambda}z) = \Re(\bar{\Lambda}z_o)$	line $\Re(\bar{\Lambda}w) = P_1$
circle $\lvert z-z_2\rvert = r$	line $\Re(\bar{\Lambda}w) = P_2$
circle $\lvert z-z'\rvert = r'$	line $\Re(\bar{\Lambda}w) = P' = P_1 + \frac{r}{r'}(P_2-P_1)$

<u>Examples</u>:

(i) $w = -a + \frac{1}{z}$; a real.

line $y = 0$, circle $\lvert z + \frac{1}{2}i\rvert = \frac{1}{2}$	lines $v = 0$, and $v = 1$, respectively

(ii) $w = -a + \frac{2ir}{z}$; a real.

line $x = 0$, circle $\lvert z-r\rvert = r > 0$	lines $v = 0$, and $v = 1$, respectively.

5.7 Two circles in contact on two parallel lines.

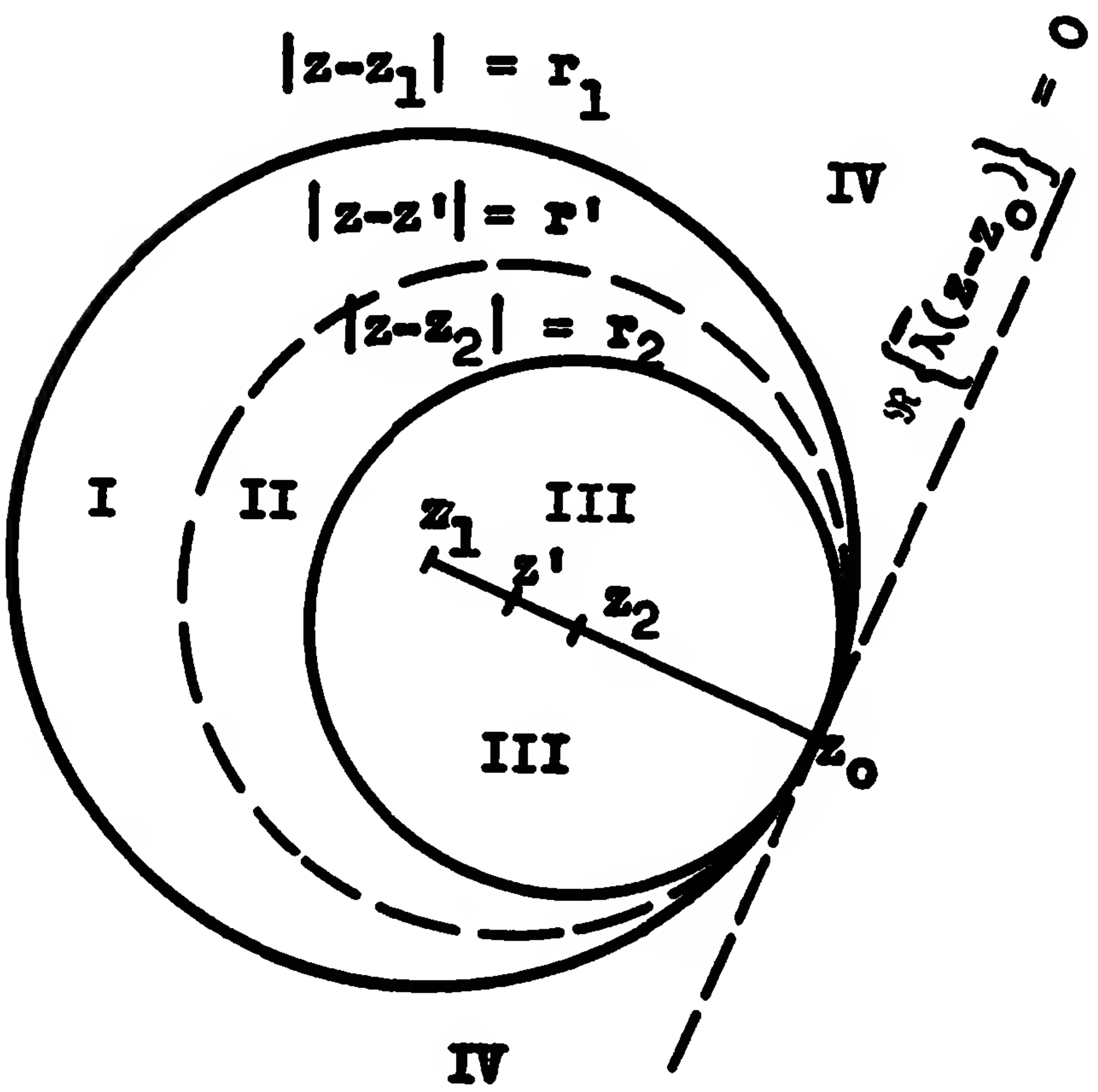
z - plane	w - plane

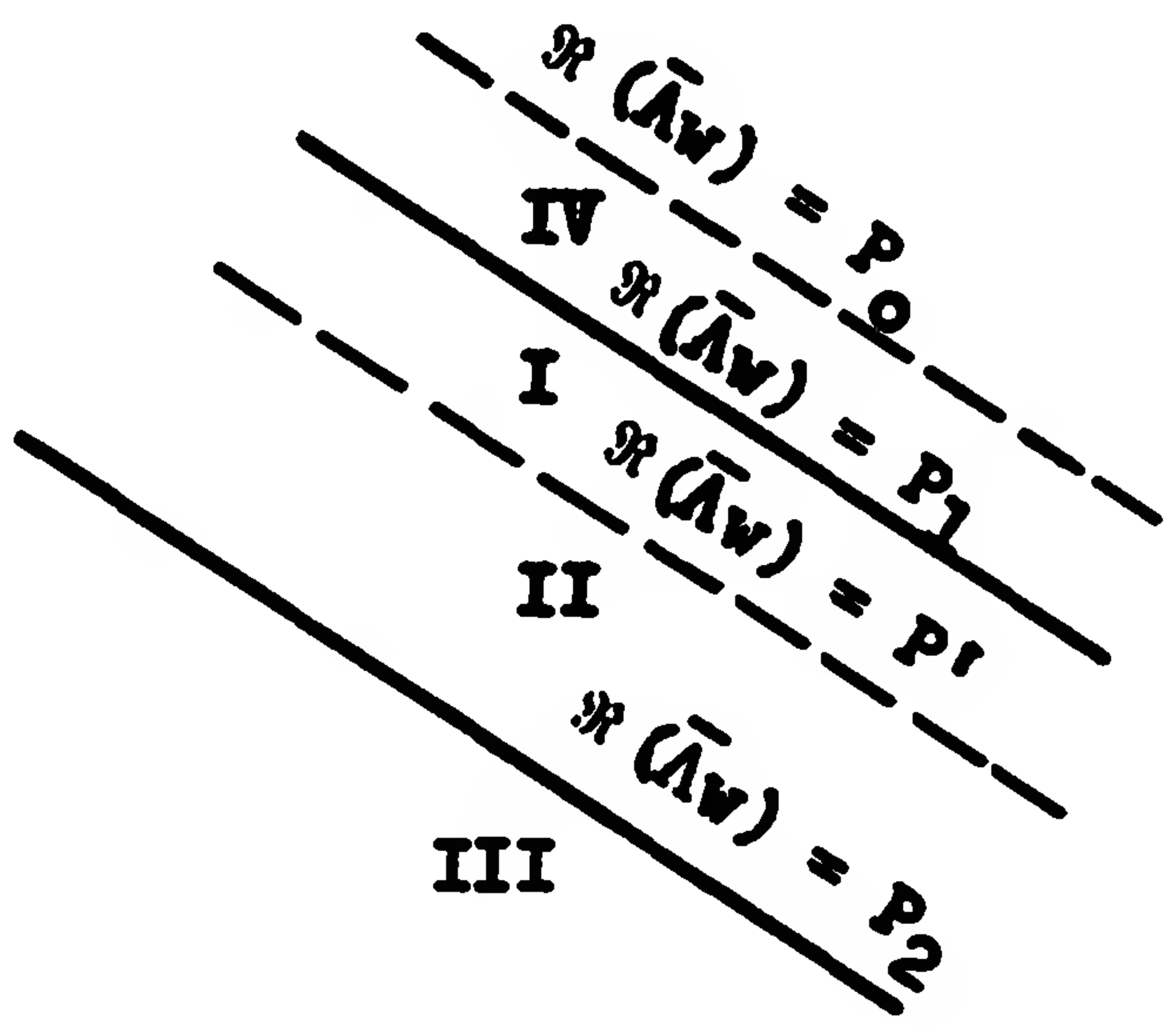
(i) Inner contact

Transformation: $\bar{\Lambda} w = P_1 + ai + \gamma \dfrac{z+z_o-2z_1}{z_o-z}$, a real,

where $\gamma = \dfrac{P_2-P_1}{r_1-r_2}\, r_2$;

z - plane	w - plane
given $r_1 > 0$, $r_2 > 0$, z_1, z_2, with $\lvert z_1-z_2 \rvert = \lvert r_1-r_2 \rvert$,	and Λ ; P_1, P_2 (real).





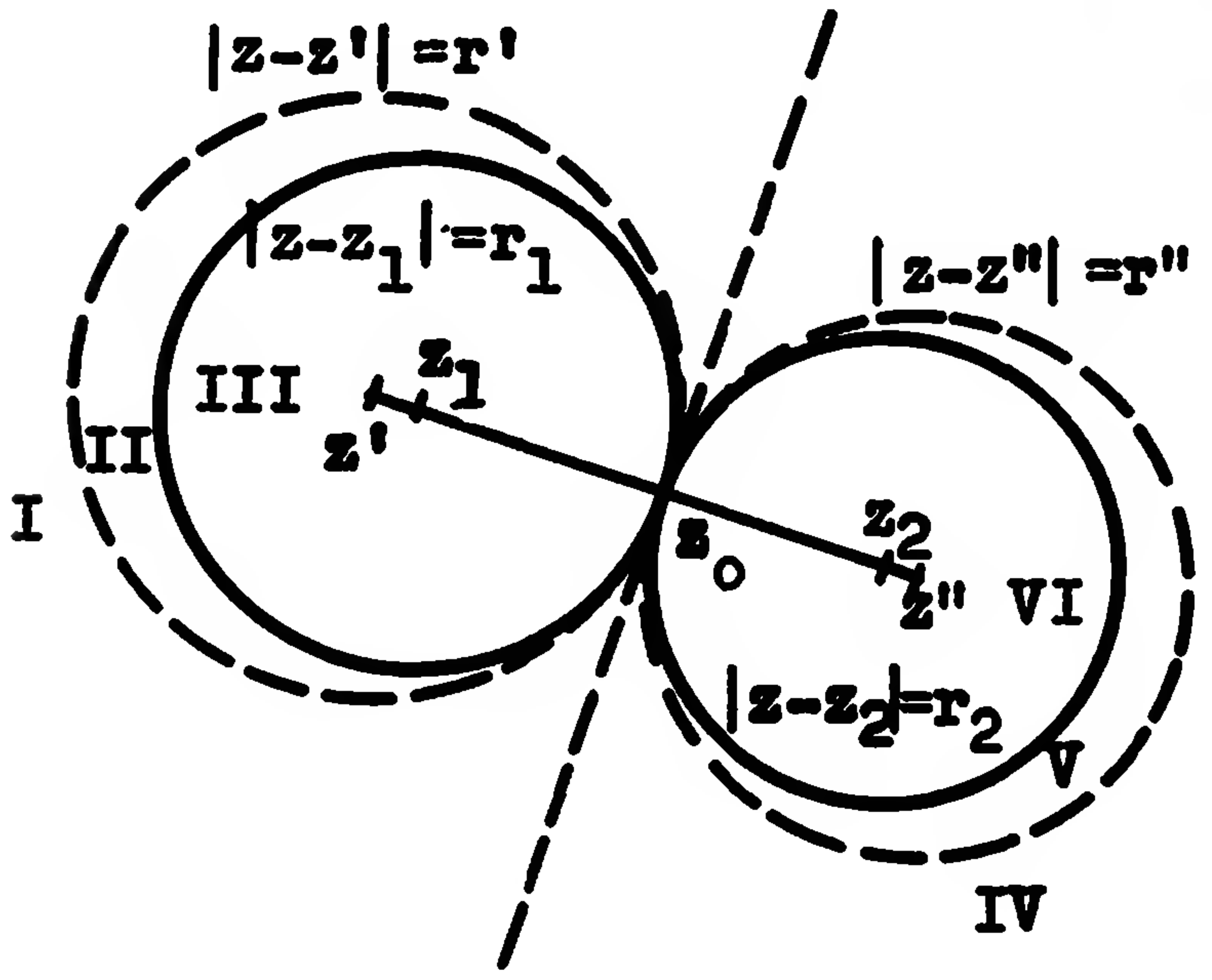
Set $z_o = \dfrac{r_1 z_2 - r_2 z_1}{r_1 - r_2}$.

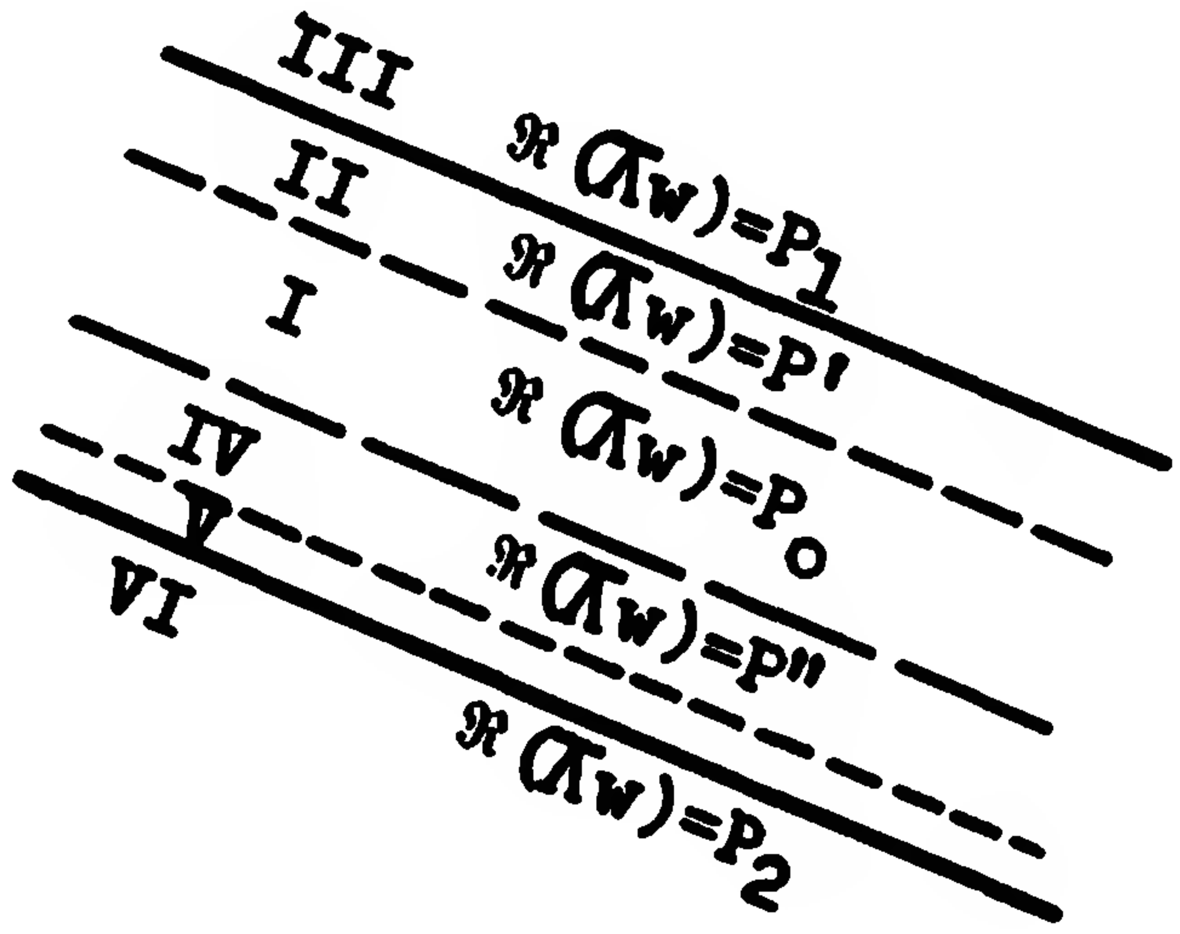
z - plane	w - plane
circle $\lvert z-z_1 \rvert = r_1$	line $\Re(\bar{\Lambda}w) = P_1$
circle $\lvert z-z_2 \rvert = r_2$	line $\Re(\bar{\Lambda}w) = P_2$
circle $\lvert z-z' \rvert = r'$ touching at z_o	line $\Re(\bar{\Lambda}w) = P'$; $P' = P_1 + (P_2-P_1)\dfrac{r_2}{r'}\dfrac{r'-r_1}{r_2-r_1}$
line $\Re\left\{\bar{\lambda}(z-z_o)\right\} = 0$, where $\lambda = z_1-z_2$; touching at z_o	line $\Re(\bar{\Lambda}w) = P_o$; $P_o = \dfrac{P_1 r_1 - P_2 r_2}{r_1-r_2}$

z - plane	w - plane

(ii) <u>Outer contact:</u> $|z_1-z_2| = r_1+r_2$.

Transformation: $\overline{\Lambda}w = P_1 + ai + \gamma \dfrac{z+z_o-2z_1}{z_o-z}$; a real; $\gamma = \dfrac{P_1-P_2}{r_1+r_2} r_2$.

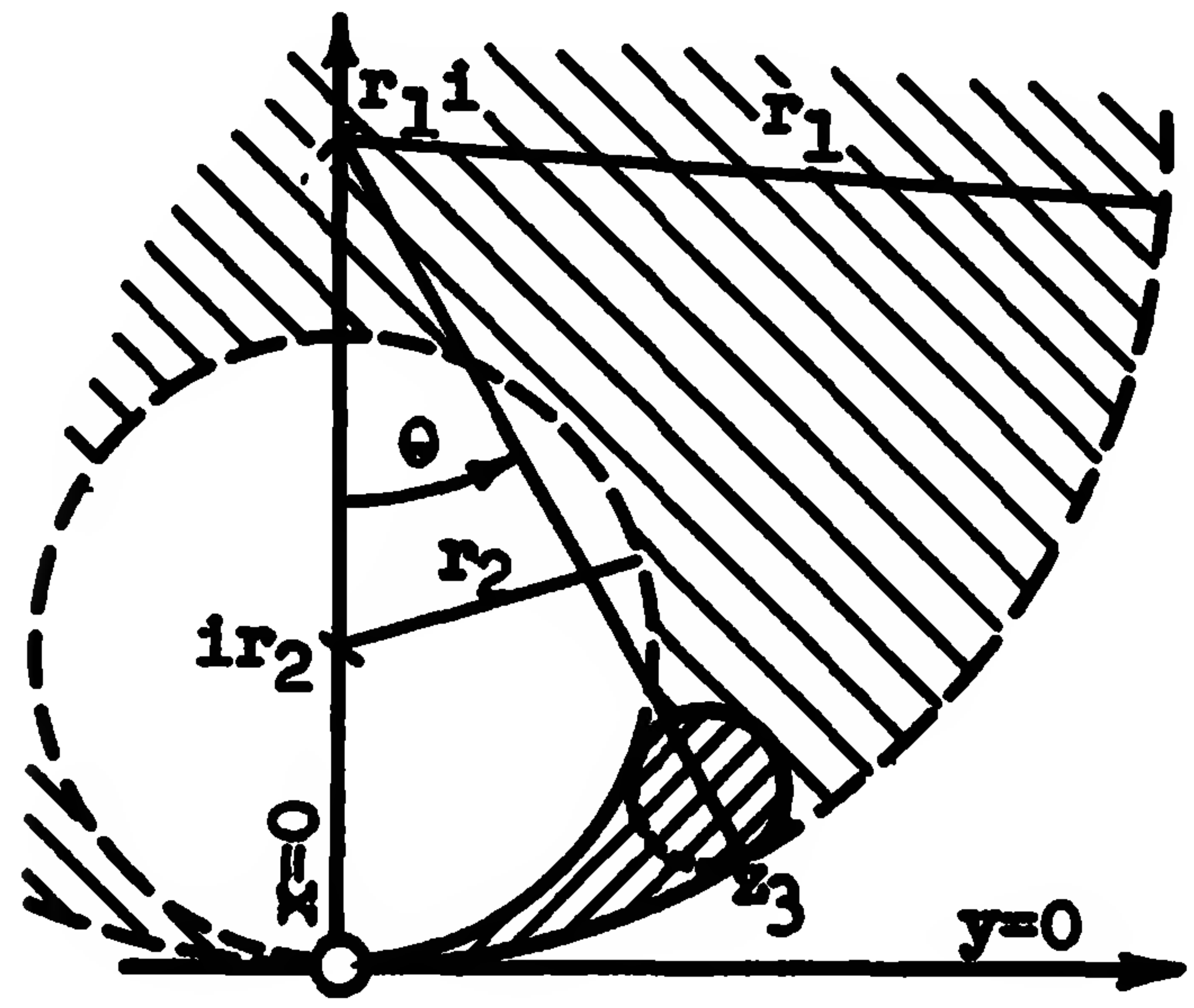
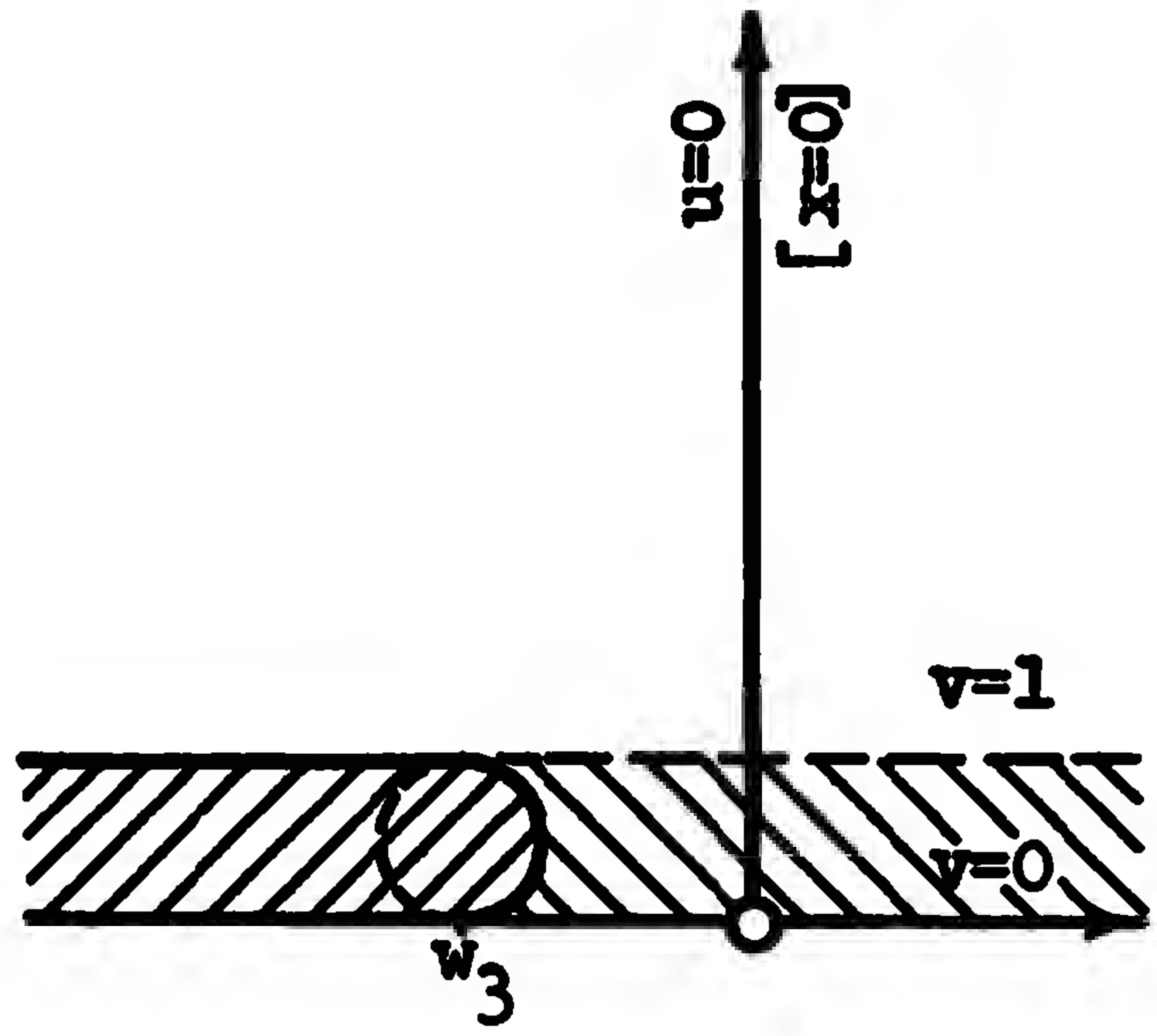




Set $z_o = \dfrac{r_1z_2+ r_2z_1}{r_1+r_2}$.

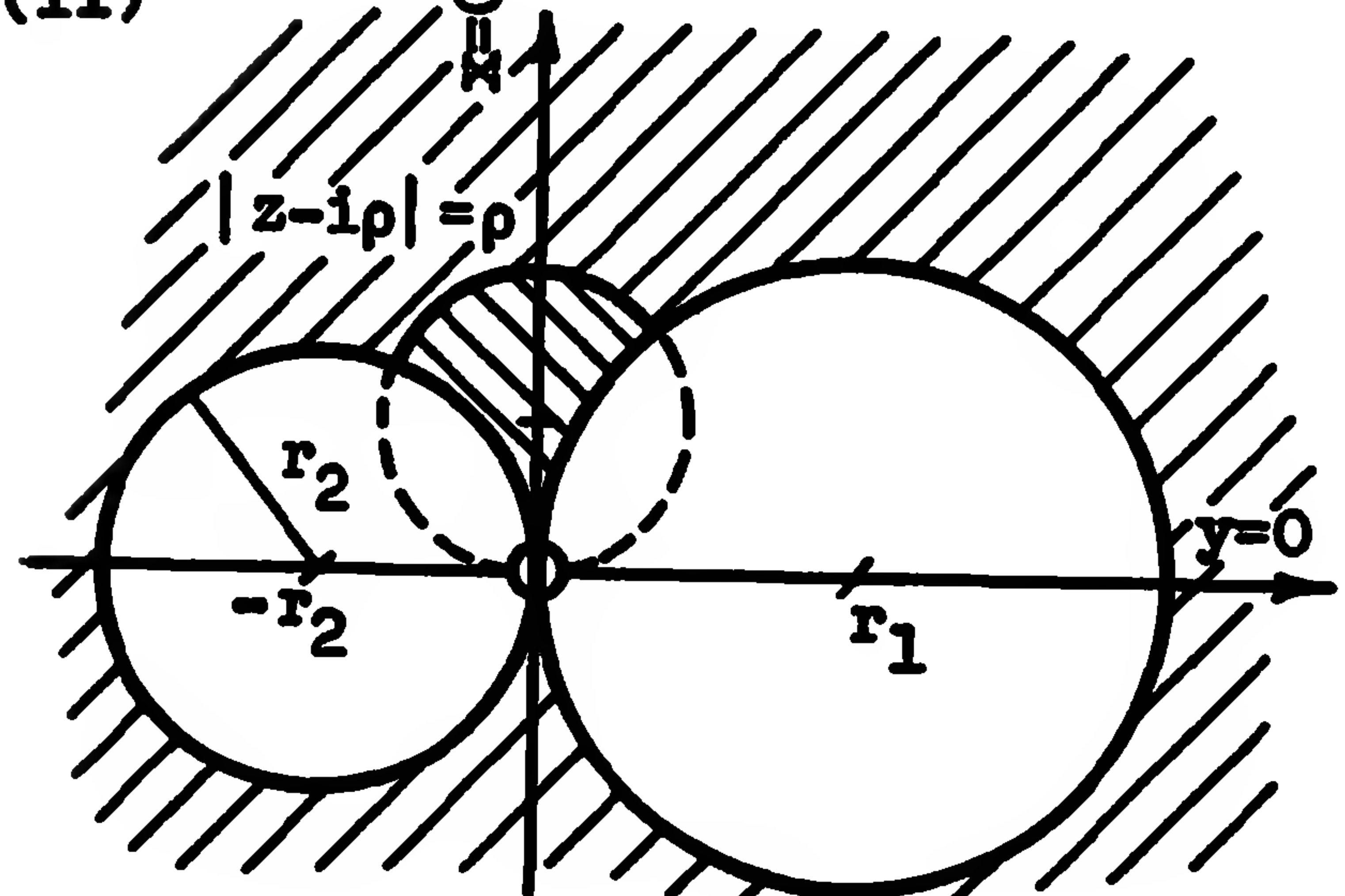
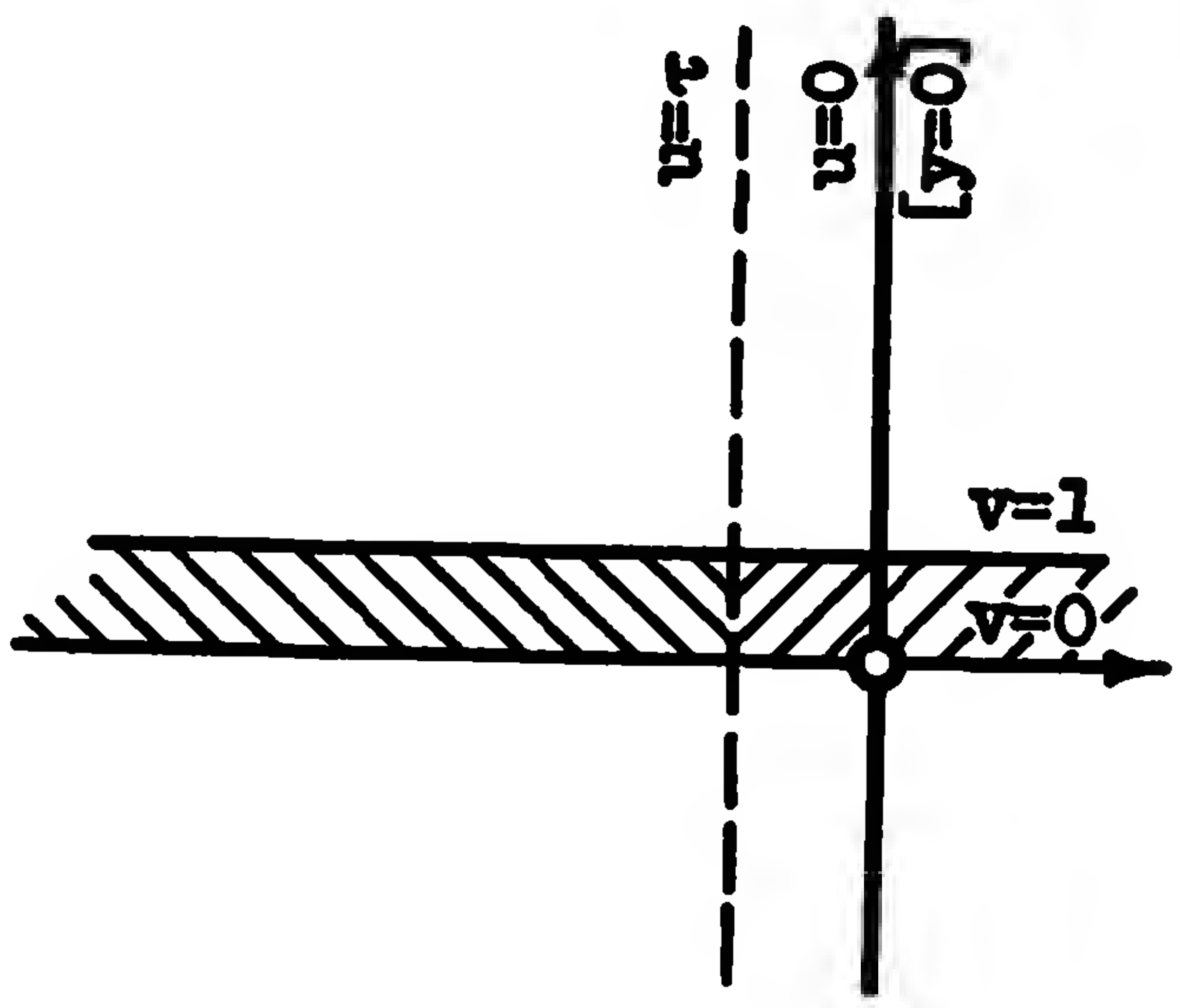
circle $\lvert z-z_1\rvert = r_1$	line $\Re(\overline{\Lambda}w) = P_1$
circle $\lvert z-z_2\rvert = r_2$	line $\Re(\overline{\Lambda}w) = P_2$
circle $\lvert z-z'\rvert = r'$ touching at z_o	line $\Re(\overline{\Lambda}w) = P' = P_2+(P_1-P_2)\dfrac{r_1}{r'} \times \dfrac{r'+r_2}{r_1+r_2}$
circle $\lvert z-z''\rvert = r''$ touching at z_o	line $\Re(\overline{\Lambda}w) = P'' = P_1+(P_2-P_1)\dfrac{r_2}{r''} \times \dfrac{r''+r_1}{r_1+r_2}$
line $\Re\left\{\overline{\lambda}(z-z_o)\right\} = 0$, with $\lambda = z_1-z_2$, touching at z_o	line $\Re(\overline{\Lambda}w) = P_o = \dfrac{P_1r_1+P_2r_2}{r_1+r_2}$

Examples:

z - plane	w - plane
(i) 	

$$\boxed{w = \frac{r_2}{r_2-r_1}\left(\frac{2r_1}{z} + i\right)}$$

points $z = 0;\ \infty;\ 2ir_1$	points $w = \infty;\ ir_2/(r_2-r_1);\ 0$
circle $\lvert z-ir_1\rvert = r_1$	line $v = 0$
circle $\lvert z-ir_2\rvert = r_2$	line $v = 1$
line $x = 0$	line $u = 0$
line $y = 0$	line $v = r_2/(r_2-r_1)$
point $z_3 = ir_1(1-e^{i\theta})$	point $w_3 = \dfrac{r_2}{r_2-r_1}\cot\dfrac{\theta}{2}$
small circle touching both circles and passing through z_3	circle $\lvert w - (\frac{1}{2}i + w_3)\rvert = \frac{1}{2}$

(ii) 	

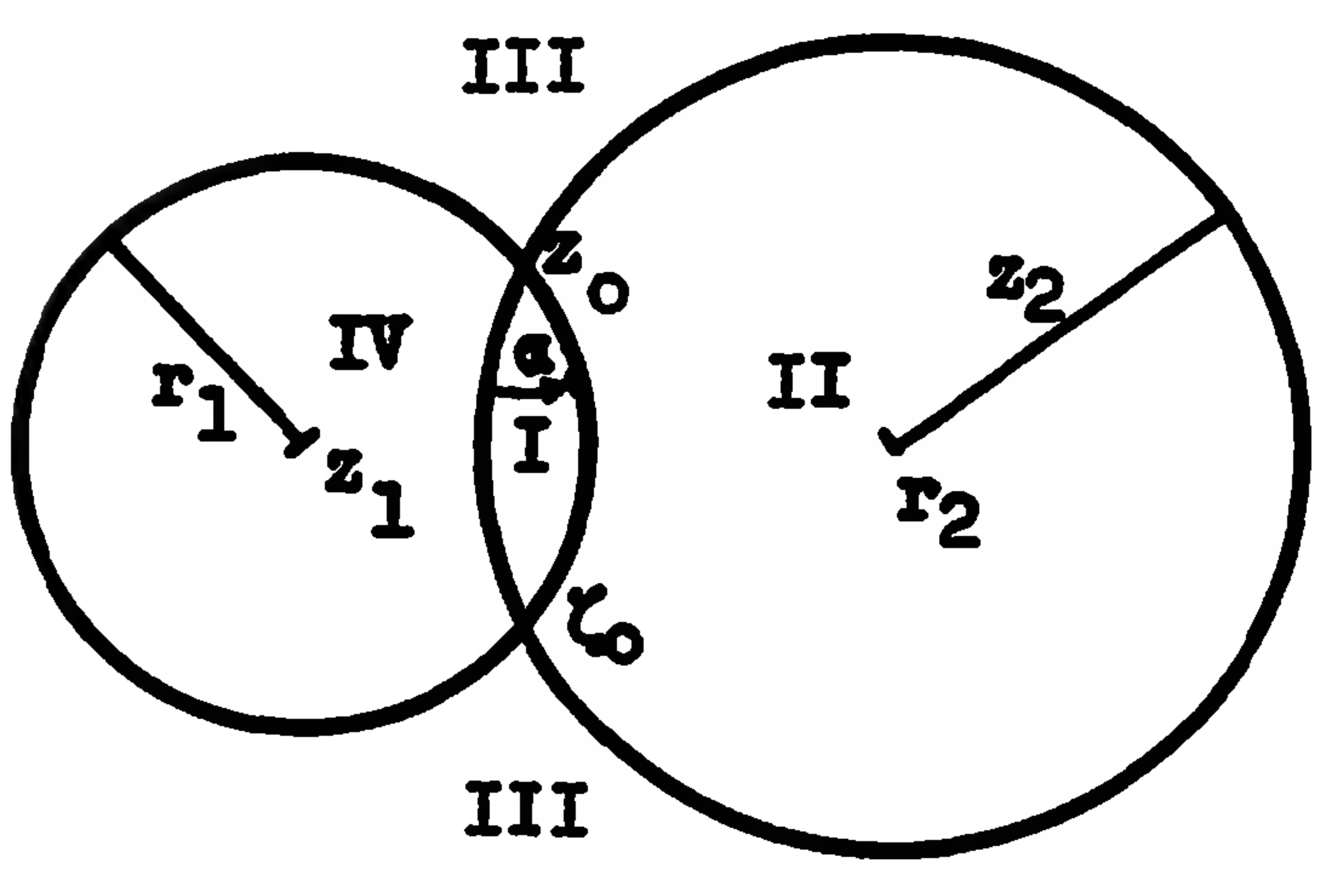
z - plane	w - plane
	$\boxed{w = \frac{ir_2}{r_1+r_2}(1 - \frac{2r_1}{z})}$
points $z = 0;\ \infty;\ 2r_1$	points $w = \infty;\ ir_2/(r_1+r_2);\ 0$
circle $\lvert z-r_1 \rvert = r_1$	line $v = 0$
circle $\lvert z+r_2 \rvert = r_2$	line $v = 1$
circle $\lvert z-i\rho \rvert = \rho$	line $u = \tau$, where $\tau = \frac{-r_1 r_2}{\rho(r_1+r_2)}$
lines $x = 0;\ \ y = 0$	lines $v = r_2/(r_1+r_2);\ \ u = 0.$

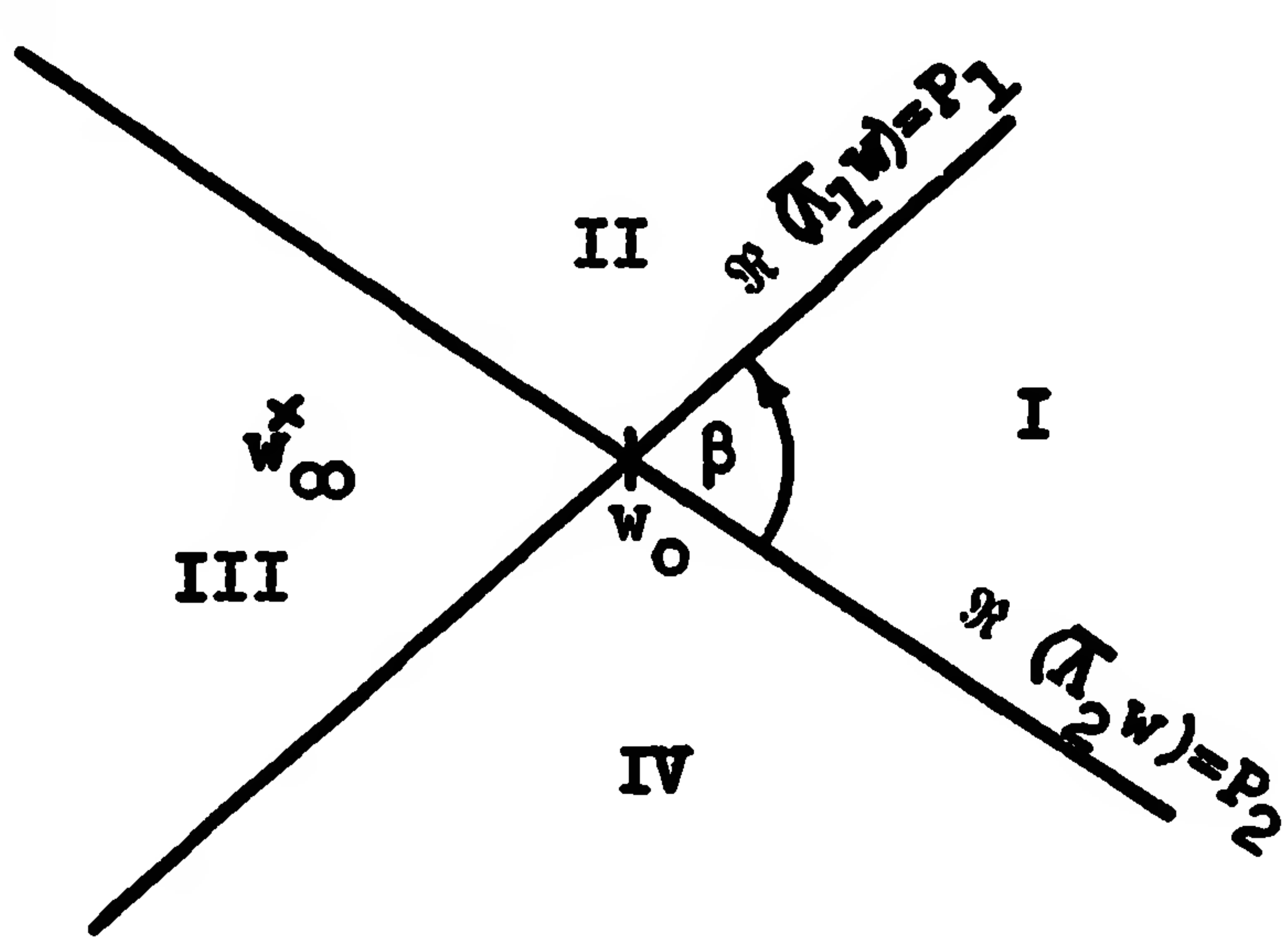
5.8 **Two "circles", intersecting at two points, on two intersecting straight lines.**

z - plane	w - plane

Given $z_1;\ r_1 > 0;\ z_2;\ r_2 > 0$ and $\Lambda_1,\ P_1;\ \Lambda_2,\ P_2$ [P_1, P_2 real].

But $\alpha = \beta$, i.e. $\frac{\bar{\Lambda}_1}{\bar{\Lambda}_2}\frac{z_o-z_1}{z_o-z_2}$ real.

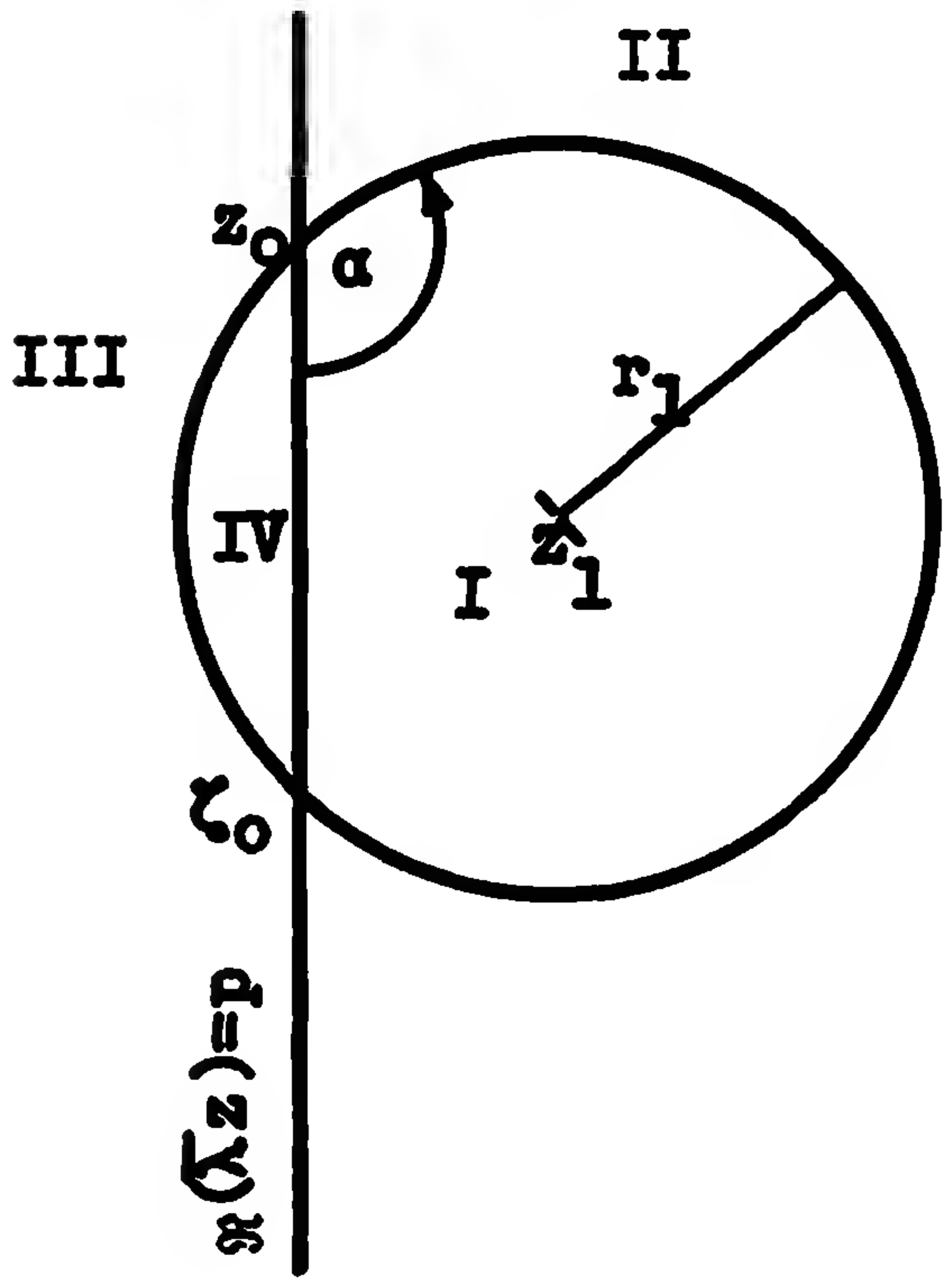




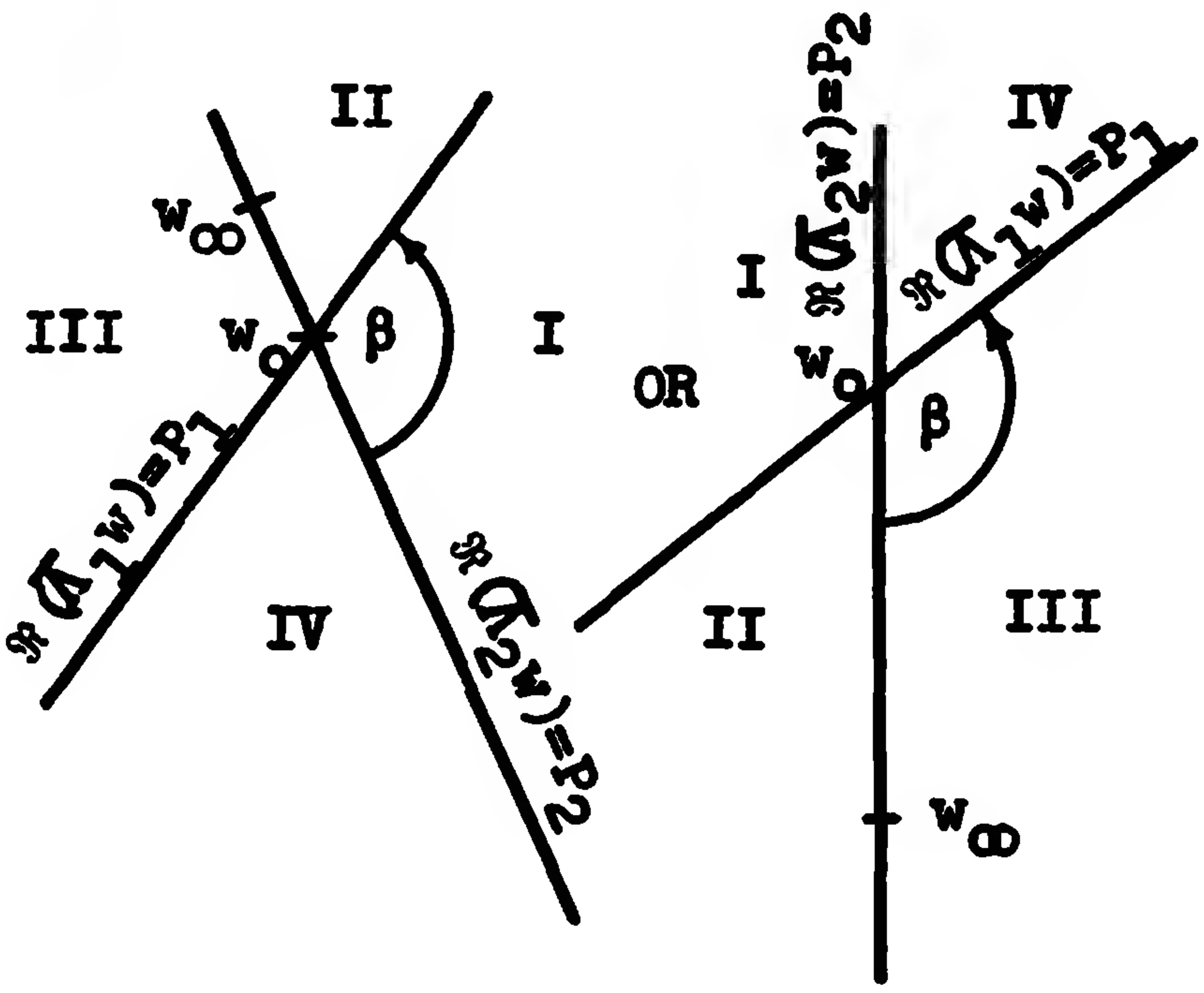
$\lvert r_1-r_2 \rvert < \lvert z_1-z_2 \rvert < r_1+r_2$

z - plane	w - plane
OR	
Given z_1; $r_1 > 0$; λ, p [p real] and	Λ_1, P_1, Λ_2, P_2 [P_1, P_2 real]

But $\alpha = \beta$, i.e. $\dfrac{\bar{\Lambda}_1}{\bar{\Lambda}_2}\,\dfrac{z_1 - z_0}{\lambda}$ real.



$|\Re(\bar{\lambda}z_1) - p| < r_1|\lambda|$



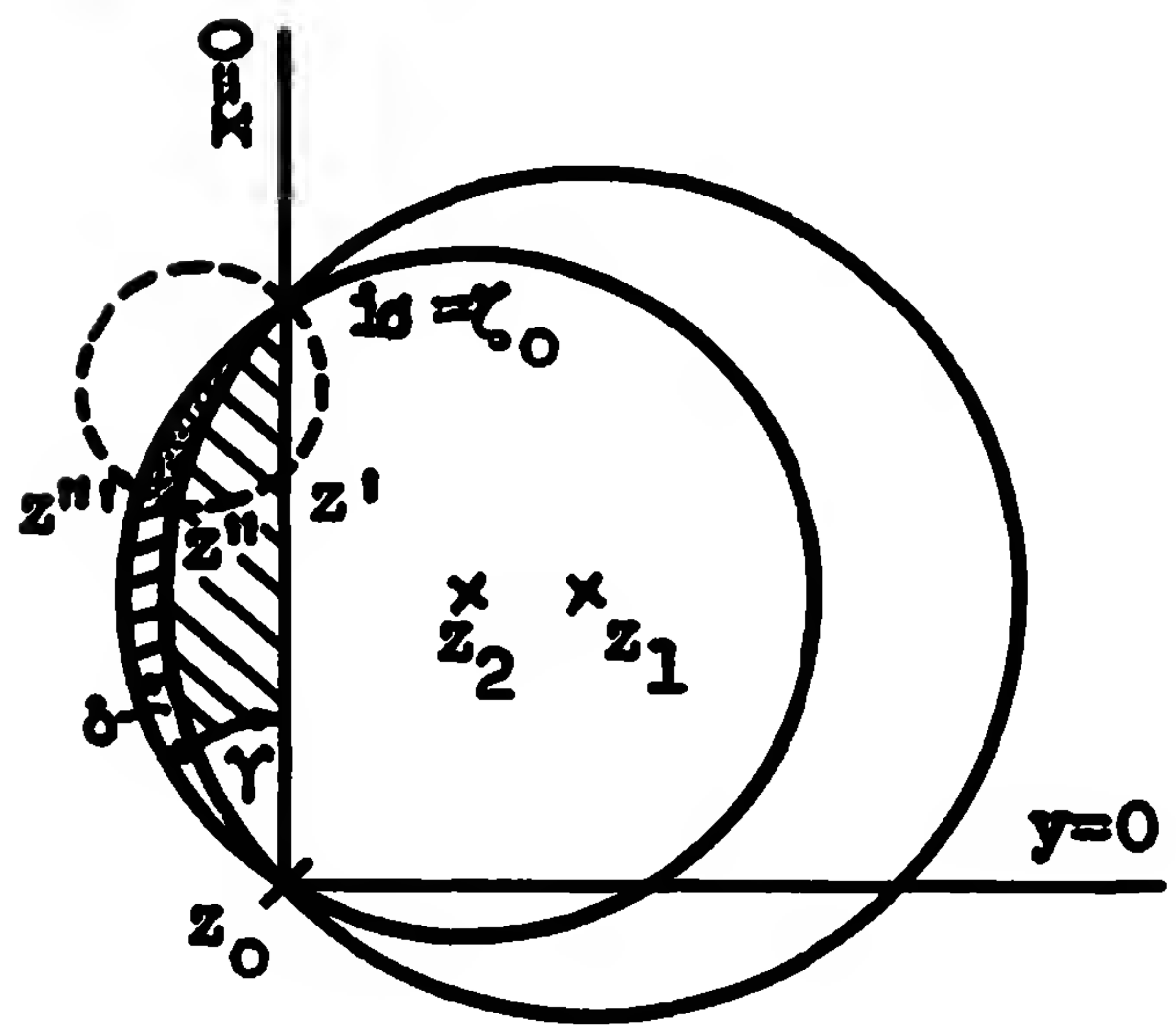
z_0, ζ_0 are the points of intersection of the given "circles" in the z-plane; w_0 is the point of intersection of $\Re(\bar{\Lambda}_1 w) = P_1$ and $\Re(\bar{\Lambda}_2 w) = P_2$.

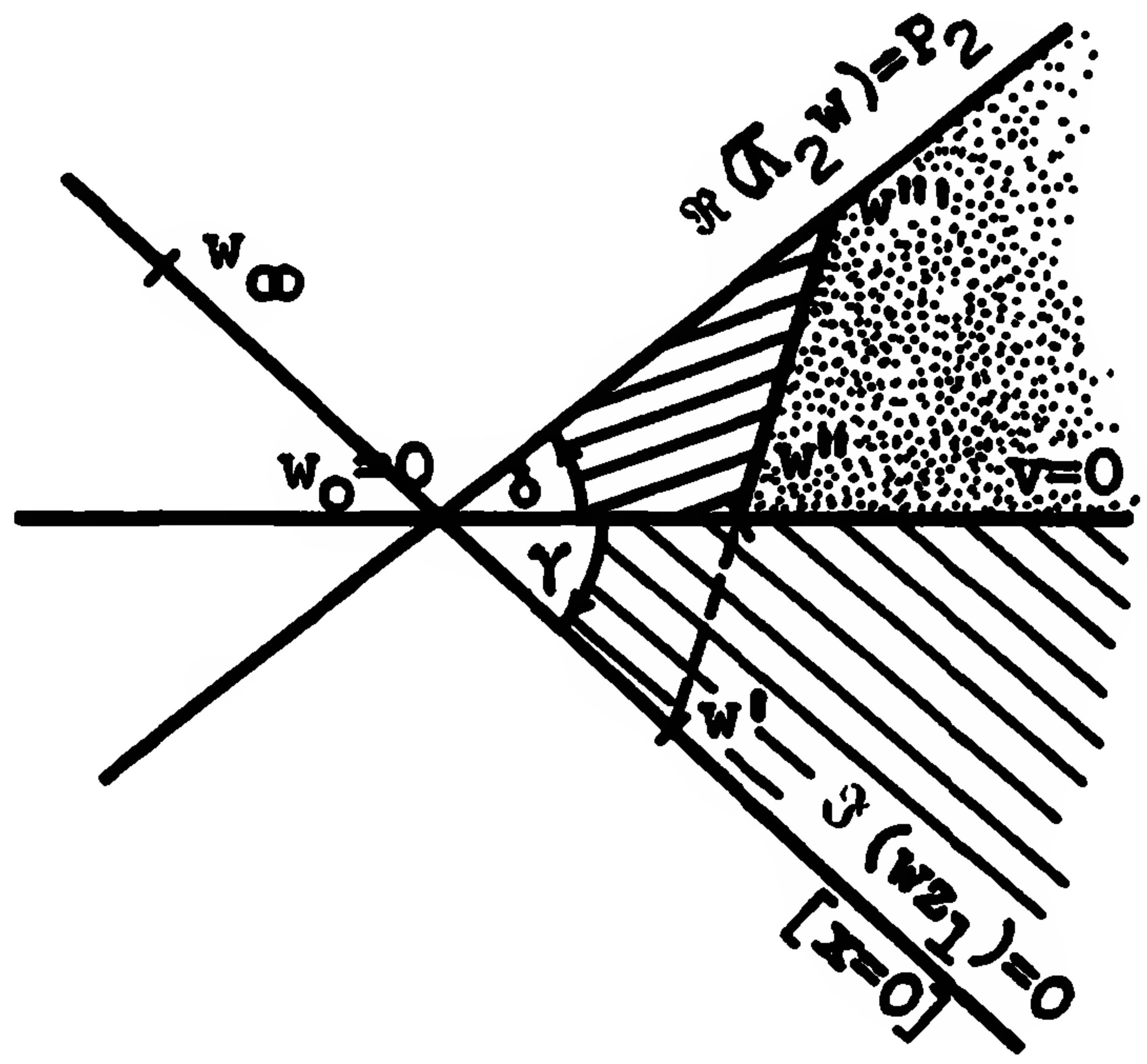
Transformation required: $w = w_0 + K\Lambda_1 \dfrac{(z - z_0)(z_1 - \zeta_0)}{(z - \zeta_0)(z_0 - \zeta_0)}$ [K real, $\neq 0$]

circle $\lvert z - z_1 \rvert = r_1$	line $\Re(\bar{\Lambda}_1 w) = P_1$
circle $\lvert z - z_2 \rvert = r_2$; or line $\Re(\bar{\lambda}z) = p$, respectively	line $\Re(\bar{\Lambda}_2 w) = P_2$
points $z = z_0$; ζ_0 $z = \infty$	points $w = w_0$; ∞ $w_\infty = w_0 + K\Lambda_1 \dfrac{z_1 - \zeta_0}{z_0 - \zeta_0}$
any circle passing through z_0, ζ_0	line passing through w_0

z - plane	w - plane
any circle orthogonal to $\lvert z-z_1 \rvert = r_1$, and to $\lvert z-z_2 \rvert = r_2$ or to $\Re(\bar{\lambda} z) = p$	circle with centre w_o

<u>**Example:**</u>





$$w = \frac{K}{\sigma}\,\frac{z(i\sigma - z_1)}{z - i\sigma}, \quad \sigma > 0, \quad K > 0.$$

z - plane	w - plane
points $\zeta_o = i\sigma$; $z_o = 0$; z'; z''; z'''	points $w = \infty$; $w_o = 0$; w'; w''; w'''
$z = \infty$	$w_\infty = \frac{K}{\sigma}(i\sigma - z_1) = -\frac{K}{\sigma}\bar{z}_1$
circle $\lvert z-z_1\rvert = \lvert z_1\rvert$ $[\Im(z_1) = \frac{1}{2}\sigma]$	line $v = 0$
circle $\lvert z-z_2\rvert = \lvert z_2\rvert$ $[\Im(z_2) = \frac{1}{2}\sigma]$	line $\Re(\bar{\Lambda}_2 w) = P_2$
line $x = 0$	line $\Im(wz_1) = 0$
circle (z', z'', z''', ζ_o)	line (w', w'', w''')
curvilinear triangle z_o, z', z'',	rectilinear triangle w_o, w', w''
curvilinear triangle z_o, z'', z''', each with sum of angles $= \pi$	rectilinear triangle w_o, w'', w'''

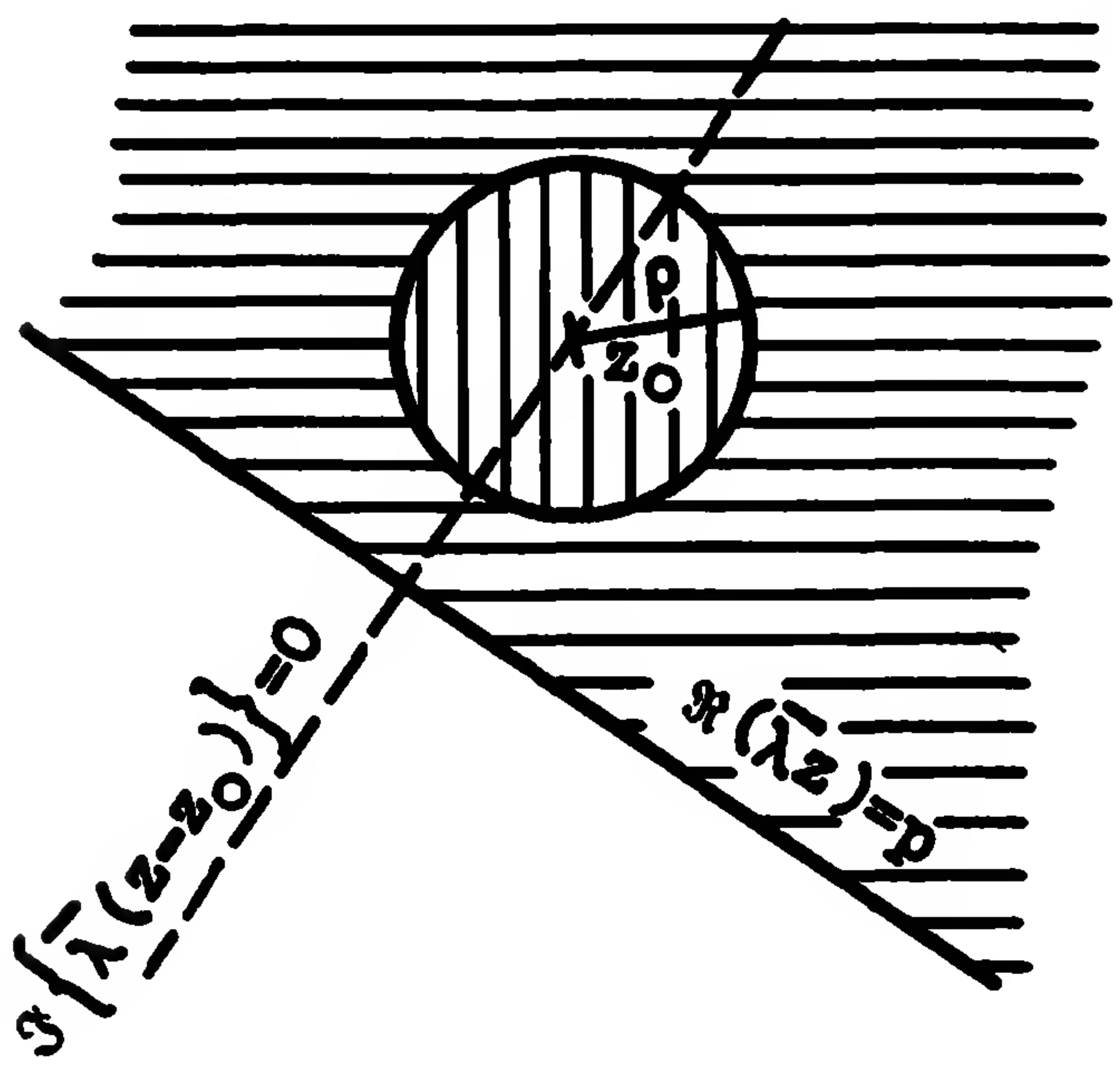
5.9 **Circle and straight line, without common point, on two concentric circles.**

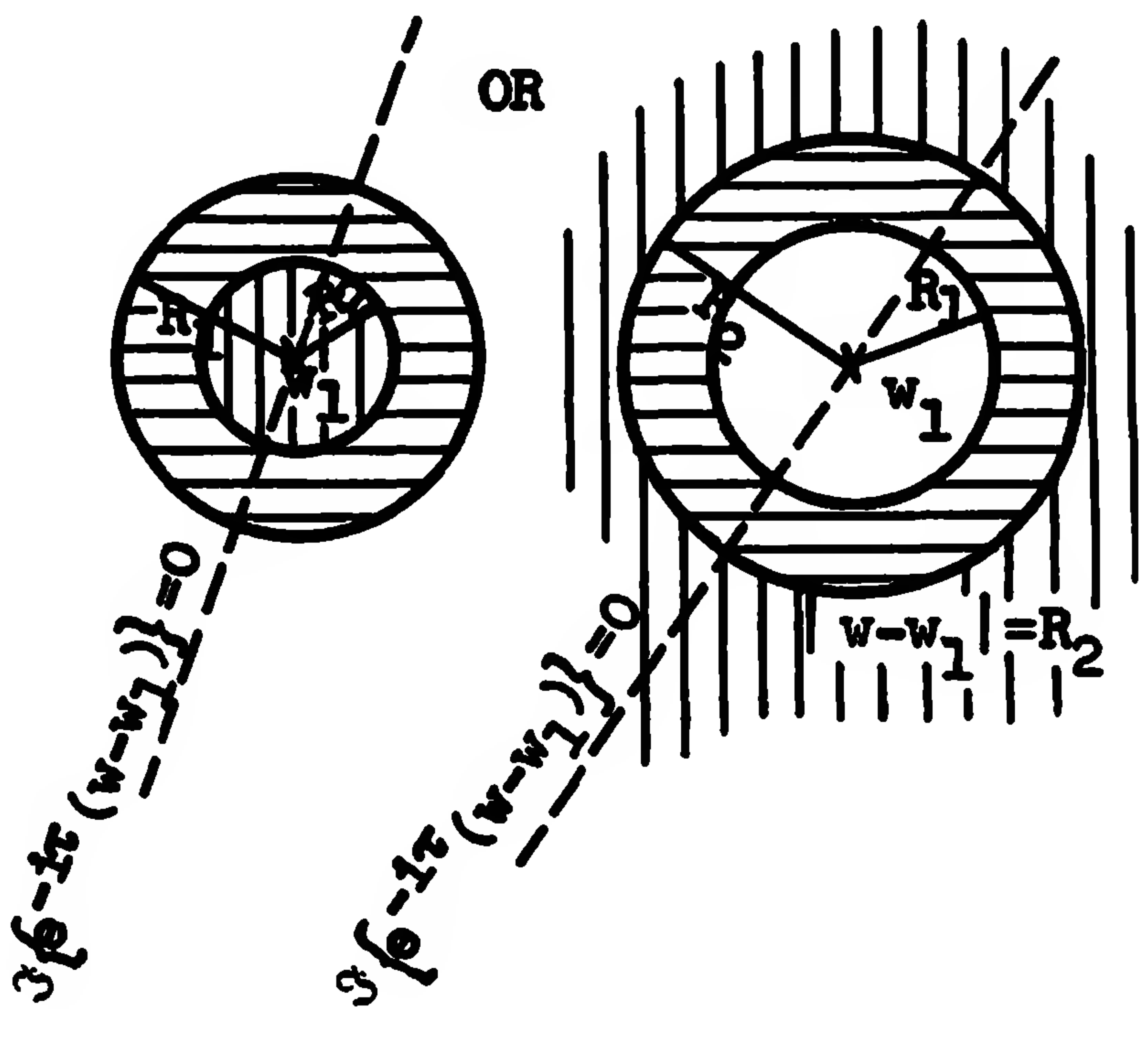
z - plane	w - plane

Transformation: $w-w_1 = R_1 e^{i\tau}\dfrac{(z-z_o)|\lambda| - \lambda(A+\sigma)}{(z-z_o)|\lambda| - \lambda(A-\sigma)}$ where τ real, $A = \dfrac{p-\Re(\bar{\lambda} z_o)}{|\lambda|}$,

$$\sigma = \sqrt{A^2-\rho^2} > 0 \quad \text{or} \quad \sigma = -\sqrt{A^2-\rho^2}$$

Given z_o; $\rho > 0$; λ; p [real], and w_1; $R_1 > 0$; $R_2 > 0$. But $R_1\rho = R_2|A+\sigma|$





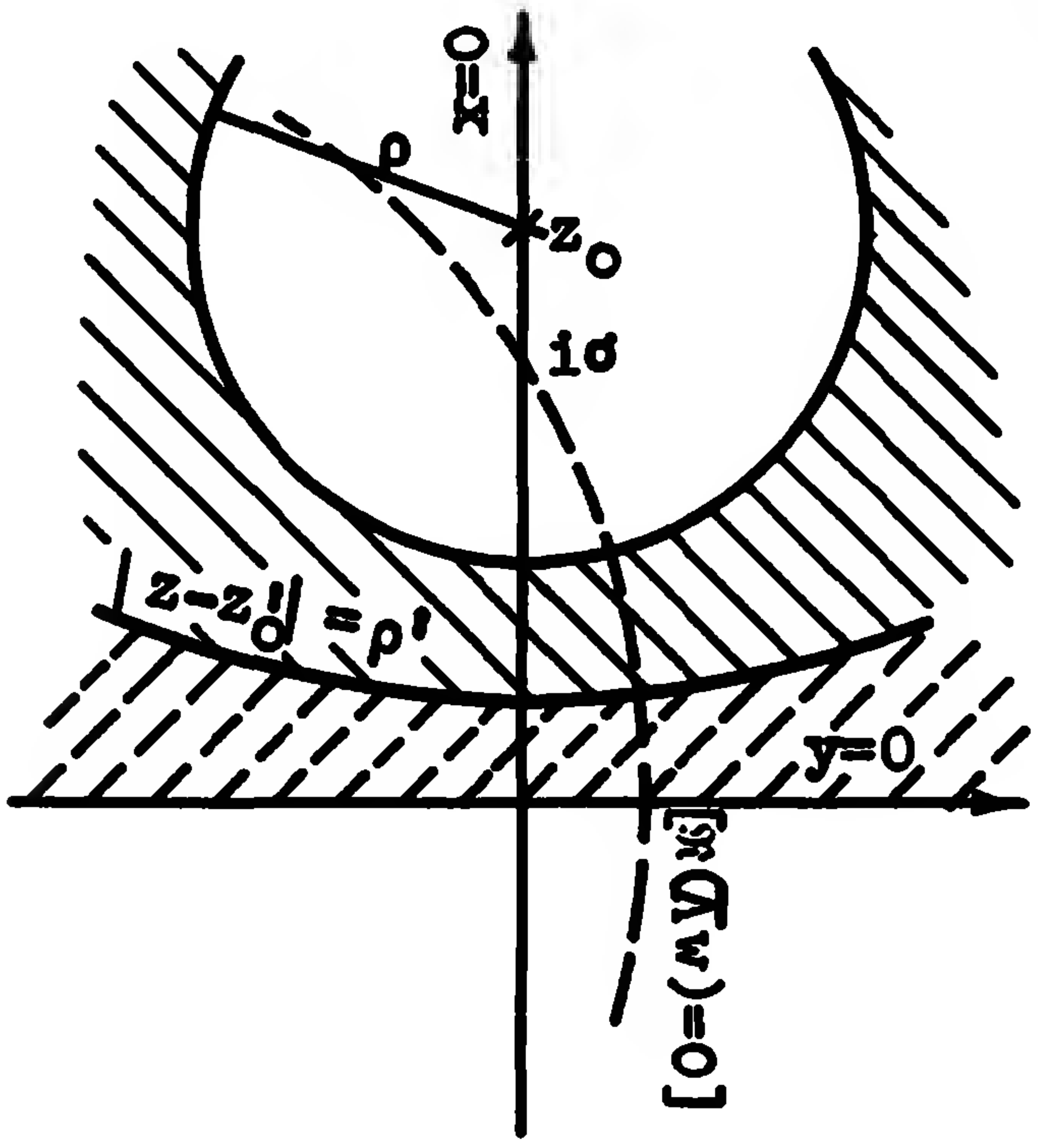
$|A+\sigma| > \rho$ $\qquad$ $|A+\sigma| < \rho$

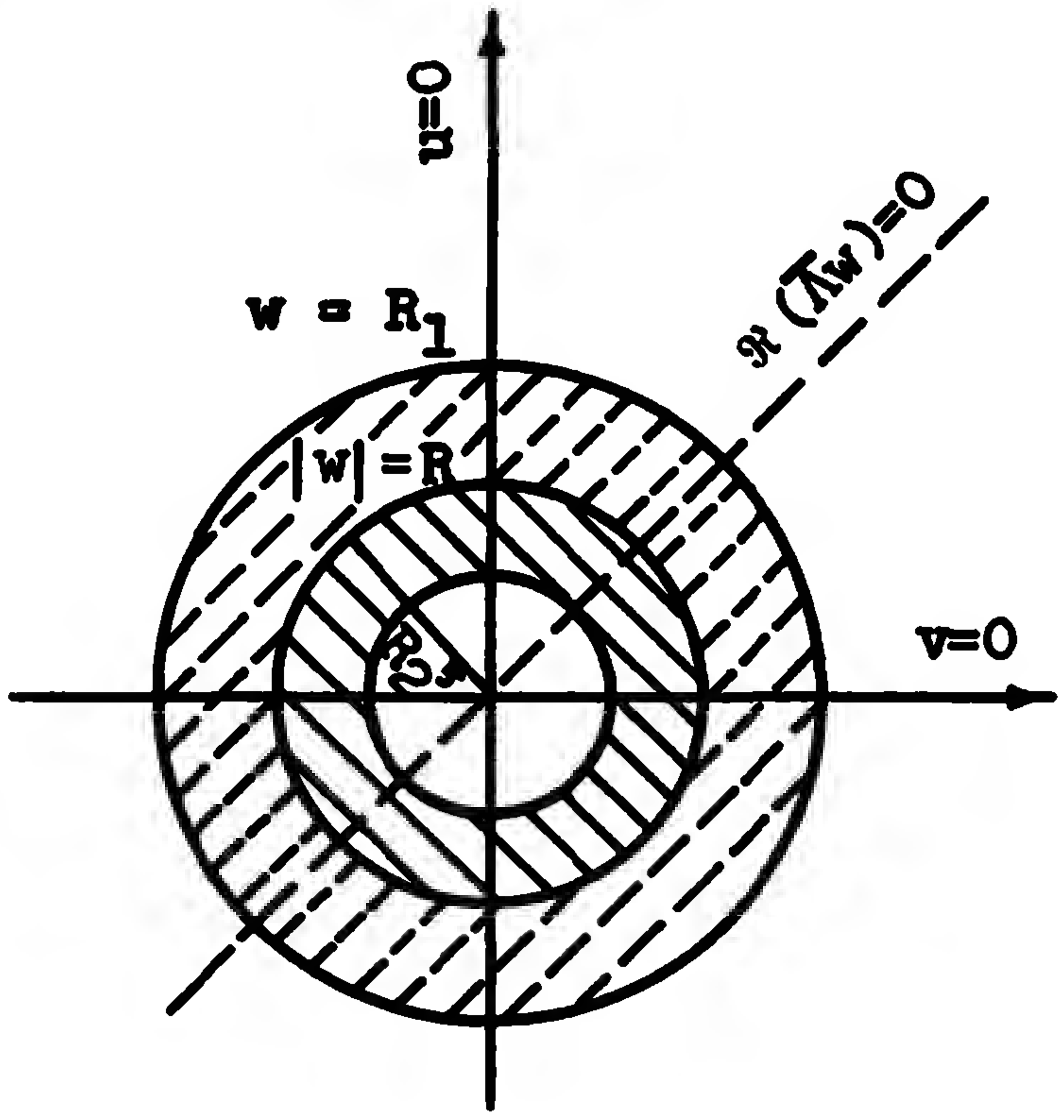
z - plane	w - plane
line $\Re(\bar{\lambda} z) = p$	circle $\lvert w-w_1 \rvert = R_1$
circle $\lvert z-z_o \rvert = \rho$	circle $\lvert w-w_1 \rvert = R_2$
line $\Im\{\bar{\lambda}(z-z_o)\} = 0$	line $\Im\{e^{-i\tau}(w-w_1)\} = 0$

Example:

z - plane	w - plane

$$w = R_1 \frac{z-i\sigma}{z+i\sigma}, \quad \sigma = \sqrt{k^2-\rho^2} > 0; \qquad k > \rho > 0.$$





z - plane	w - plane
$z = 0, \ \infty; \ i\sigma; \ -i\sigma$	$w = -R_1; \ R_1; \ 0; \ \infty.$
line $x = 0$	line $v = 0$
line $y = 0$	circle $\lvert w \rvert = R_1$
circle $\lvert z-ik \rvert = \rho$ $[z_o = ik]$	circle $\lvert w \rvert = R_2 = \dfrac{R_1\rho}{\lvert k+\sigma \rvert}$
circle $\lvert z-z_o' \rvert = \rho'$, where $z_o' = i\sigma \dfrac{R_1^2+R^2}{R_1^2-R^2}$, $\rho' = \dfrac{2\sigma R_1 R}{\lvert R_1^2-R^2 \rvert}$, $R \neq R_1$	circle $\lvert w \rvert = R$
circle $\lvert z \rvert = \sigma$	line $u = 0$
circle $\left\lvert z - \sigma \dfrac{\Im(\Lambda)}{\Re(\Lambda)} \right\rvert = \sigma \left\lvert \dfrac{\Lambda}{\Re(\Lambda)} \right\rvert$, where $\Re(\Lambda) \neq 0$, intersecting $x = 0$ at $\pm i\sigma$	line $\Re(\bar{\Lambda} w) = 0$

5.10 **Two circles, without common point, on two concentric circles.**

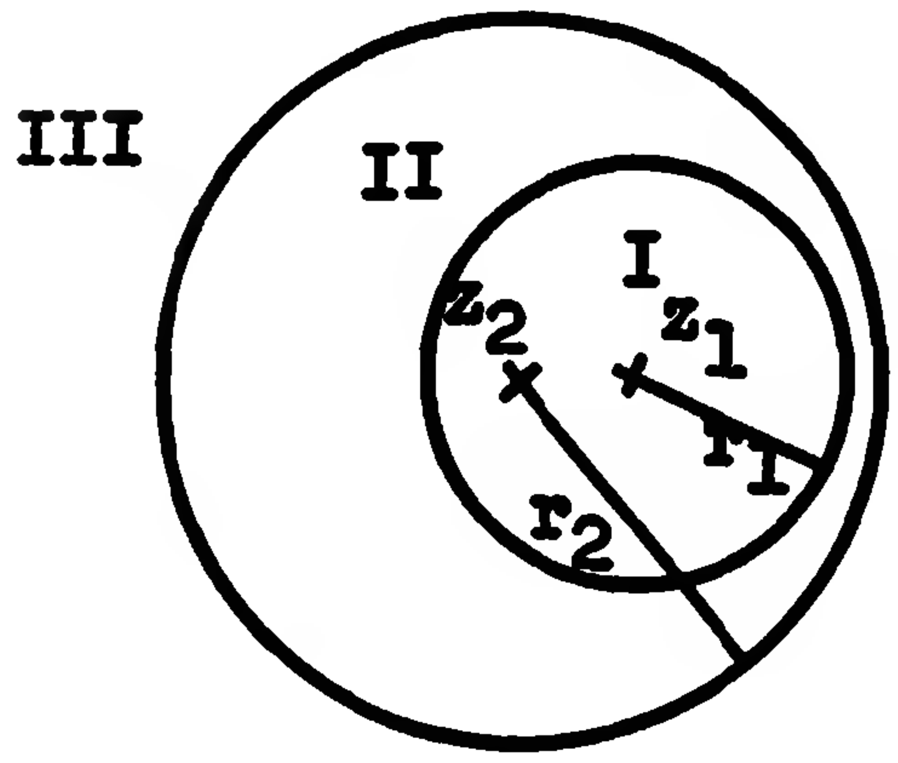
z - plane	w - plane

Transformation: $w-w_o = t\,\dfrac{R_1}{r_1}\,e^{i\theta}\,\dfrac{d(z-z_1)-s(z_2-z_1)}{d(z-z_1)-t(z_2-z_1)}$; θ real, $d = |z_2-z_1| > 0.$

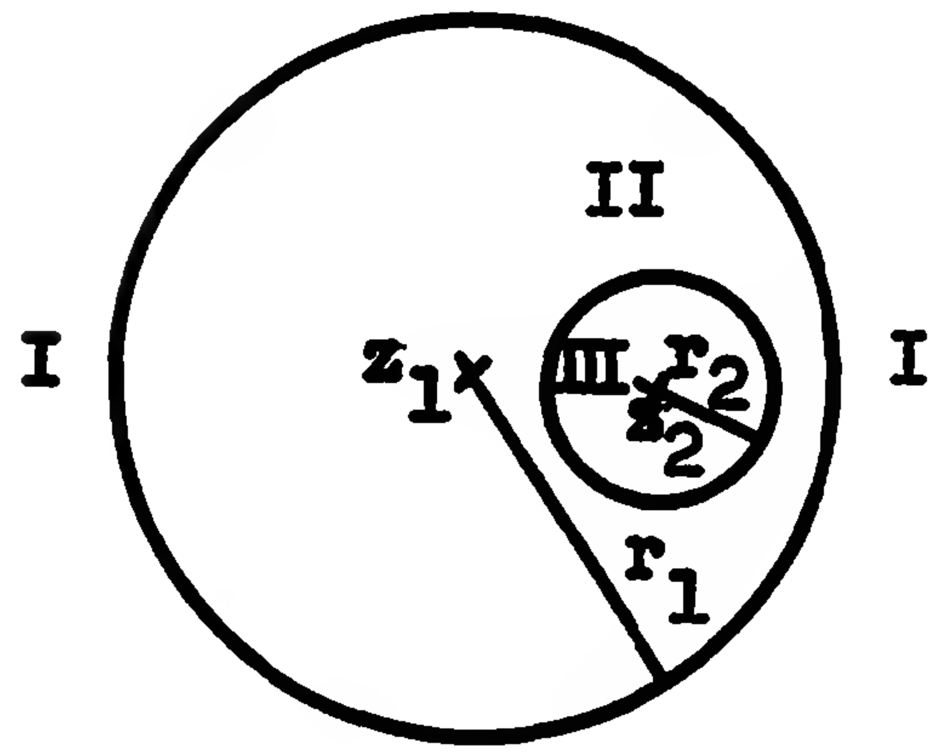
s and t are the roots of the equations $\boxed{\begin{aligned} st &= r_1^2 \\ (d-s)(d-t) &= r_2^2 \end{aligned}}$;

they are real.

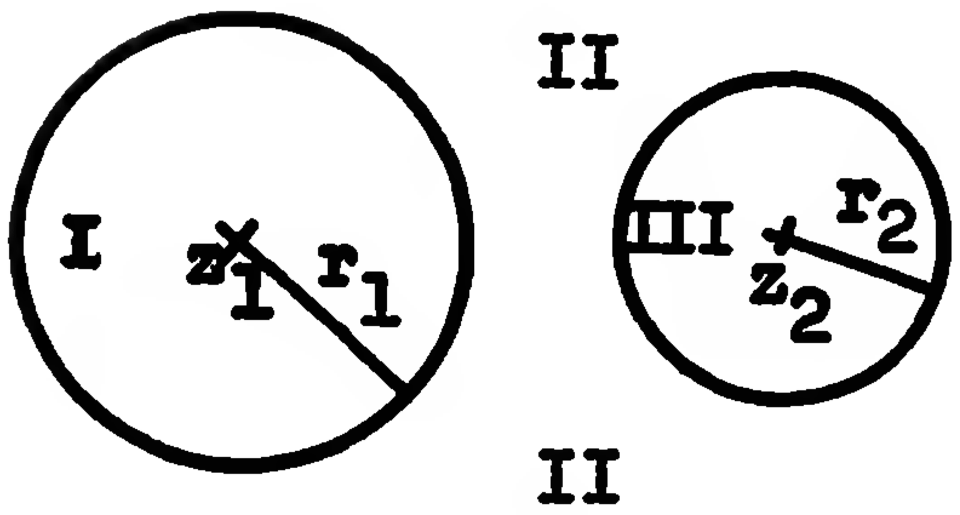
Given $z_1, z_2, r_1 > 0$, $r_2 > 0$; $z_1 \neq z_2$, and w_o, $R_1 > 0$, $R_2 > 0$; but $\dfrac{R_2}{R_1} = \dfrac{r_2}{r_1}\left|\dfrac{t}{d-t}\right|$



OR



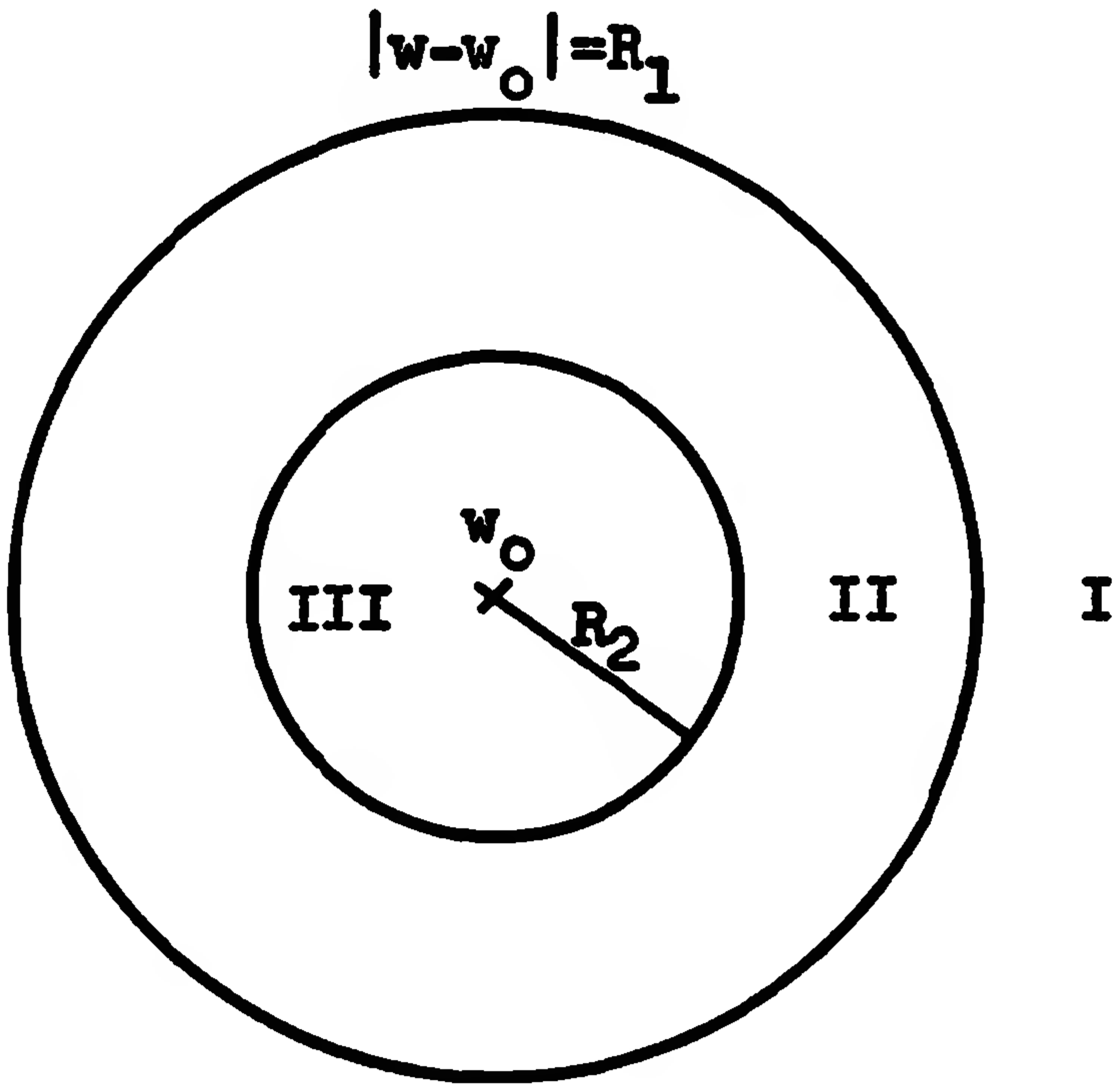
OR



circle $|z-z_1| = r_1$

circle $|z-z_2| = r_2$

radical axis of these circles



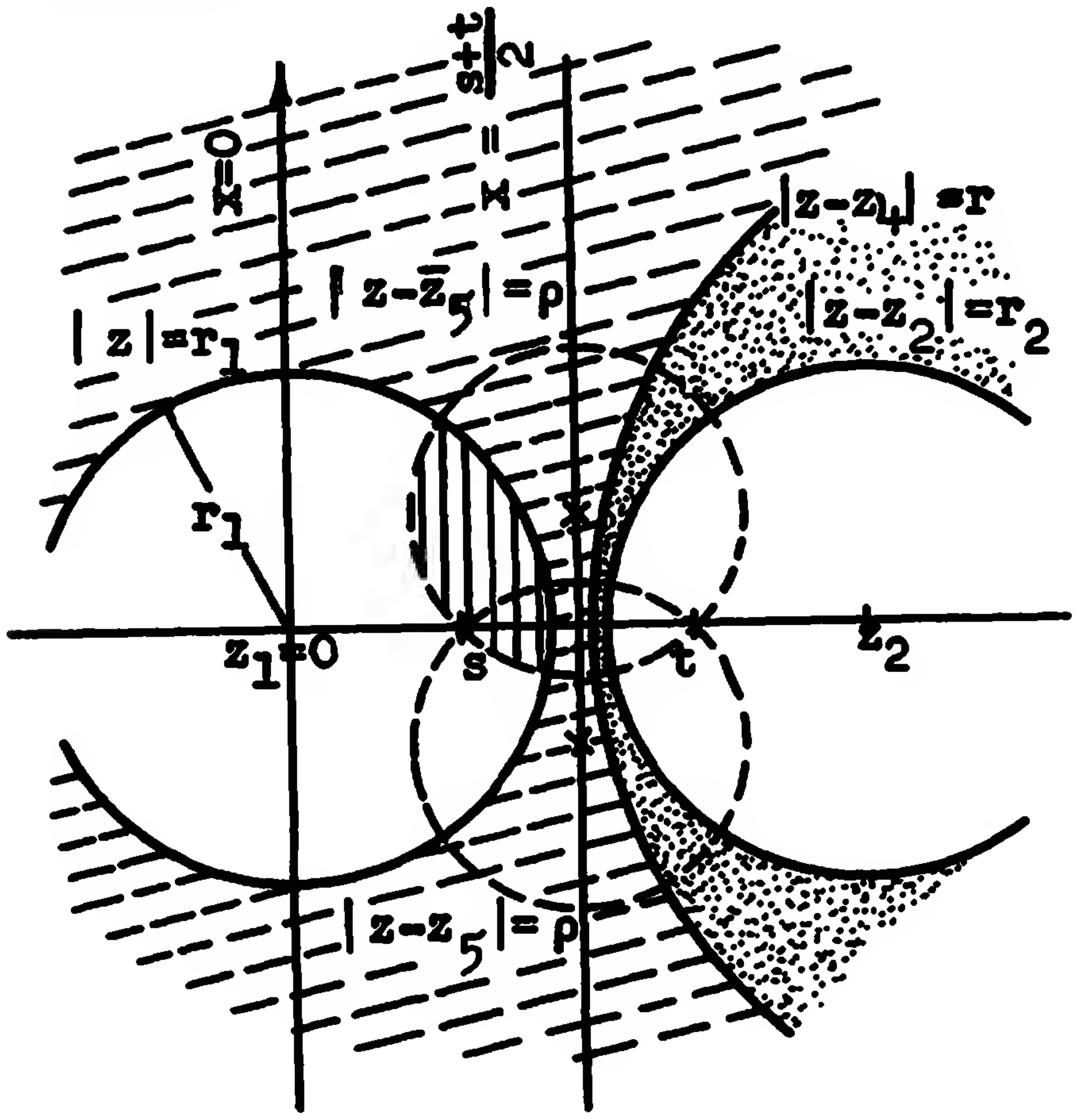
circle $|w-w_o| = R_1$

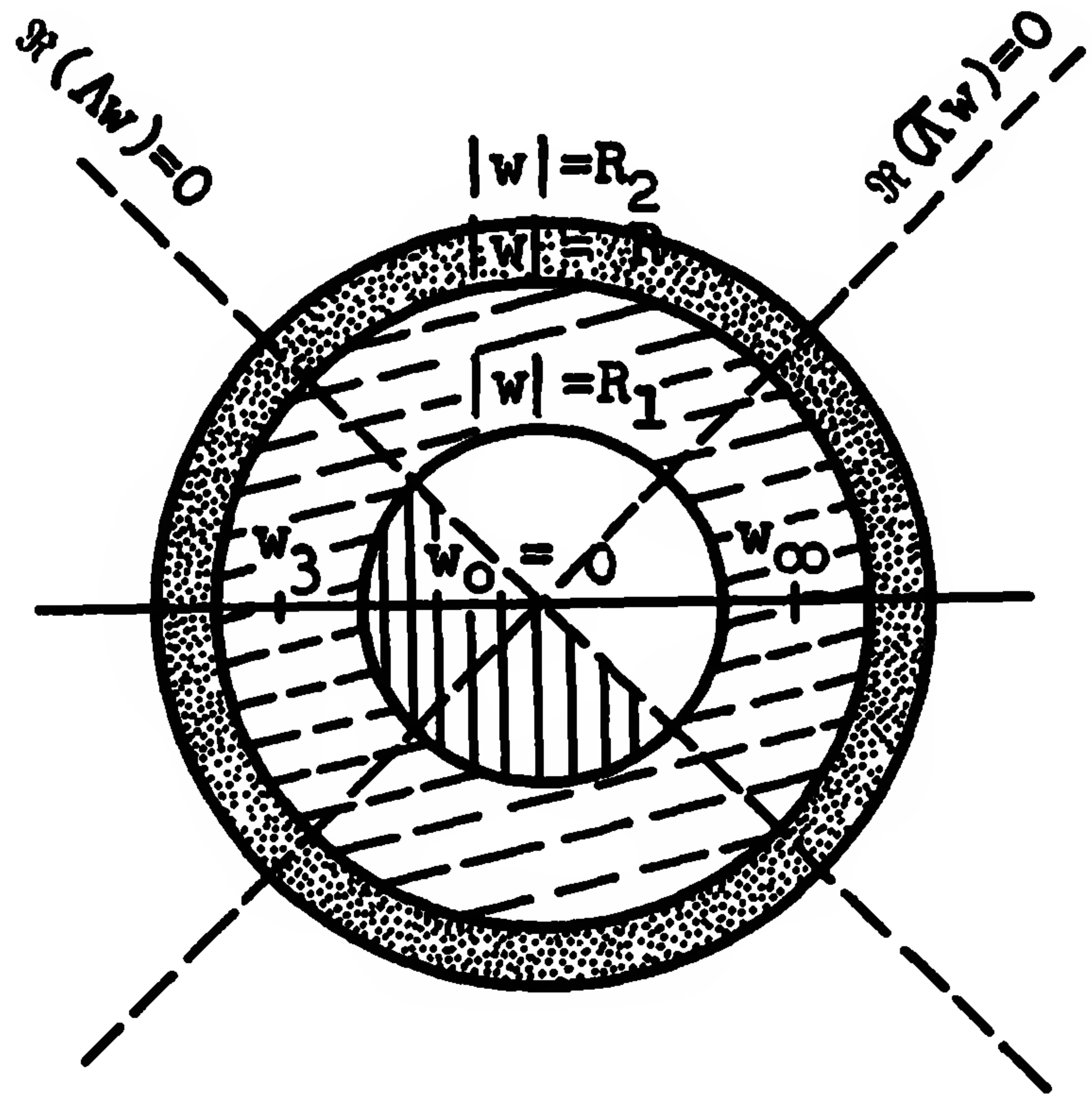
circle $|w-w_o| = R_2$

circle $|w-w_o| = R_1|t|/r_1$

z - plane	w - plane
$z'_\infty = z_1 + \frac{t}{d}(z_2-z_1)$	$w = \infty$
$z_o = z_1 + \frac{s}{d}(z_2-z_1)$	$w = w_o$
any "circle" passing through z'_∞ and z_o	line passing through w_o

Example:





$$w = \frac{tR_1}{r_1}\,\frac{z-s}{z-t};$$

s and t are the roots of $\boxed{\begin{aligned} st &= r_1^2 \\ (z_2-s)(z_2-t) &= r_2^2 \end{aligned}}$

circle $\lvert z-z_{\substack{1\\2}}\rvert = r_{\substack{1\\2}}$ $\begin{bmatrix} z_1 = 0 \\ z_2 > 0 \end{bmatrix}$	circle $\lvert w\rvert = R_{\substack{1\\2}}$; $\frac{R_2}{R_1} = \frac{r_2}{r_1}\,\frac{t}{z_2-t}$
points $z_o = s$; $z'_\infty = t$; $z = \infty$; $z_3 = \frac{s+t}{2}$	points $w_o = 0$; ∞; $w_\infty = \frac{tR_1}{r_1}$; $w_3 = -\frac{tR_1}{r_1}$

z - plane	w - plane
circle $\lvert z-z_4\rvert = r$ where $z_4 = r_1^2 \frac{R_1^2-R^2}{tR_1^2-sR^2}$; $r = \frac{r_1R_1R\,\lvert t-s\rvert}{\lvert sR^2-tR_1^2\rvert}$	circle $\lvert w\rvert = R \neq w_\infty$
line $x = \frac{s+t}{2}$	circle $\lvert w\rvert = \frac{R_1t}{r_1} = w_\infty$
circle $\lvert z-z_5\rvert = \rho$ where $z_5 = \frac{t\Lambda+s\bar{\Lambda}}{2\Re(\Lambda)}$, $\rho = \left\lvert\frac{\Lambda(s-t)}{2\Re(\Lambda)}\right\rvert$, $\Re(\Lambda) \neq 0$	line $\Re(\bar{\Lambda}w) = 0$
circle $\lvert z-\bar{z}_5\rvert = \rho$	line $\Re(\Lambda w) = 0$
line $y = 0$	line $v = 0$
curvilinear rectangular quadrilateral formed by $\lvert z\rvert = r_1$, $\lvert z-z_2\rvert = r_2$ and $\lvert z-z_5\rvert = \rho$	half-ring on the left of $\Re(\bar{\Lambda}w) = 0$ between $\lvert w\rvert = R_1$ and $\lvert w\rvert = R_2$
domain $\lvert z-z_2\rvert \leq r_2$	domain $\lvert w\rvert \geq R_2$

APPENDIX

An example for the use of formulae of Part I in combinations

Problem: On what curves of the w-plane are

(i) the line segments $x = p$, $-\pi \leq y < \pi$ [p real]

(ii) the lines $y = q$ $[-\pi \leq q < \pi]$

of the z-plane mapped by

$$w = \frac{4e^z - 3i}{5e^z + 6i} \text{ ?}$$

Problem (i): The transformation

$$\xi = e^z \qquad [\xi = \sigma + i\tau; \quad \sigma, \tau \text{ real}]$$

maps $x = p$, $-\pi \leq y < \pi$, on the circle $|\xi| = e^p$ in a one-one correspondence (see III, §10.1). The bilinear transformation $w = \dfrac{4\xi - 3i}{5\xi + 6i}$ has the critical points $\xi'_\infty = -\dfrac{6}{5}i$ and ∞, while $w_\infty = \dfrac{4}{5}$.

A.) $p \neq \log \frac{6}{5}$; the circle $|\xi| = e^p$ does not pass through ξ'_∞. The transformation then maps this circle on the circle

$$|w - w_0| = R; \quad w_0 = \frac{18 + 20e^{2p}}{25e^{2p} - 36}; \quad R = \frac{39e^p}{|36 - 25e^{2p}|}$$

in the w-plane (cf. §3.3) in a one-one correspondence.

B.) $p = \log \frac{6}{5}$; the circle $|\xi| = e^p$ is, in this case, mapped on a straight line $\Re(\bar{\Lambda} w) = P$, the formula of §3.3 gives $\Lambda = -\frac{13}{10}$; $P = -\frac{39}{200}$. Hence, $w = (4e^z - 3i)/(5e^z + 6i)$ maps the line segment $x = \log \frac{6}{5}$, $-\pi \leq y < \pi$ on the line $u = \frac{3}{20}$. An easy calculation shows that the part $x = \log \frac{6}{5}$,

$-\pi \leq y < -\frac{\pi}{2}$ is mapped on $u = \frac{3}{20}$, $\frac{13}{20} < v < \infty$; $x = \log\frac{6}{5}$, $-\frac{\pi}{2} < y < \pi$ is mapped on $u = \frac{3}{20}$, $-\infty < v < \frac{13}{20}$, as corresponding points are $z = \log\frac{6}{5} \mp i\pi$; $z = \log\frac{6}{5} - \frac{i\pi}{2}$; $z = \log\frac{6}{5}$; $z = \log\frac{6}{5} + i\frac{\pi}{2}$, and $w = \frac{3}{20} + \frac{13}{20}i$; $w = \infty$; $w = \frac{3}{20} - \frac{13}{20}i$; $w = \frac{3}{20}$, respectively.

Problem (ii): The transformation $\xi = e^z$ maps $y = q$ on the half-line

$$\xi = e^{iq}e^x, \quad -\infty < x < \infty \quad [0 < |\xi| < \infty],$$

i.e., on part of the line $\Re(\bar{\lambda}\xi) = 0$, $\lambda = ie^{iq}$. For $q = \pm\pi/2$, this line passes through $\xi'_\infty = -\frac{61}{5}$. Hence, we take

A.) $q = \pm\pi/2$. Then $\Re(\bar{\lambda}\xi) = 0$ is transformed into a line $\Re(\bar{\Lambda}w) = P$, the formula of §3.3 gives $\Lambda = \frac{39}{25}e^{-iq} = \mp i\frac{39}{25}$; $P = 0$. Hence either of the lines $y = \pi/2$, $-\infty < x < \infty$, and $y = -\pi/2$, $-\infty < x < \infty$, is transformed into part of $v = 0$; into the part $-\frac{1}{2} < u < \frac{4}{5}$, or into the remaining segments $u < -\frac{1}{2}$ and $u > \frac{4}{5}$, respectively, as a little calculation shows.

B.) $q \neq \pm\pi/2$. By the formula of §3.3, we see that the line $\Re(\bar{\lambda}\xi) = 0$ $[\lambda = ie^{iq}]$ is transformed into the circle

$$|w - w_0| = R; \quad w_0 = \frac{8e^{iq} - 5e^{-iq}}{20\cos q}, \quad R = \frac{13}{20|\cos q|}.$$

Hence $y = q$, $-\infty < x < \infty$ is mapped on one of the two arcs of this circle, bounded by the two points $w = -\frac{1}{2}$ and $w_\infty = \frac{4}{5}$. These circles, and the line $v = 0$, form a pencil, orthogonal to the set of "circles" obtained in problem (i).

PART TWO

ALGEBRAIC FUNCTIONS, AND $w = z^\alpha$ FOR REAL α.

6. THE FUNCTIONS $w = z^2$, $w = z^\alpha$, $w = az^\alpha + bz^\beta$ $(\alpha > 0 > \beta)$, $w = az^\alpha + bz^\beta$ $(\beta > \alpha > 0)$.

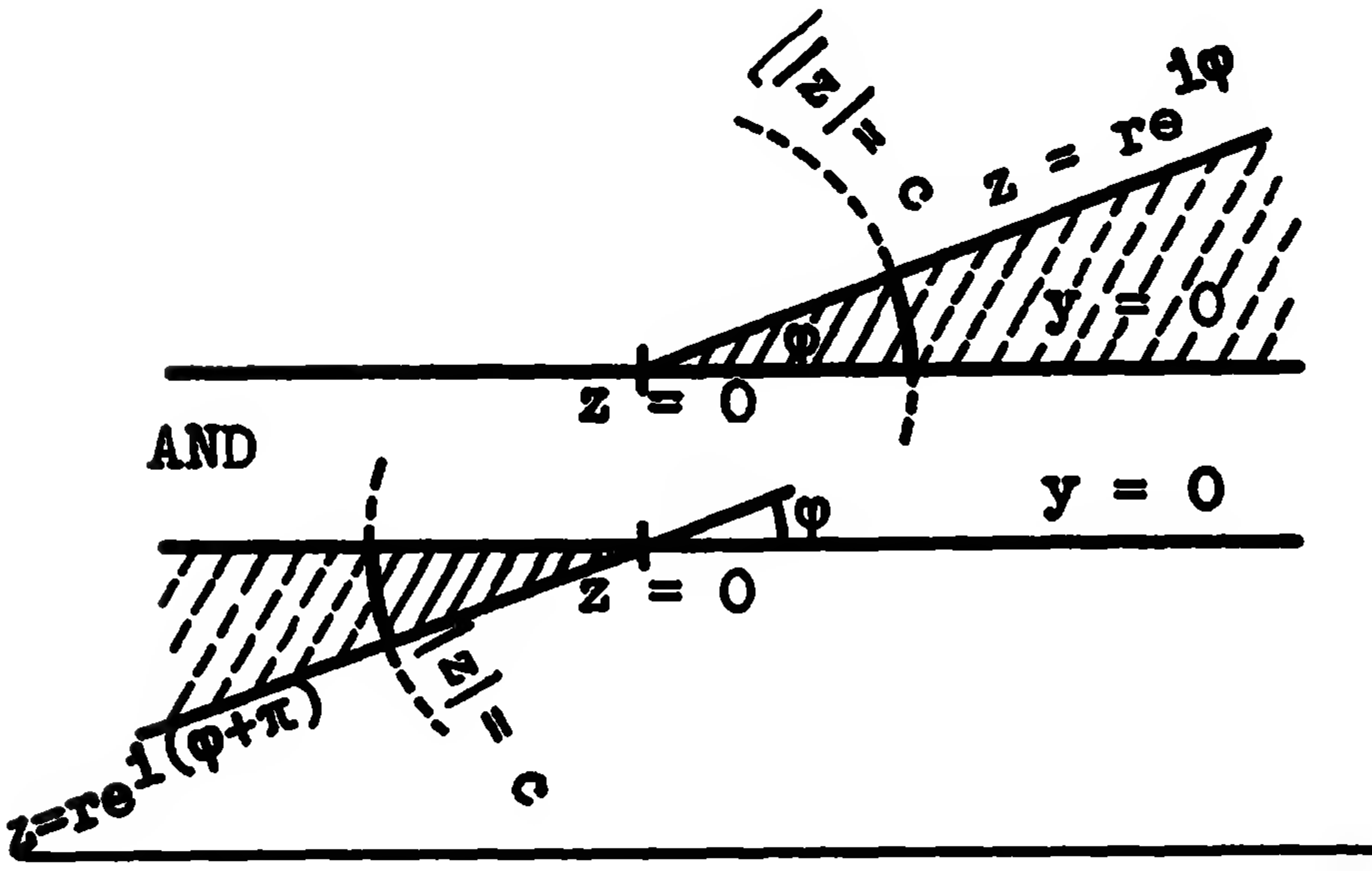
6.1 $\boxed{w = z^2}$; $\boxed{z = \sqrt{w}}$

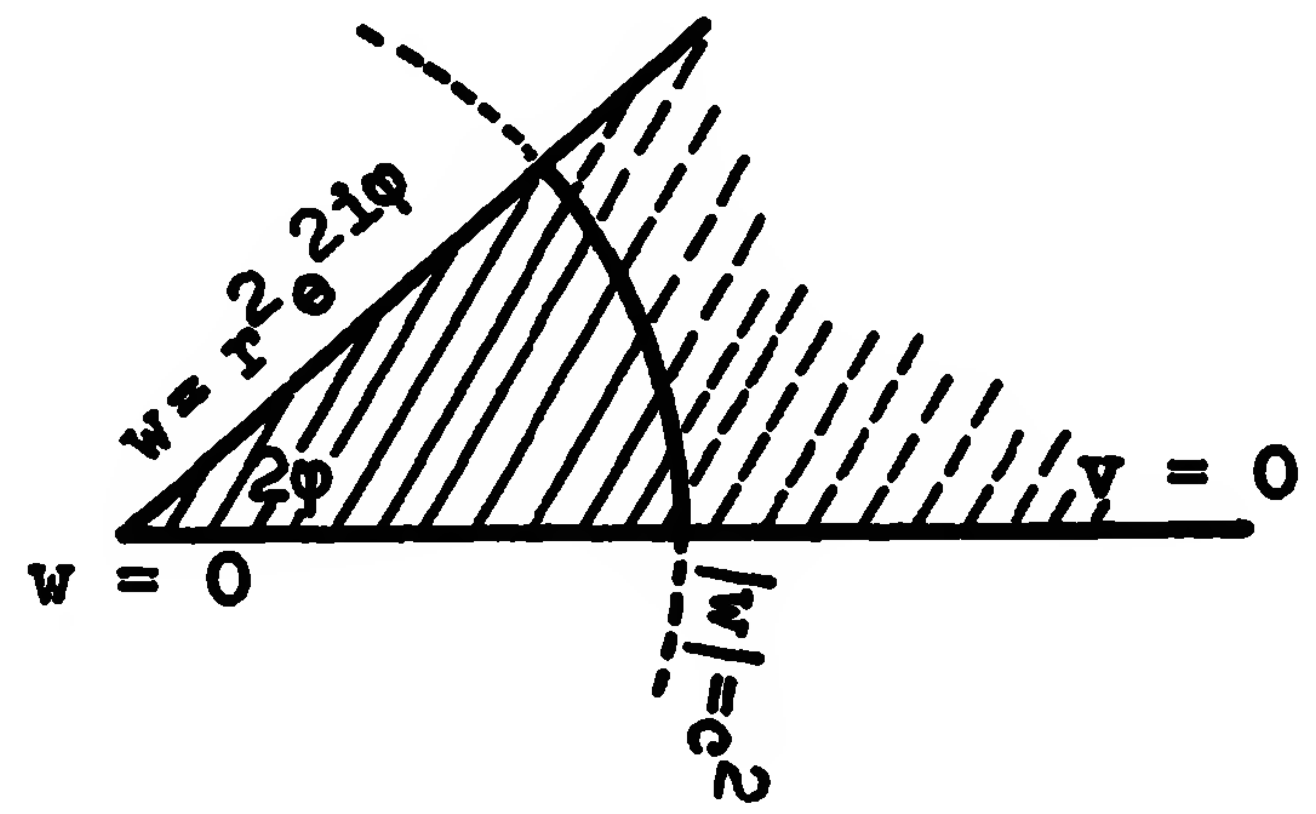
Critical points: $z = 0$ and $z = \infty$.

*Fixed points: $F_1 = 0$; $F_2 = 1$

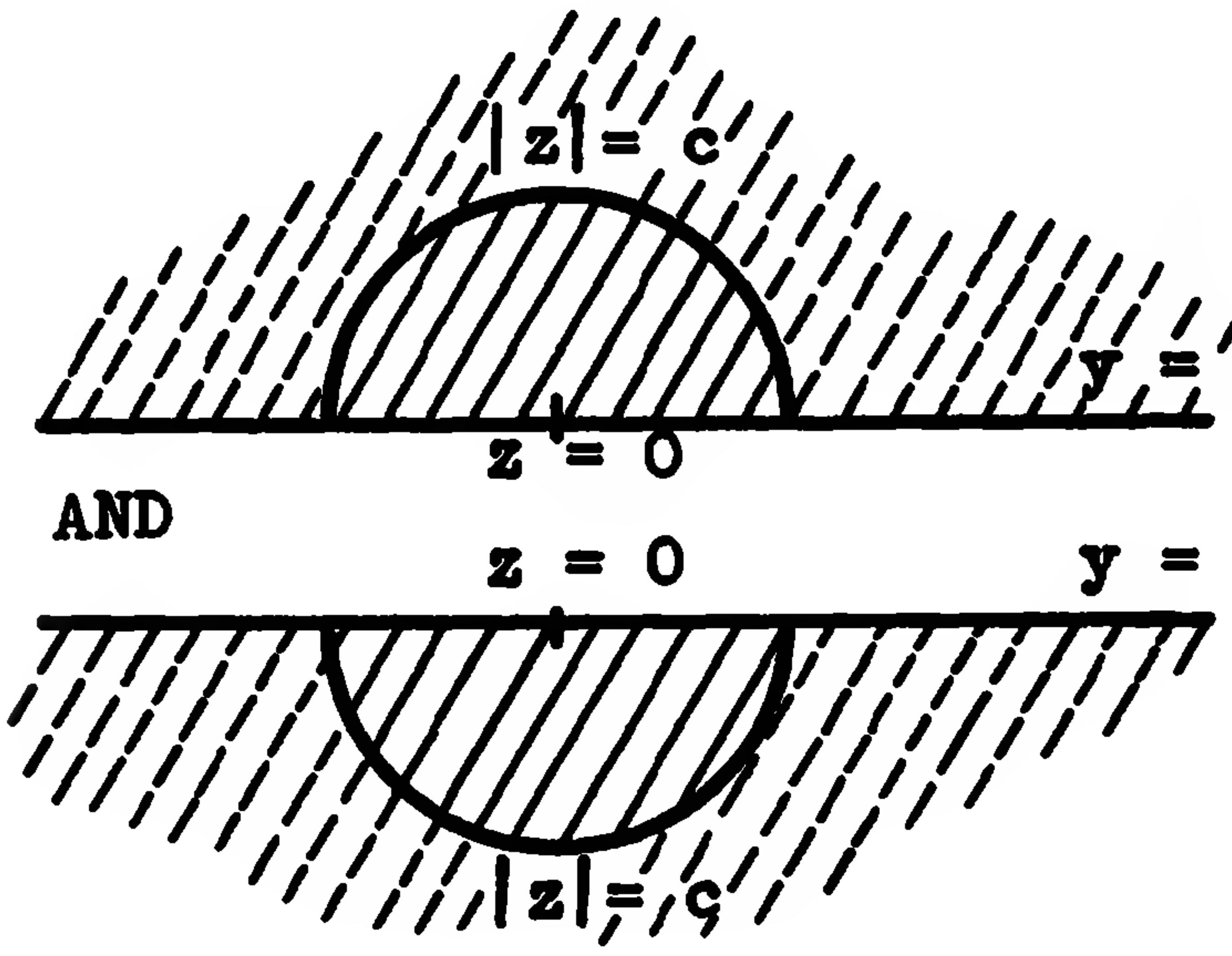
z - plane	w - plane
half-line $y = 0$; $0 \leq x < \infty$ half-line $y = 0$; $0 \geq x > -\infty$	half-line $v = 0$; $0 \leq u < \infty$
half-line $x = 0$; $0 \leq y < \infty$ half-line $x = 0$; $0 \geq y > -\infty$	half-line $v = 0$; $0 \geq u > \infty$
half-line $z = re^{i\varphi}$ half-line $z = re^{i(\varphi+\pi)}$ $\Big\}$ φ fixed, $0 \leq r < \infty$	half-line $w = Re^{2i\varphi}$ $[0 \leq R = r^2 < \infty]$

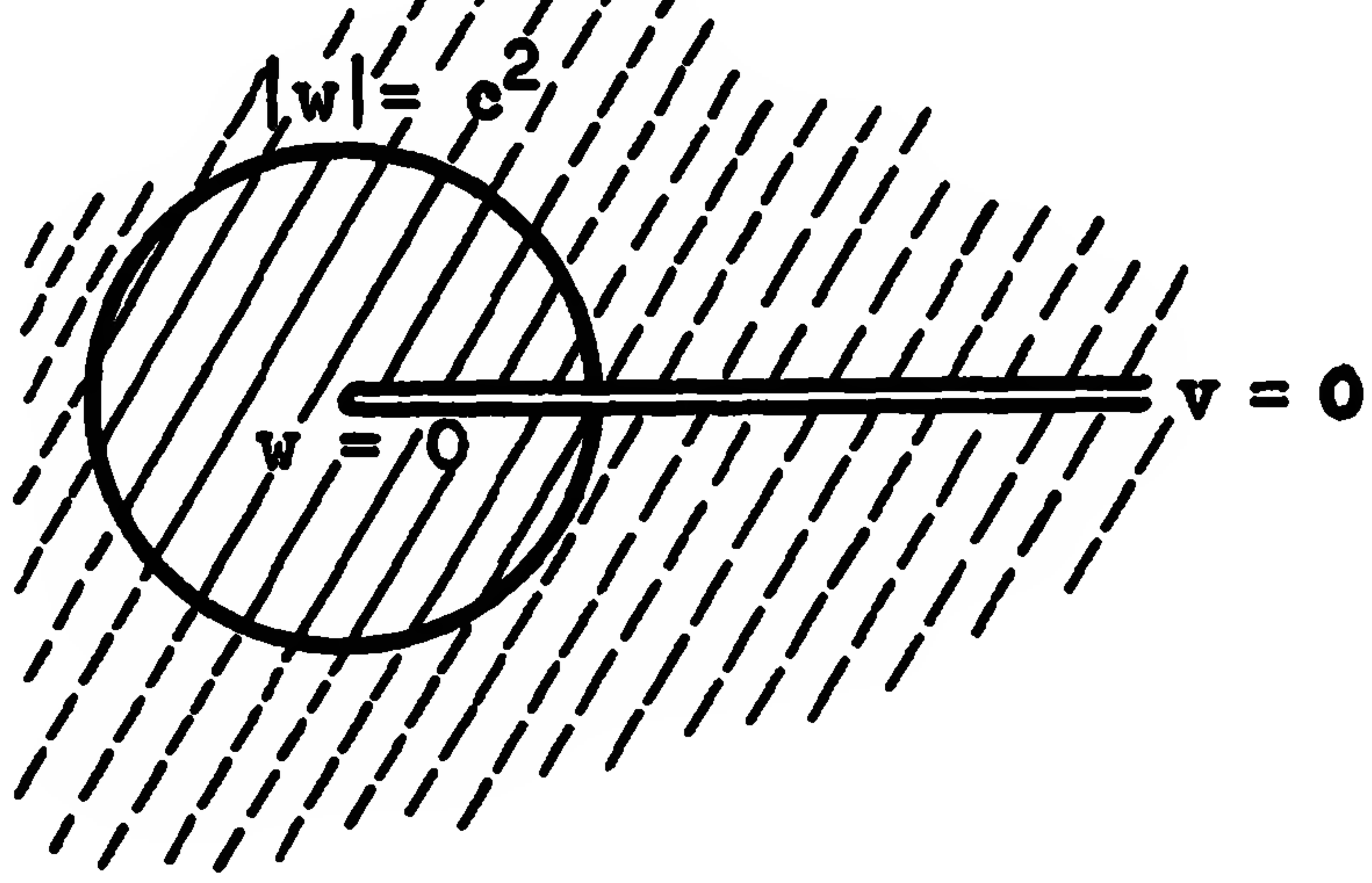
$\varphi < \pi$





$\varphi = \pi$

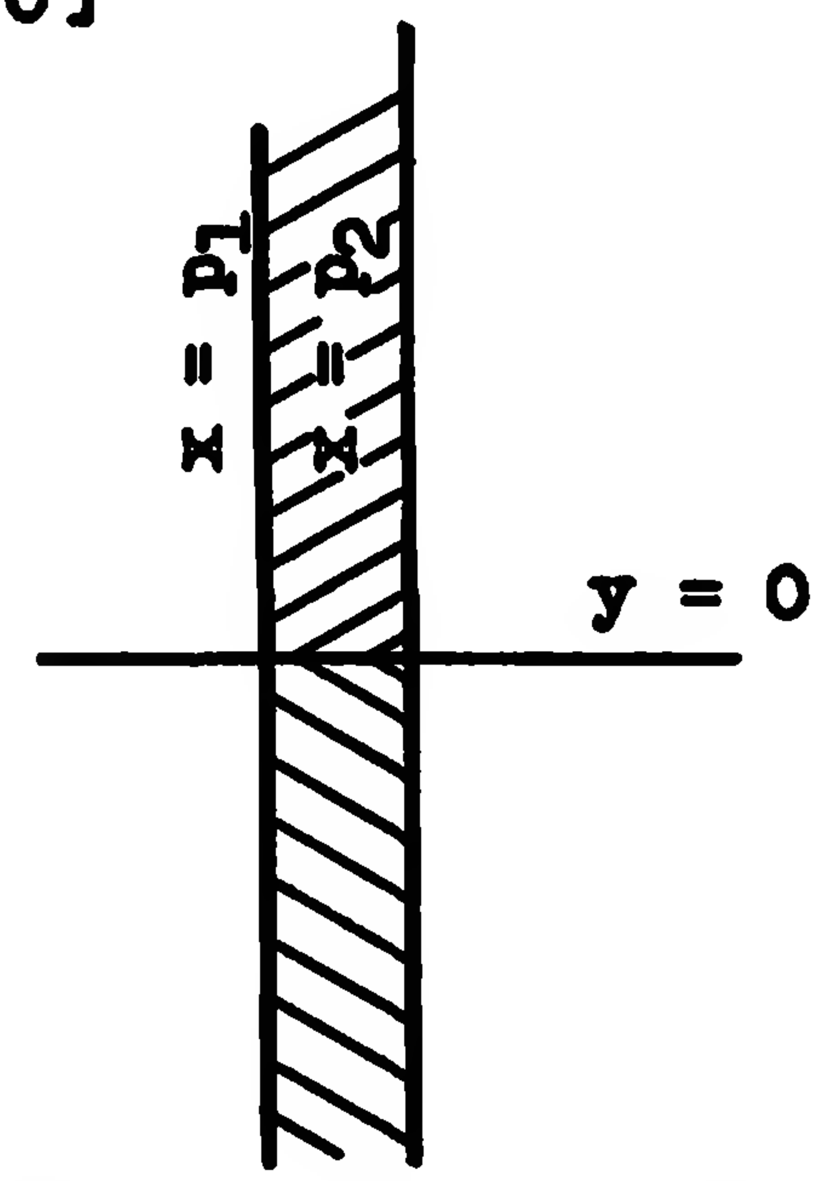


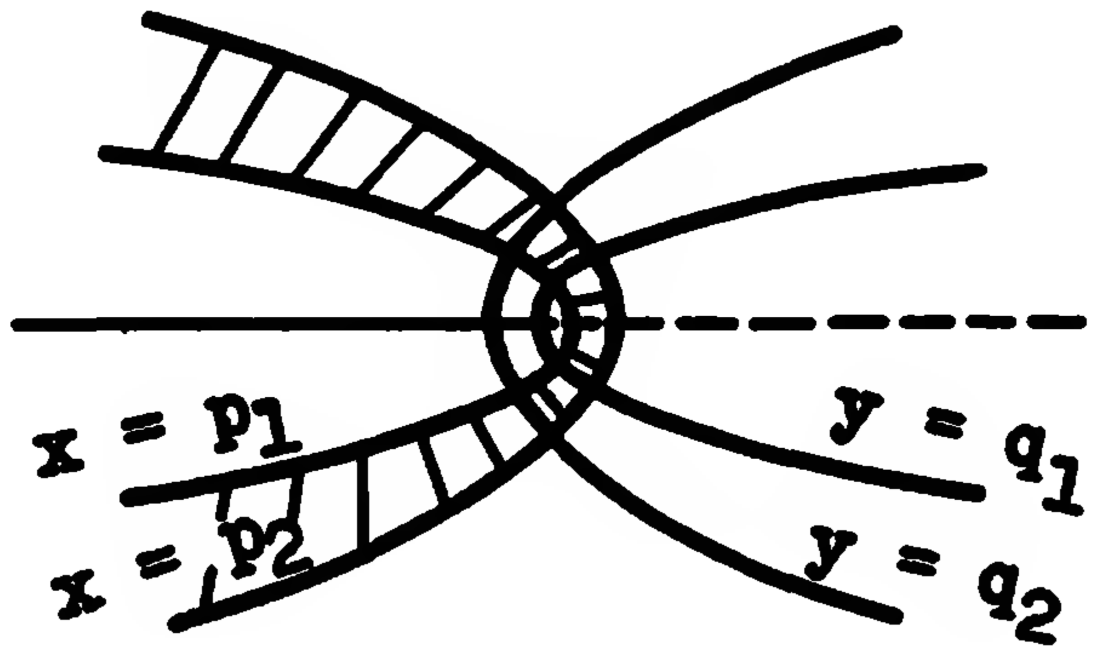


z - plane	w - plane
half-plane $y > 0$ } half-plane $y < 0$ }	cut w-plane
semi-circle $\lvert z\rvert = c,\ y \geq 0$ } semi-circle $\lvert z\rvert = c,\ y \leq 0$ } $(c > 0)$	circle $\lvert w\rvert = c^2$; point $w = c^2$ counted twice

Parabola

z - plane	w - plane
line $\Re(\bar{\lambda} z) = s \quad (s \gtrless 0)$ } line $\Re(\bar{\lambda} z) = -s$ }	parabola $\lvert w\rvert = \lvert 2s^2 - \Re(w\bar{\lambda}^2)\rvert \lvert\lambda\rvert^{-2}$, focus $w = 0$, directrix $\Re(w\bar{\lambda}^2) = 2s^2$
half-plane $\Re(\bar{\lambda} z/s) > 1$	exterior of this parabola
strip $0 < \Re(\bar{\lambda} z/s) < 1$	its interior, cut along the axis of the parabola from the focus to infinity
line $x = p \quad (p \gtrless 0)$ } line $x = -p$ }	parabola $v^2 = -4p^2(u-p^2)$
line $y = q \quad (q \gtrless 0)$ }‡ line $y = -q$ }	parabola $v^2 = 4q^2(u+q^2)$
strip bounded by $x = p_1$ and $x = p_2$ $[p_1 p_2 > 0]$	region between $v^2 = 4p_1^2(p_1^2 - u)$ and $v^2 = 4p_2^2(p_2^2 - u)$





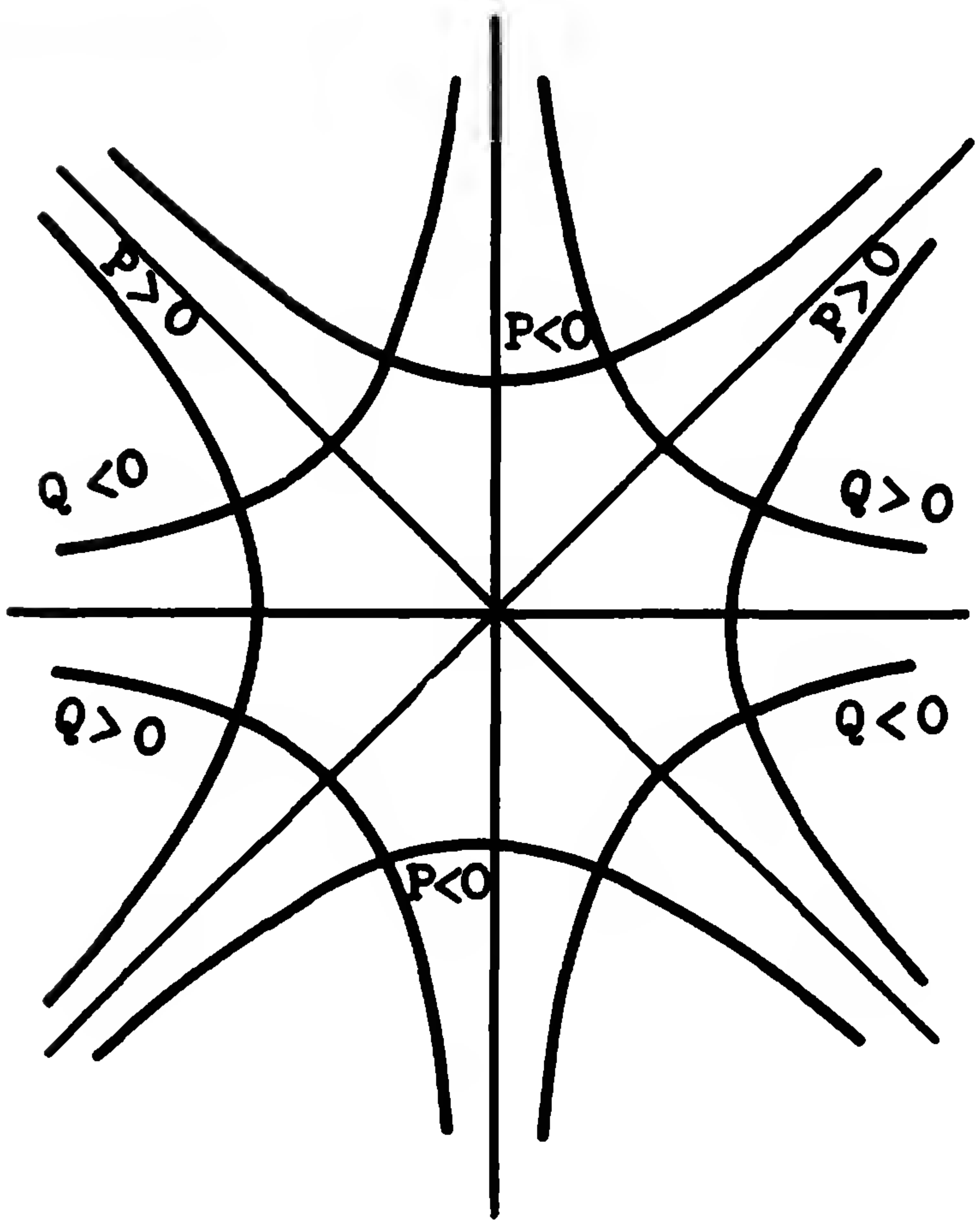
‡ The single curled bracket indicates that each of the two lines of the z-plane (in this case both $y = q$ and $y = -q$) is mapped on the curve of the w-plane [here on $v^2 = 4q^2(u + q^2)$] in a one-one correspondence.

Rectangular hyperbola $\Re(c^{-2}z^2) = \frac{1}{2}$

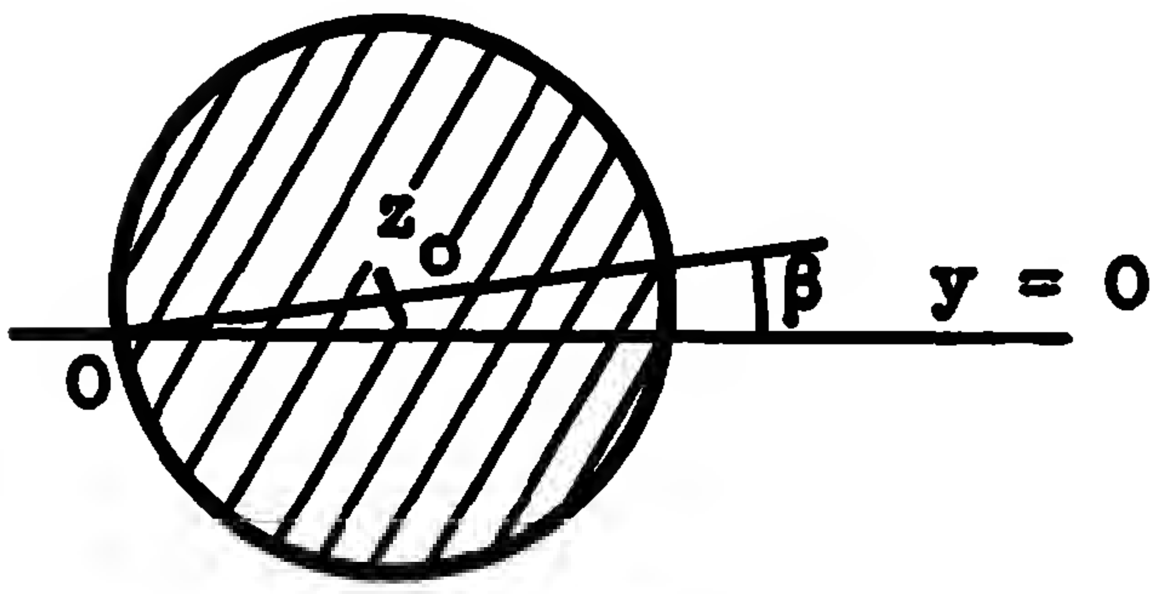
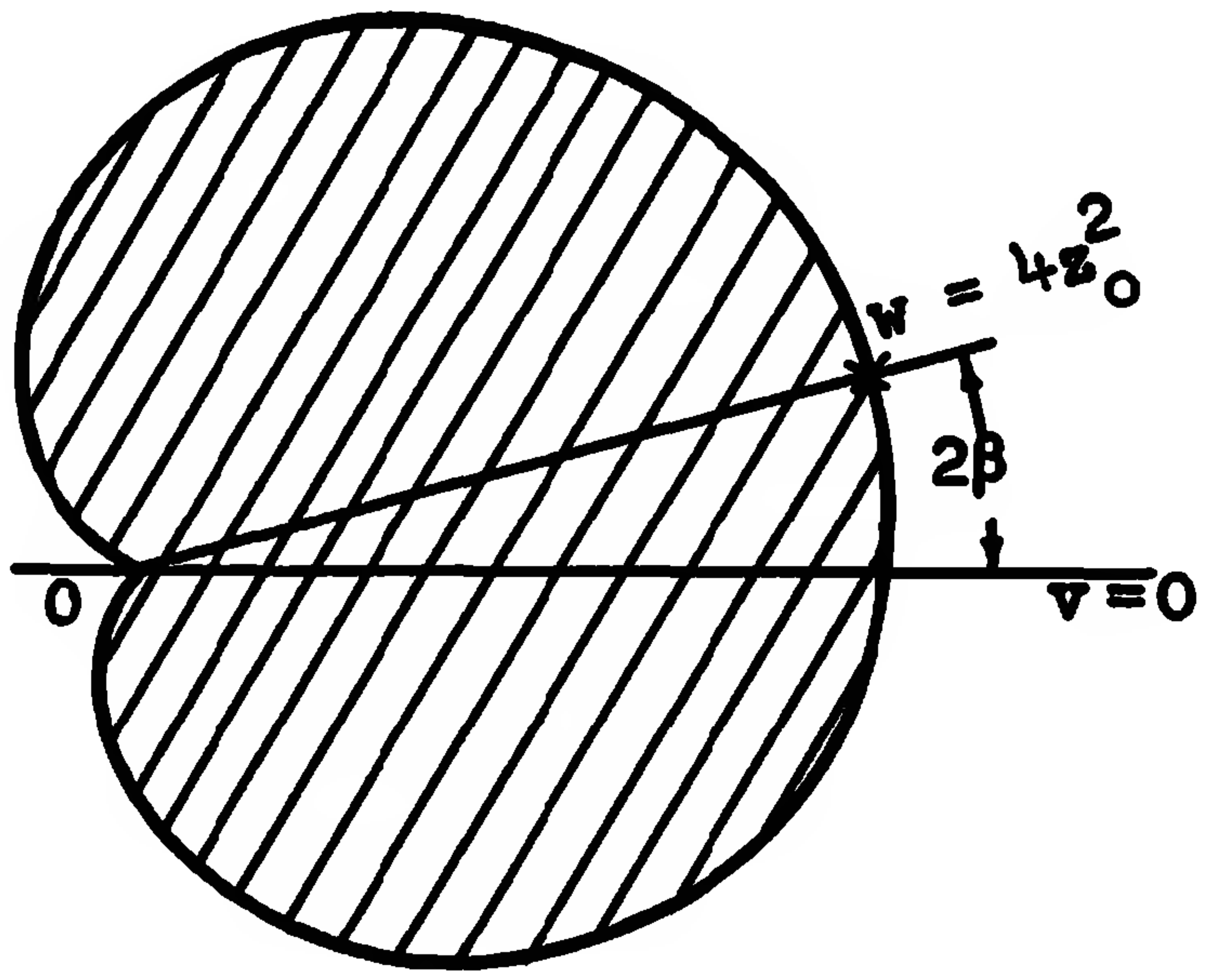
Foci $\pm c$; asymptotes $\Re(c^{-2}z^2) = 0$.

z-plane	w-plane
either branch of hyperbola	line $\Re(c^{-2}w) = \frac{1}{2}$
interior of either branch	half-plane $\Re(c^{-2}w) > \frac{1}{2}$
region bounded by a branch of hyperbola and adjacent parts of the asymptotes	strip $0 < \Re(c^{-2}w) < \frac{1}{2}$
region bounded by adjacent branches of $\Re(c_1^{-2}z^2) = \frac{1}{2}$, $\Re(c_2^{-2}z^2) = \frac{1}{2}$; $c_2/c_1 \neq \pm 1$, but real	strip bounded by $\Re(c_1^{-2}w) = \frac{1}{2}$, $\Re(c_2^{-2}w) = \frac{1}{2}$
either branch of $x^2-y^2 = P$ $(P \gtrless 0)$	line $u = P$ (i.e., $c = \sqrt{2P}$)
either branch of $2xy = Q(Q \gtrless 0)$	line $v = Q$ (i.e., $c = \sqrt{2iQ}$)

Hyperbolae mapped on $u = P$ or $v = Q$:

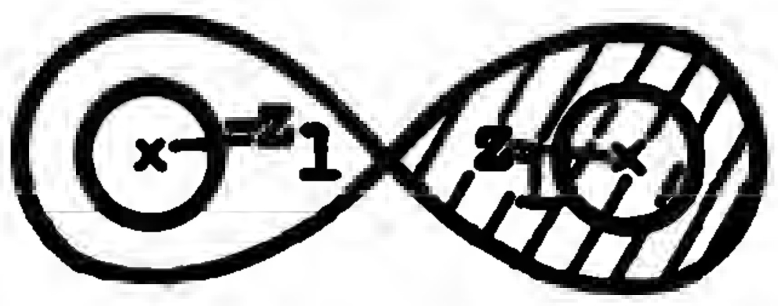
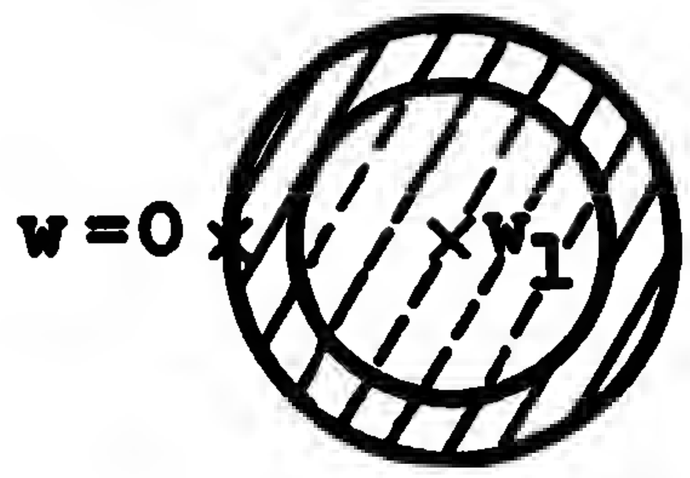


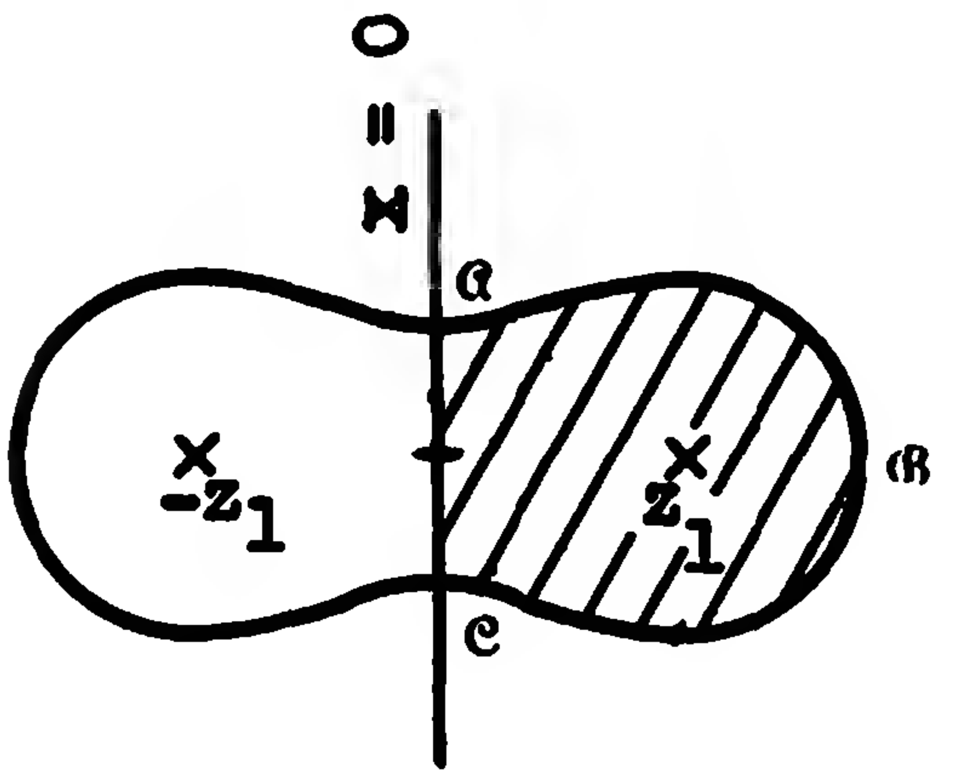
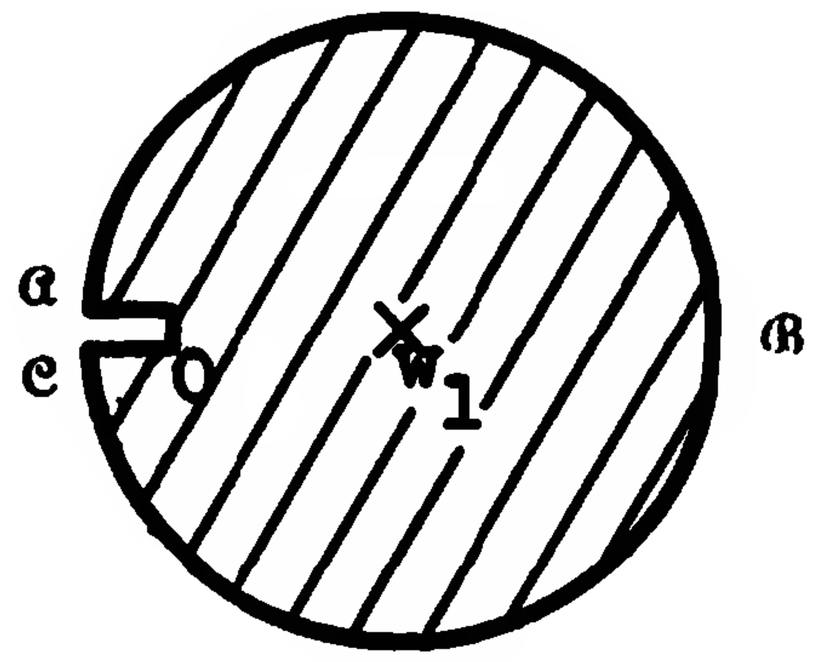
Cardioid; limaçon

z - plane	w - plane
circle $\lvert z-z_0\rvert=\lvert z_0\rvert$ circle $\lvert z+z_0\rvert=\lvert z_0\rvert$ 	cardioid $w = 2z_0^2(1+\cos\theta)e^{i\theta}$ $(-\pi \leq \theta < \pi)$ 
circle $\lvert z-z_0\rvert = c$ circle $\lvert z+z_0\rvert = c$ $(c > 0,\ z_0 \neq 0)$	limaçon $w-w_1 = 2c(z_0 + c\cos\theta)e^{i\theta}$; $w_1 = z_0^2 - c^2$; $-\pi \leq \theta < \pi$.

Cassinians

(in the figures w_1 is taken as a real and positive number)

z - plane	w - plane
Case (i) either part of $\lvert z-z_1\rvert\lvert z+z_1\rvert = C$, foci $z_1 = \pm\sqrt{w_1}$ 	circle $\lvert w-w_1\rvert = C$ $[C \leq \lvert w_1\rvert]$  outer curve: $C = \lvert w_1\rvert$ inner curve: $C < \lvert w_1\rvert$

z - plane	w - plane; $w_1 \neq 0$
Case (ii)	
$\lvert z-z_1 \rvert \lvert z+z_1 \rvert = C$, foci $z_1 = \pm\sqrt{w_1}$	circle $\lvert w-w_1 \rvert = C$ $[C > \lvert w_1 \rvert]$, counted twice
part $x > 0$ of the interior of Cassinian; part $x < 0$ of the interior of Cassinian	interior of circle, cut from $w = 0$ to $w' = \Re(w_1) - \{C^2 - \Im(w_1)^2\}^{1/2}$, i.e. point α
	

<u>Examples</u>: <u>Exterior of parabola</u>

on (i) interior of circle (ii) upper half plane

(i) $w = 2\sqrt{\frac{K}{z}} - 1$, $K > 0$; $z = \dfrac{4K}{(w+1)^2}$

z - plane	w - plane
parabola $y^2 = 4K(K-x)$ [i.e. $r\cos^2\frac{\varphi}{2} = K$]	circle $\lvert w \rvert = 1$
region outside it, not containing focus $z = 0$	region $\lvert w \rvert < 1$
points K; $4K$; $\pm 2iK$; $2K$; ∞	points 1; 0; $\mp i$; $\sqrt{2}-1$; -1

(ii) $w = ai\,\{p^{-1/2}(iz+h+p)^{1/2} - 1\}$; $p > 0$, $a > 0$.

$z = \ell w^2 + mw + n$, where $\ell = ipa^{-2}$, $m = -2pa^{-1}$, $n = ih$

z - plane	w - plane
parabola $x^2 = 4p(y-h)$	line $v = 0$
region outside it, not containing focus $z = i(h+p)$	half-plane $v > 0$
points $i(h+p)$; ih; $i(h-p)$; ∞; $\pm 2p+i(p+h)$	points $-ai$; 0; $ai(\sqrt{2}-1)$; ∞; $\mp a$
half-line $x = 0$, $-\infty < y < h$	half-line $u = 0$, $\infty > v > 0$.

$\boxed{w = az^2 + bz + c)}$ $a \neq 0$; $z = \dfrac{-b \pm (4aw + b^2 - 4ac)^{1/2}}{2a}$

This is a combination of $\xi = \zeta^2$, and of the two linear transformations

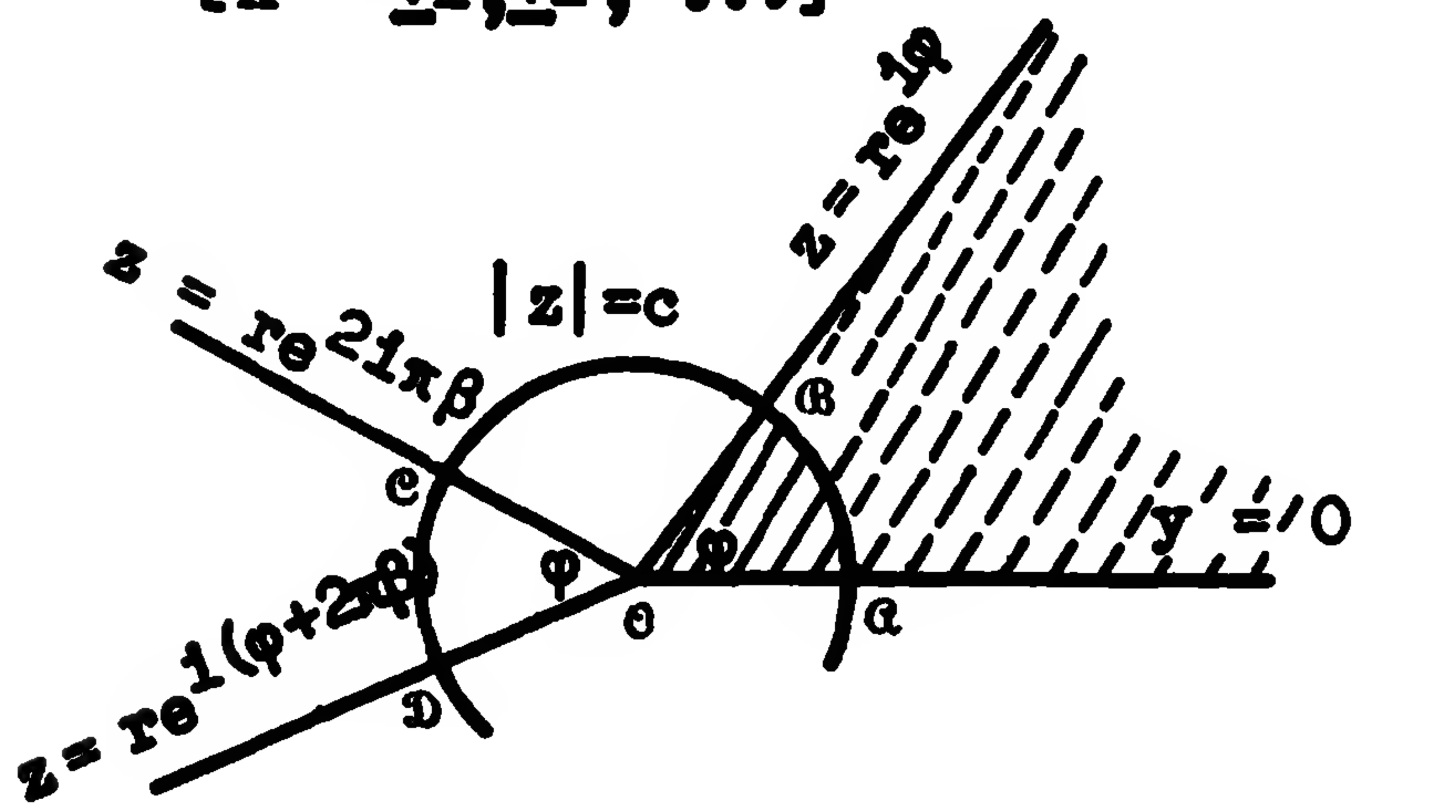
$\zeta = z + \dfrac{b}{2a}$, $w = a\xi + \dfrac{4ac-b^2}{4a}$ [see §2.2].

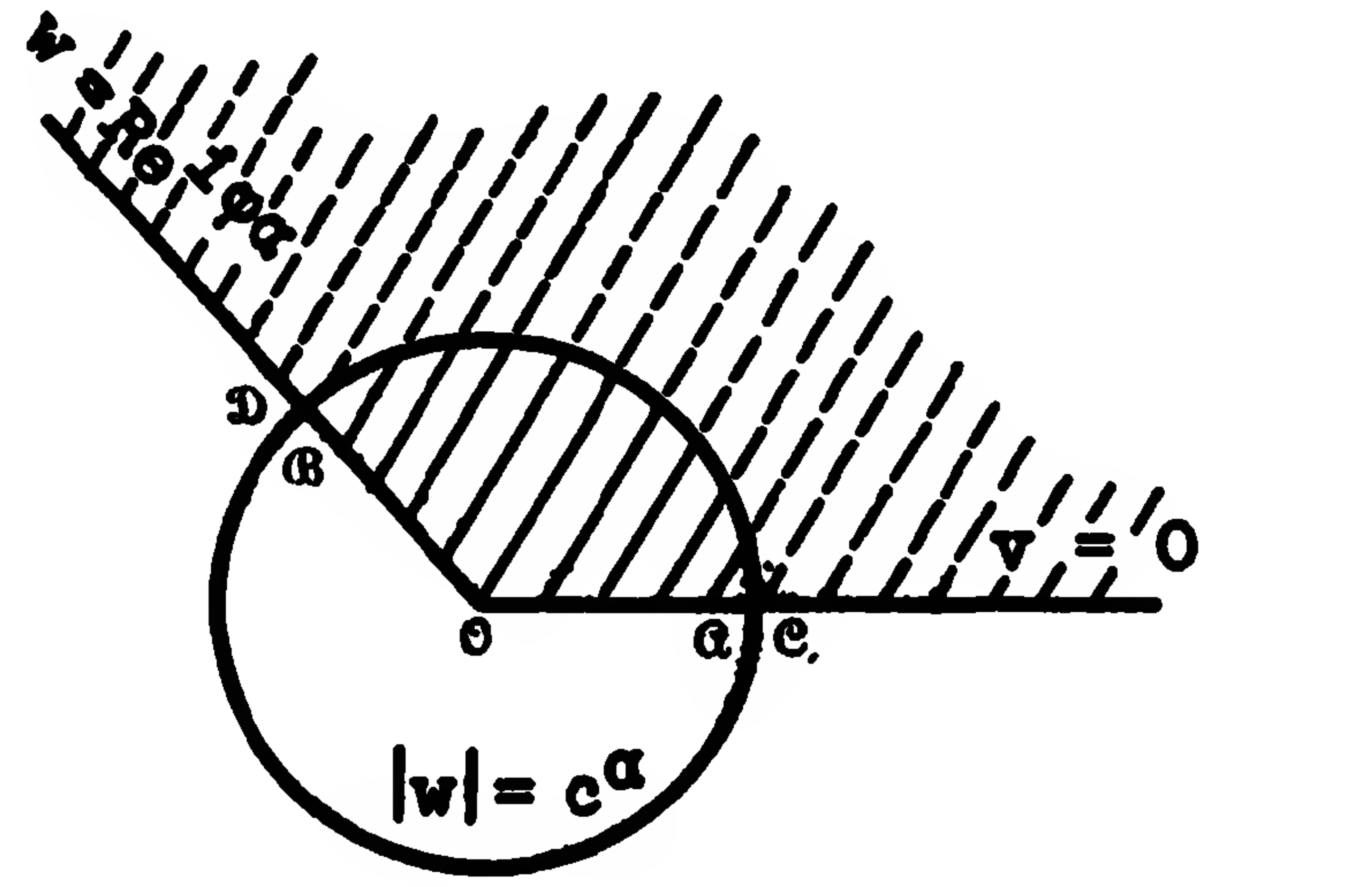
Fixed points: $*F_{\substack{1\\2}} = \dfrac{1 - b \pm \sqrt{(1-b)^2 - 4ac}}{2a}$

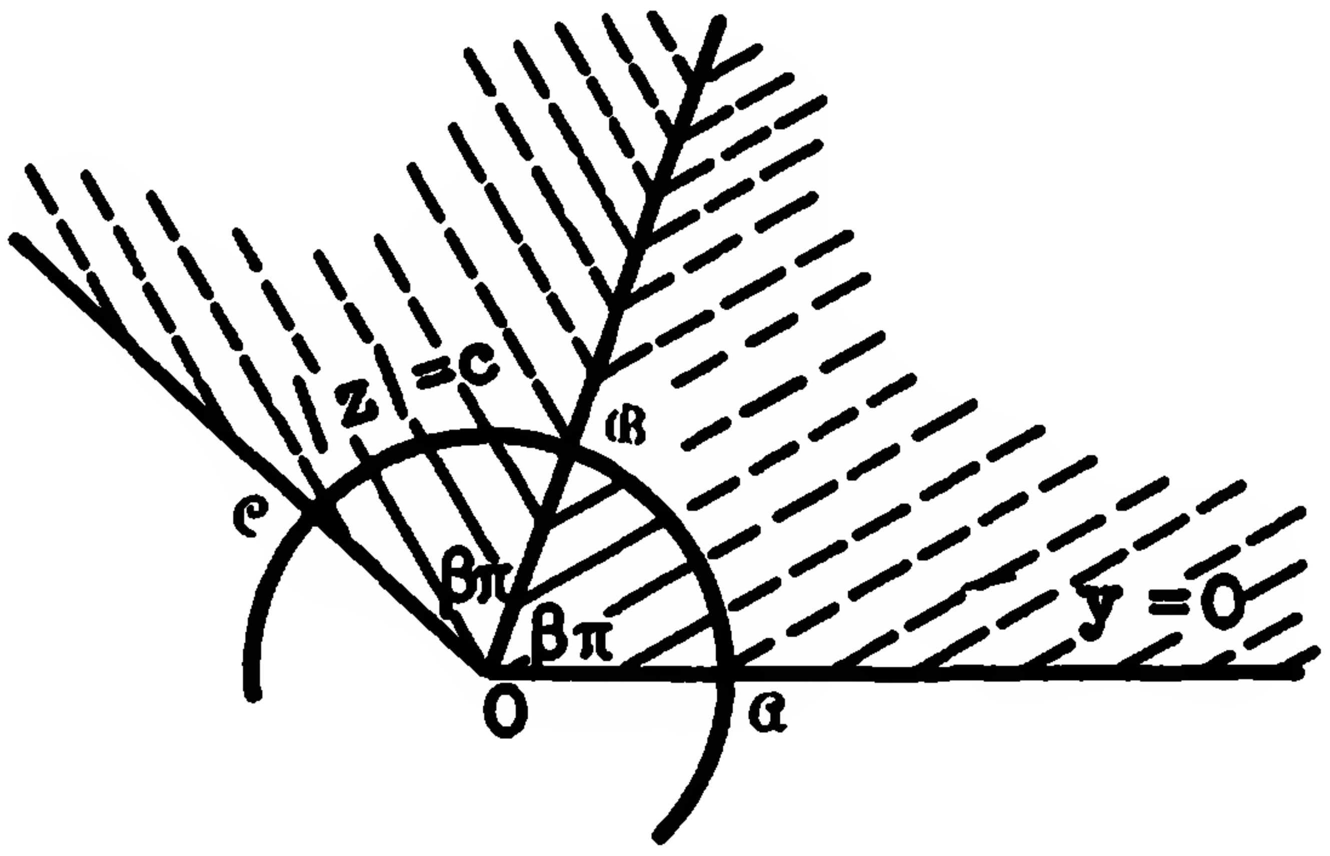
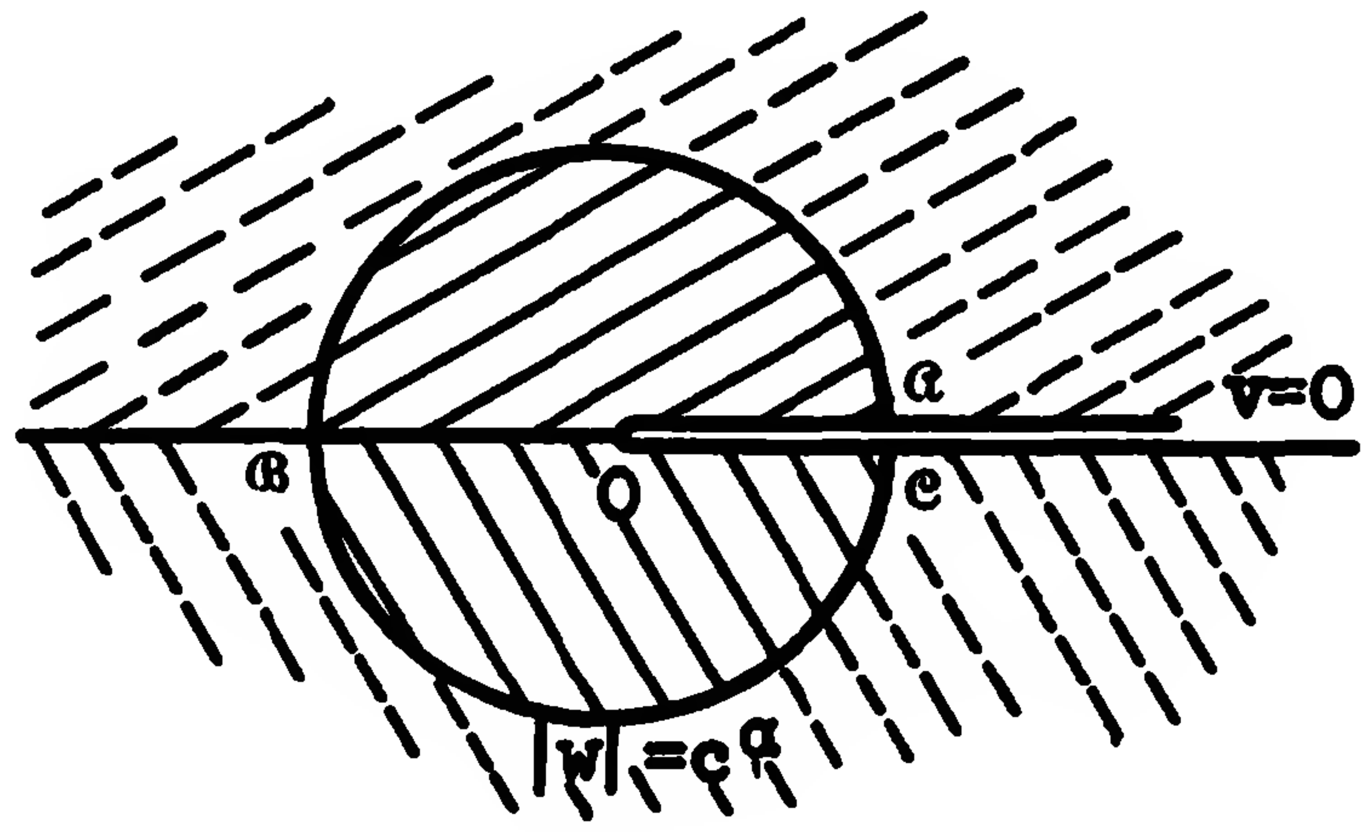
6.2 $\boxed{w = z^\alpha,\ \alpha > 0}$ $z = w^\beta$, $\beta = \dfrac{1}{\alpha}$. Angle on half-plane

Critical points: $z = 0$ and $z = \infty$.

z - plane; $z = re^{i\varphi}$ $(r > 0)$	w - plane; $w = Re^{i\theta}$ $(R > 0)$
half-line $y = 0$; $0 \leq x < \infty$	half-line $v = 0$, $0 \leq u < \infty$
half-line $z = re^{i\varphi}$ } φ fixed; $0 < r < \infty$	half-line $w = Re^{i\varphi\alpha}$, $0 < R < \infty$; $R = r^\alpha$
half-line $z = re^{i(\varphi+2k\pi\beta)}$ [$k = \pm 1, \pm 2, \ldots$]	



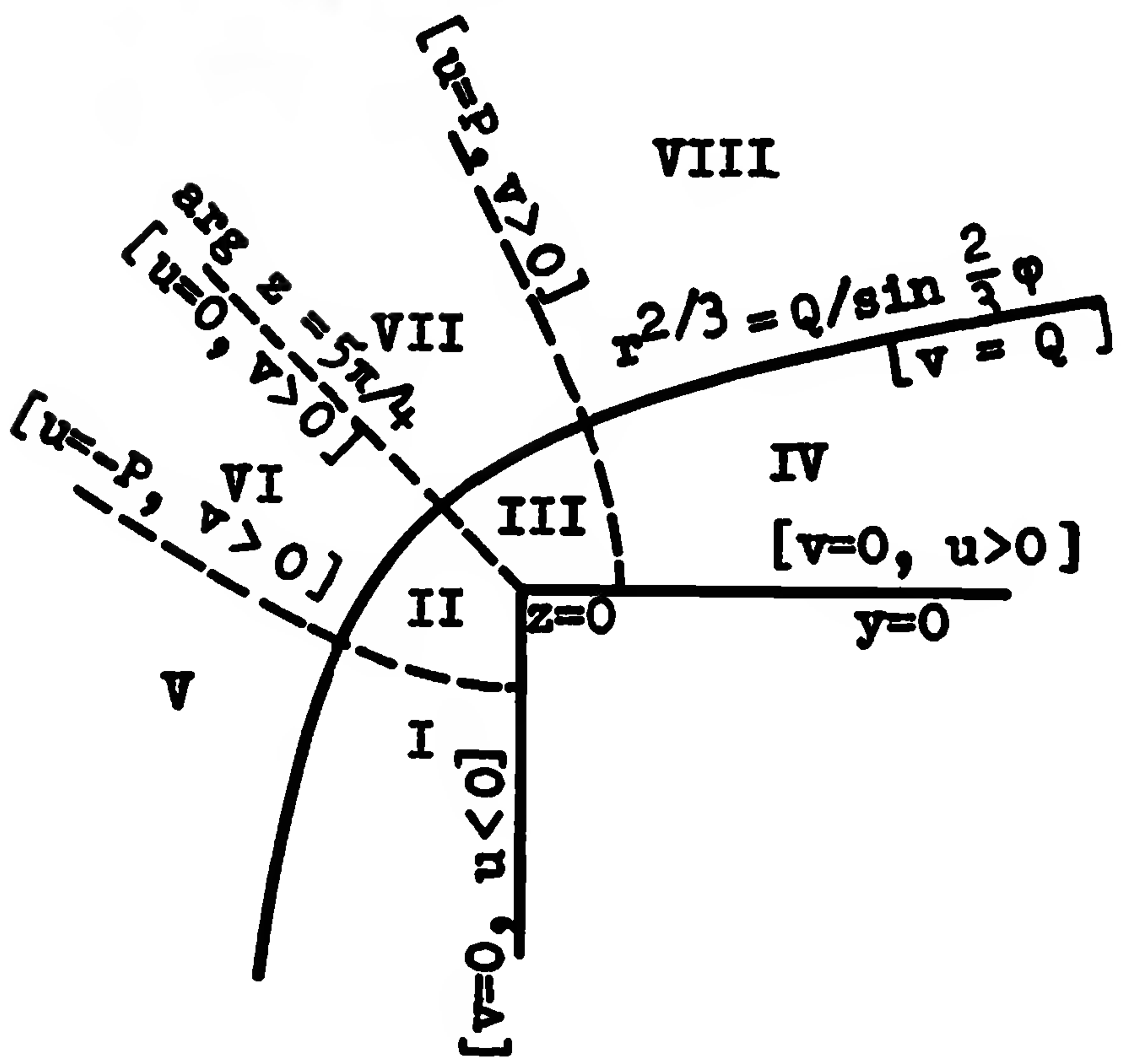
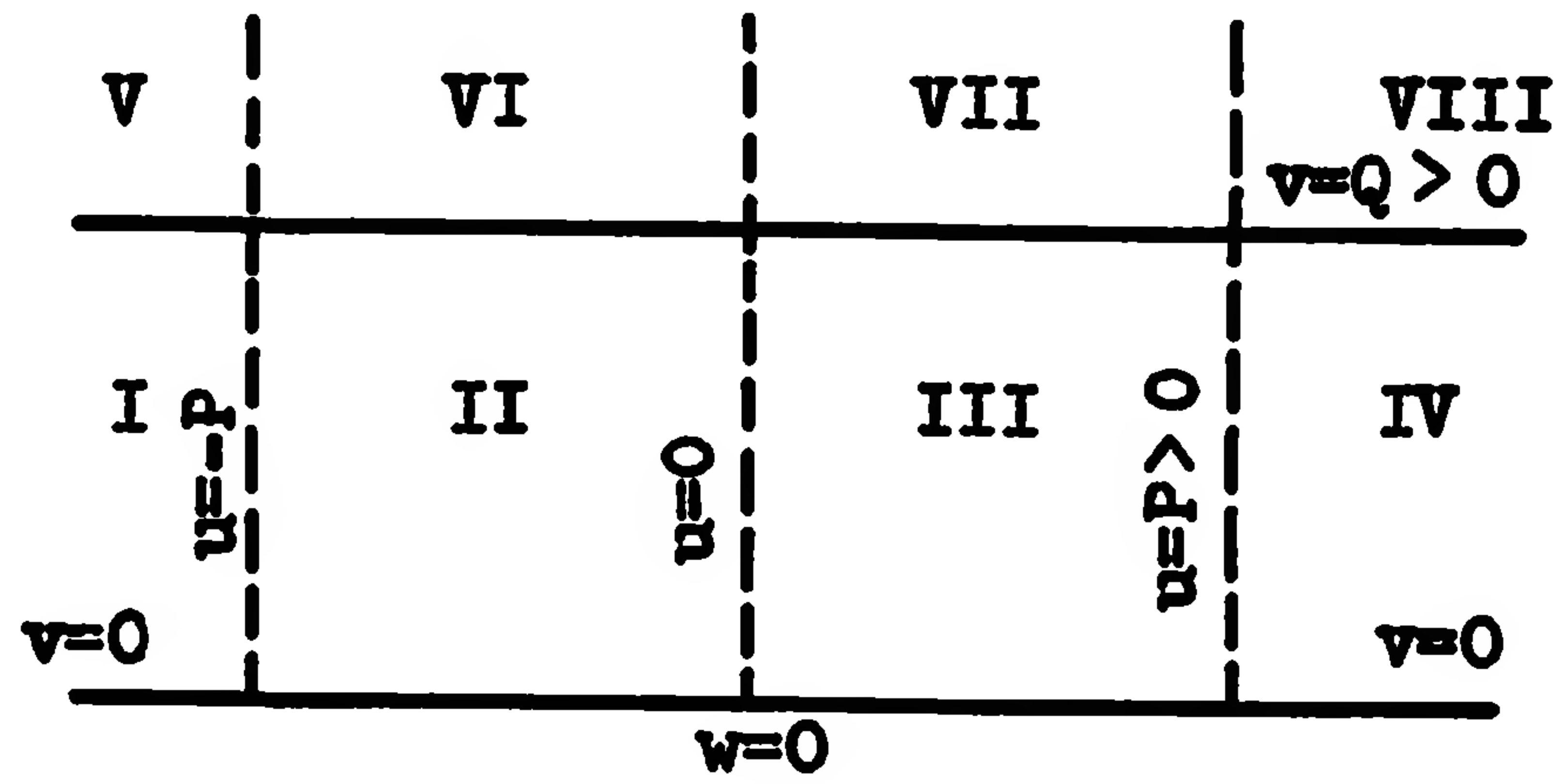


z - plane; $z = re^{i\varphi}$ $(r > 0)$	w - plane; $w = Re^{i\theta}$ $(R > 0)$
	
angle $0 < \varphi < \beta\pi$	half-plane $y > 0$
angle $\beta\pi < \varphi < 2\beta\pi$	half-plane $y < 0$

<u>Generalised parabolae and hyperbolae</u>

z - plane; $z = re^{i\varphi}$	w - plane; $w = Re^{i\theta}$
line $\Re(ze^{-i\psi}) = s$; $s \gtrless 0$.	curve $R^{\beta} = s/\cos(\beta\theta - \psi)$, where $\alpha(\pi k - \frac{\pi}{2} + \psi) < \theta < \alpha(\pi k + \frac{\pi}{2} + \psi)$.
line $x = p$ $(p \gtrless 0)$	curve $R^{\beta} = p/\cos\beta\theta$
line $y = q$ $(q \gtrless 0)$	curve $R^{\beta} = q/\sin\beta\theta$
curve $r^{\alpha} = S/\cos(\alpha\varphi - \chi)$, where $\beta(\pi k - \frac{\pi}{2} + \chi) < \varphi < \beta(\pi k + \frac{\pi}{2} + \chi)$.	line $\Re(we^{-i\chi}) = S$; $S \gtrless 0$.
curve $r^{\alpha} = P/\cos\alpha\varphi$	line $u = P$; $P \gtrless 0$.
curve $r^{\alpha} = Q/\sin\alpha\varphi$	line $v = Q$; $Q \gtrless 0$.

k any even number for $s > 0$ or $S > 0$; k odd for $s < 0$ or $S < 0$. There is a finite number of curves, corresponding to a given straight line of the other plane, if and only if α is a rational number. All the curves pass through the point at infinity. For $s = 0$, or $S = 0$, see the lines $z = re^{i\varphi}$, φ fixed.

z - plane; $z = re^{i\varphi}$	w - plane
Example: $\alpha = 2/3$; $w = z^{2/3}$. 	

Asymptotes

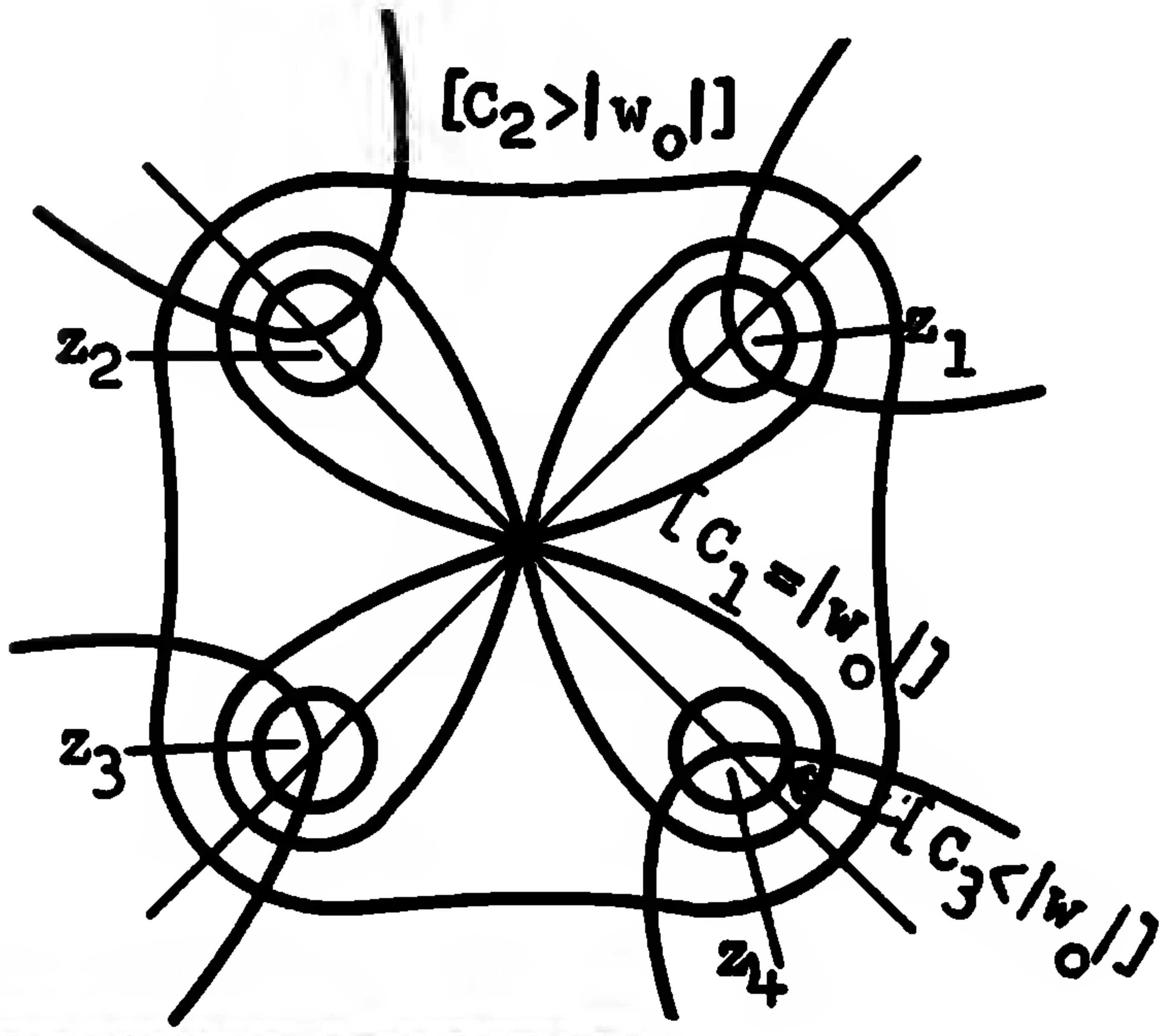
curve	case $0 < \alpha < 1$	case $\alpha > 1$
$r^{\alpha} = S/\cos(\alpha\varphi - \chi)$	no asymptote	line $z = re^{i\beta(\pi k + \frac{\pi}{2} + \chi)}$; $k = 0, \pm 1, \ldots$
$R^{\beta} = s/\cos(\beta\theta - \psi)$	line $w = Re^{i\alpha(k\pi + \frac{\pi}{2} + \psi)}$; $k = 0, \pm 1, \ldots$	no asymptote

Additional properties for the case when α is an integer n.

The generalised hyperbola $r^{\alpha} = S/\cos(\alpha\varphi - \chi)$ consists of exactly n branches, each of which is mapped on $\Re(\bar{\Lambda} w) = S$, where $\Lambda = e^{i\chi}$.

Generalised Cassinians

z - plane	w - plane
each part of $\lvert z-z_1\rvert \lvert z-z_2\rvert \ldots \lvert z-z_n\rvert = C$; foci $z_j = e^{2i\pi j/n} \sqrt[n]{w_0}$ $(j=1,2,\ldots,n)$	circle $\lvert w-w_0\rvert = C$; $0 < C \leq \lvert w_0\rvert$.

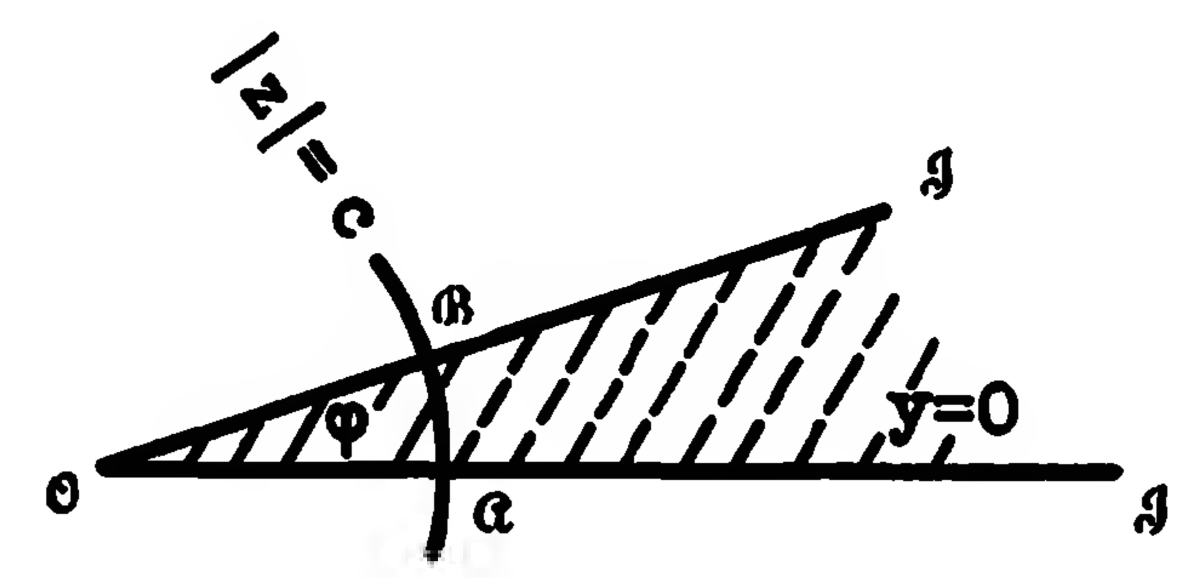
z - plane	w - plane
closed curve $\|z-z_1\|\|z-z_2\|\ldots\|z-z_n\| = C$ (approximately circle for large C)	circle $\|w-w_o\| = C$, counted n times; $C > \|w_o\| > 0$
region bounded by this curve and lines $z = re^{i\varphi}$, $z = re^{i(\varphi+2\pi/n)}$ (φ fixed)	interior of $\|w-w_o\| = C$ $(C > \|w_o\|)$, cut from 0 to the point at which the line $w = r^n e^{in\varphi}$ meets the circle
(case $n = 4$; w_o is taken as a negative number; $\Lambda = e^{i\chi}$) $[\Re(\bar{\Lambda}w) = S = \Re(\bar{\Lambda}w_o)]$ 	

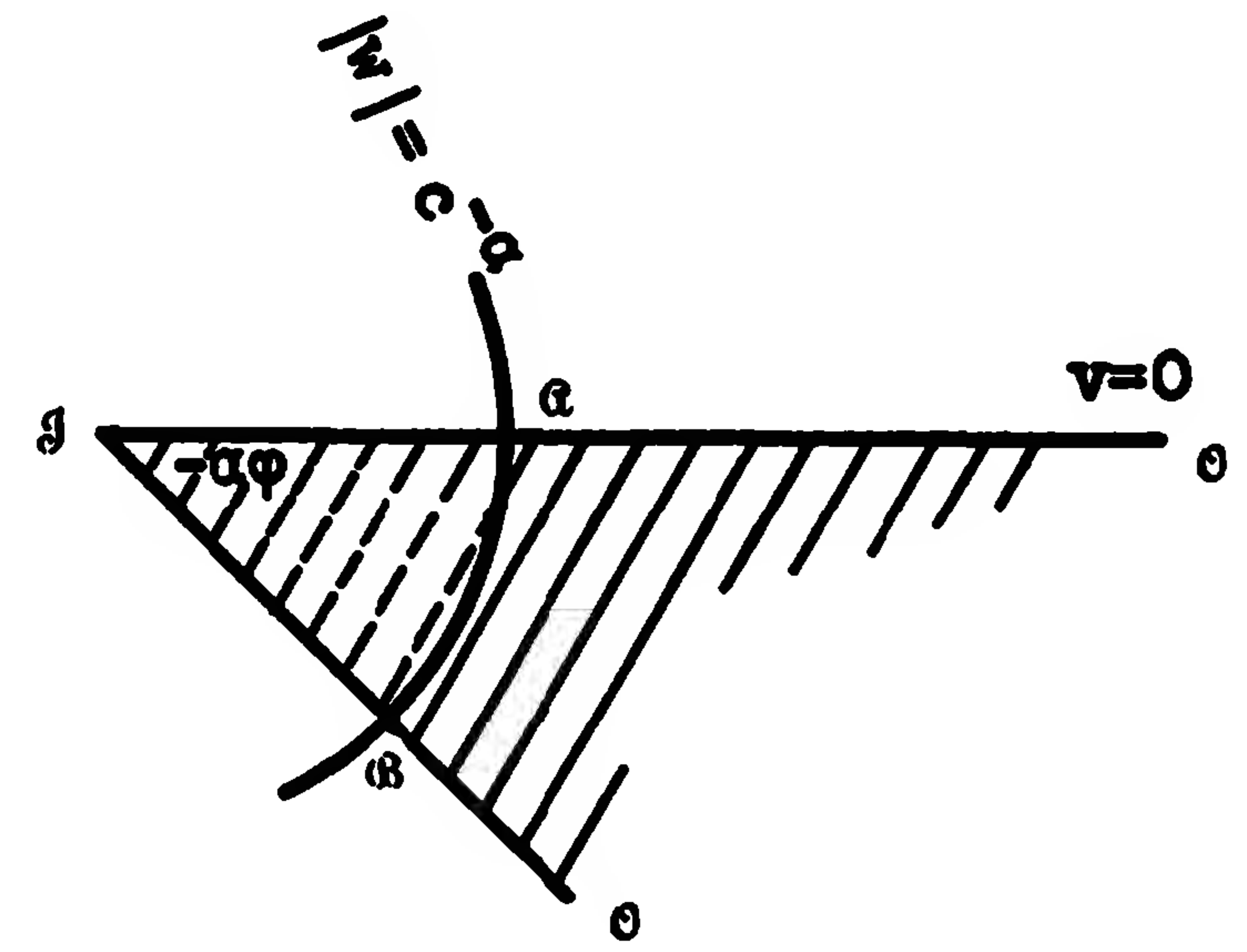
$\boxed{w = z^{-\alpha},\ \alpha > 0}$ $z = w^{-\beta}$, $\beta = 1/\alpha$. <u>Angle on half-plane.</u>

Critical points: $z = 0$; $z = \infty$

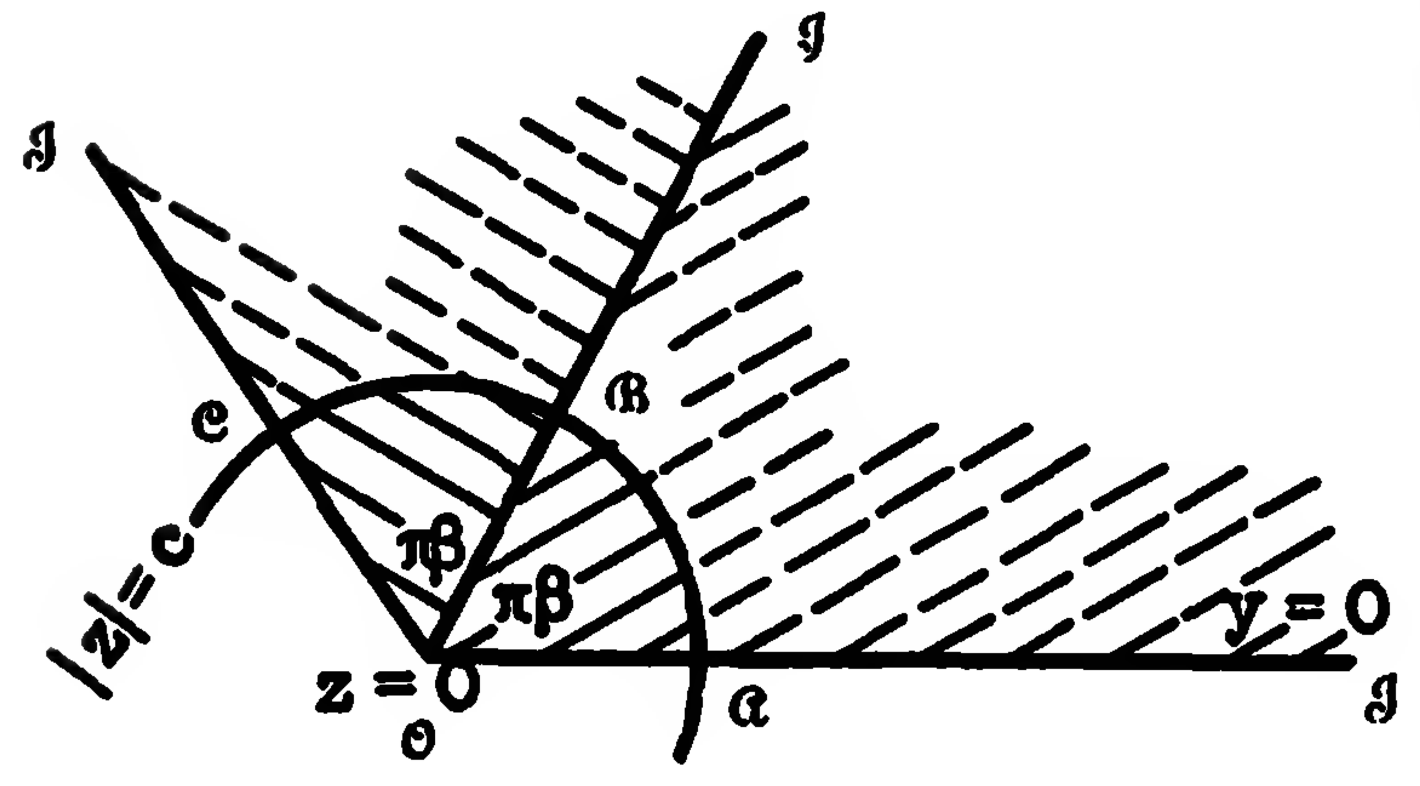
z - plane		w - plane
point $z = 0$		point $w = \infty$
point $z = \infty$		point $w = 0$
half-line $y = 0$, $0 < x < \infty$		half-line $v = 0$, $\infty > u > 0$
half-line $z = re^{i\varphi}$; $0 < r < \infty$	φ fixed	
half-line $z = re^{i(\varphi+2k\pi\beta)}$; $k = \pm 1, \pm 2, \ldots$		half-line $w = Re^{-i\varphi\alpha}$; $\infty > R = r^{-\alpha} > 0$.

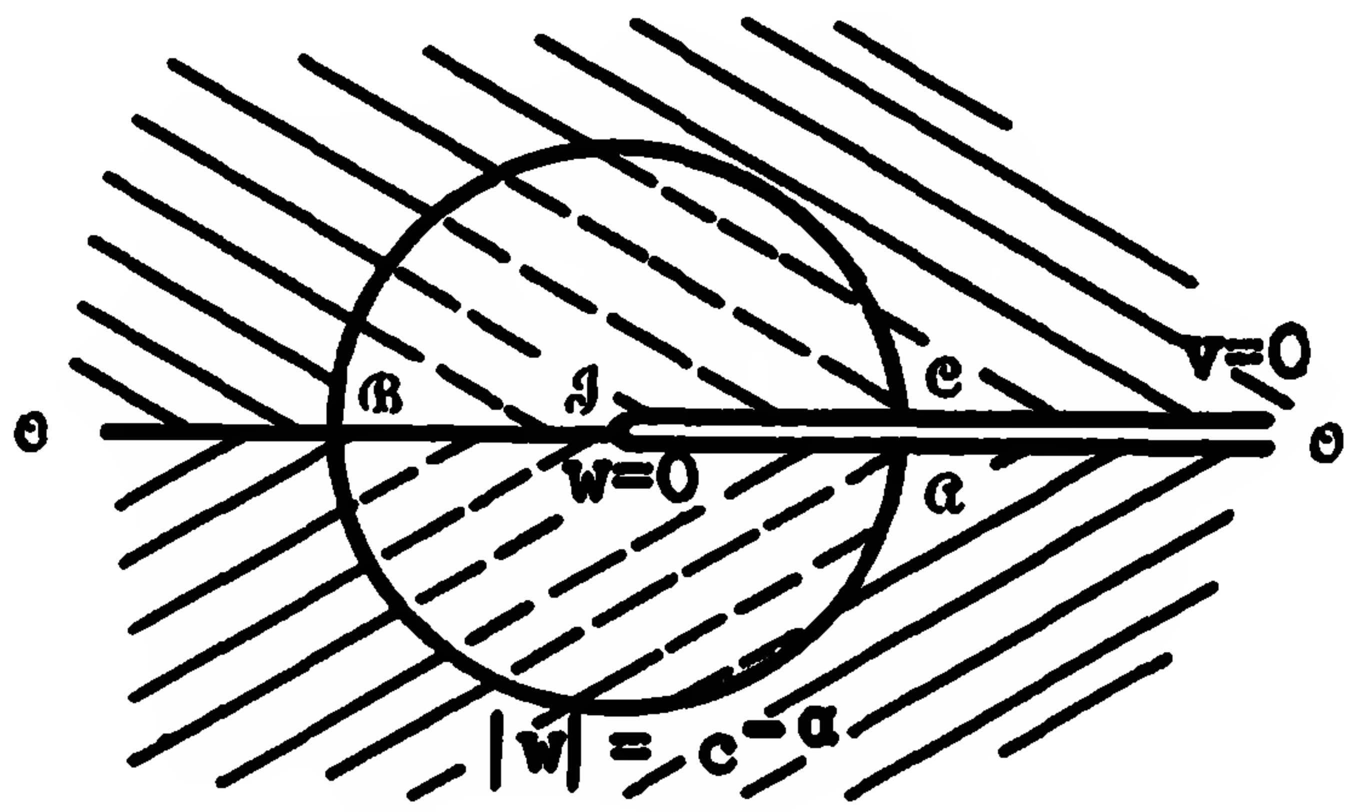
z - plane	w - plane





Case $\alpha > 1$



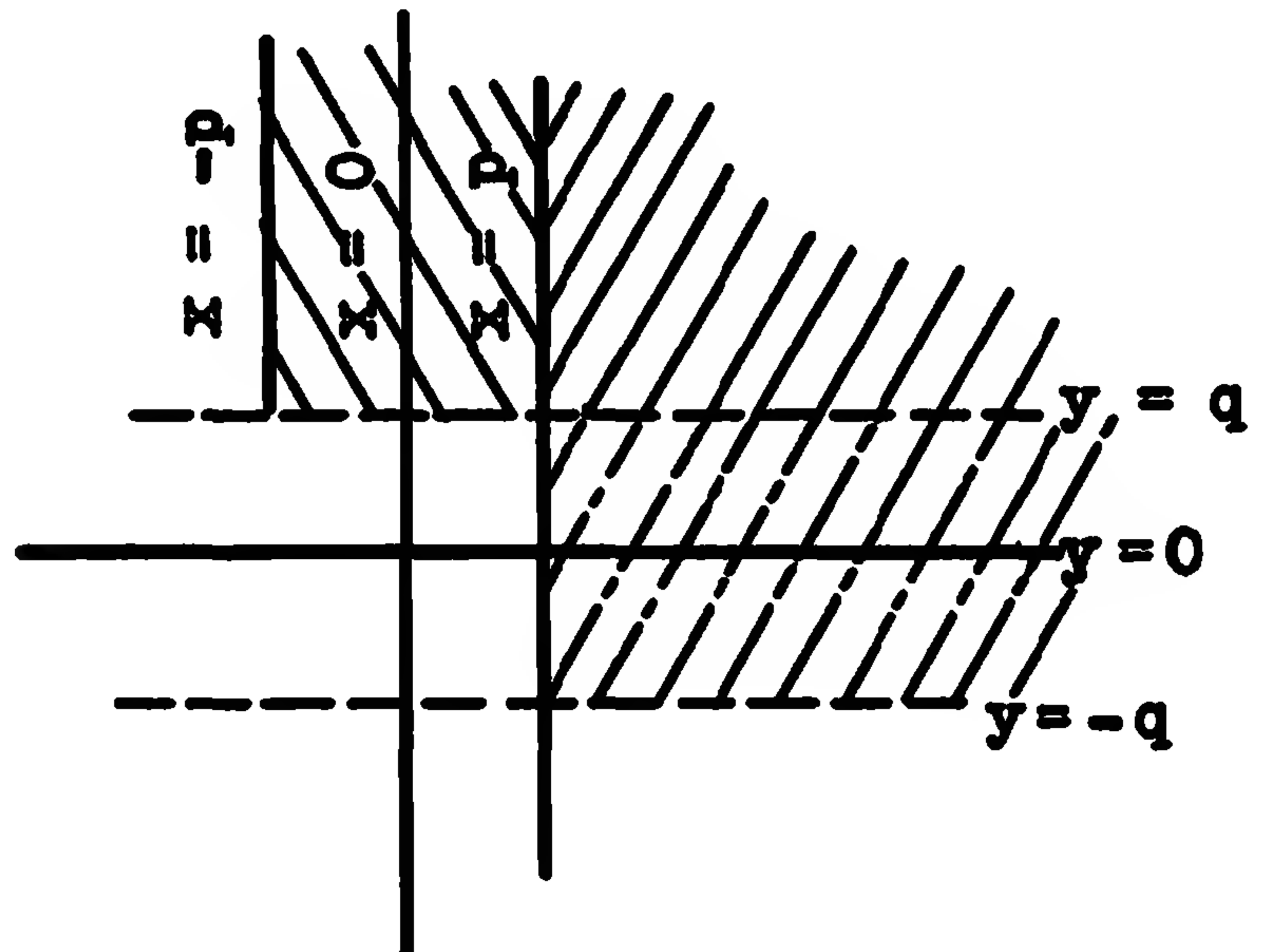


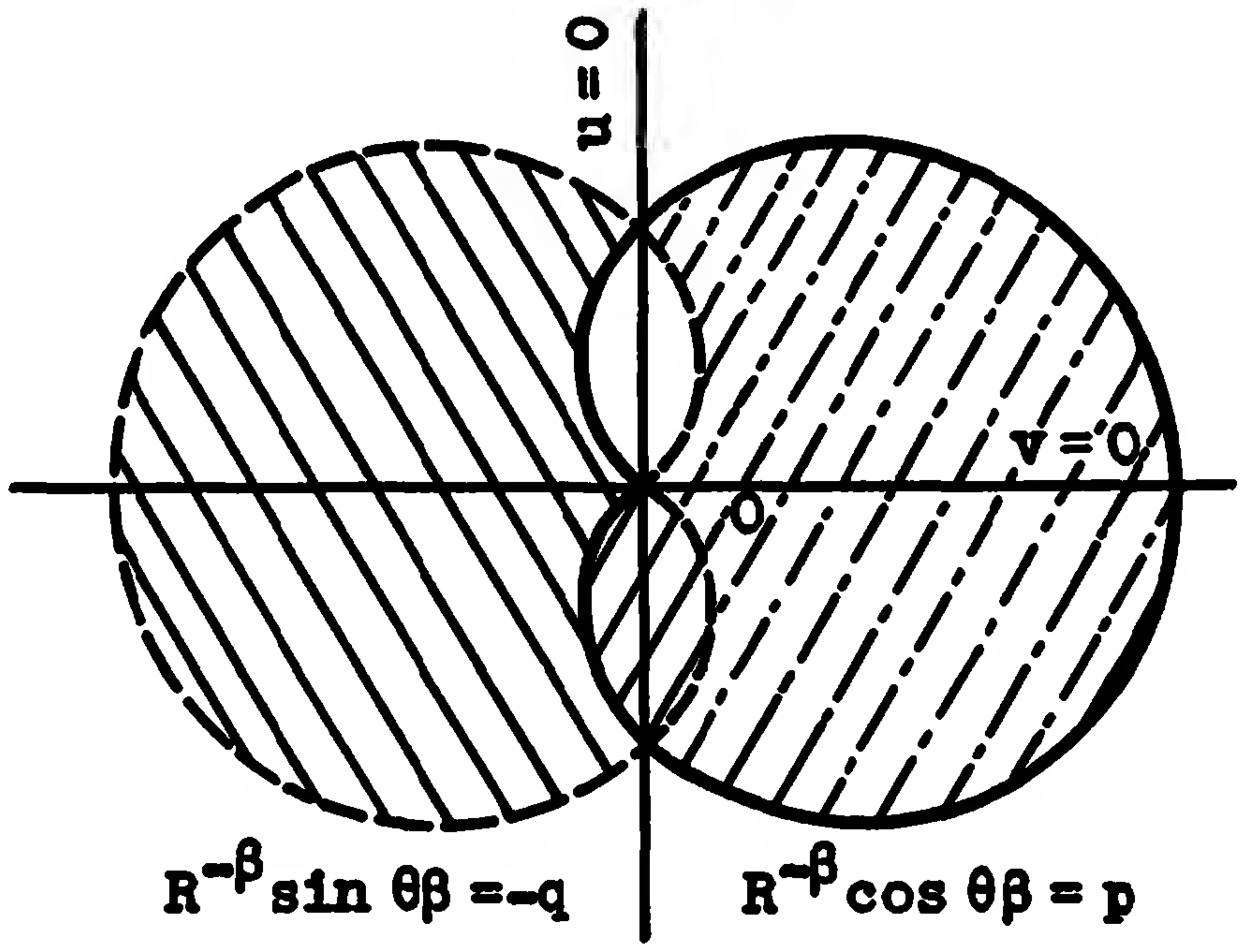
angle $0 < \varphi < \pi\beta$	half-plane $y < 0$
angle $\pi\beta < \varphi < 2\pi\beta$	half-plane $y > 0$

Cardioid and generalised cardioids ($\alpha > 1$), and lemniscates ($0 < \alpha < 1$).

In the figure, $\alpha = 2$ (cardioid), $p = q > 0$

z - plane	w - plane





z - plane	w - plane
line $\Re(e^{-i\psi}z) = s$; $s > 0$	each part of $R^{\beta} = s^{-1}\cos(\theta\beta + \psi)$; $\lvert \theta - \alpha(2k\pi - \psi) \rvert \leq \alpha\pi/2$; $k = 0, \pm 1, \pm 2, \ldots$ curves are closed, with a cusp at 0; with tangents $w = Re^{\alpha i(k\pi + \frac{\pi}{2} - \psi)}$; $k = 0, \pm 1, \pm 2, \ldots$ there are n parts if $\alpha^{-1} = n = \text{integer}$

$\boxed{w = az^{\alpha} + bz^{-\alpha}}$ $\alpha > 0$, $ab \neq 0$. For $\alpha = 1$ see also §8. This is a combination of

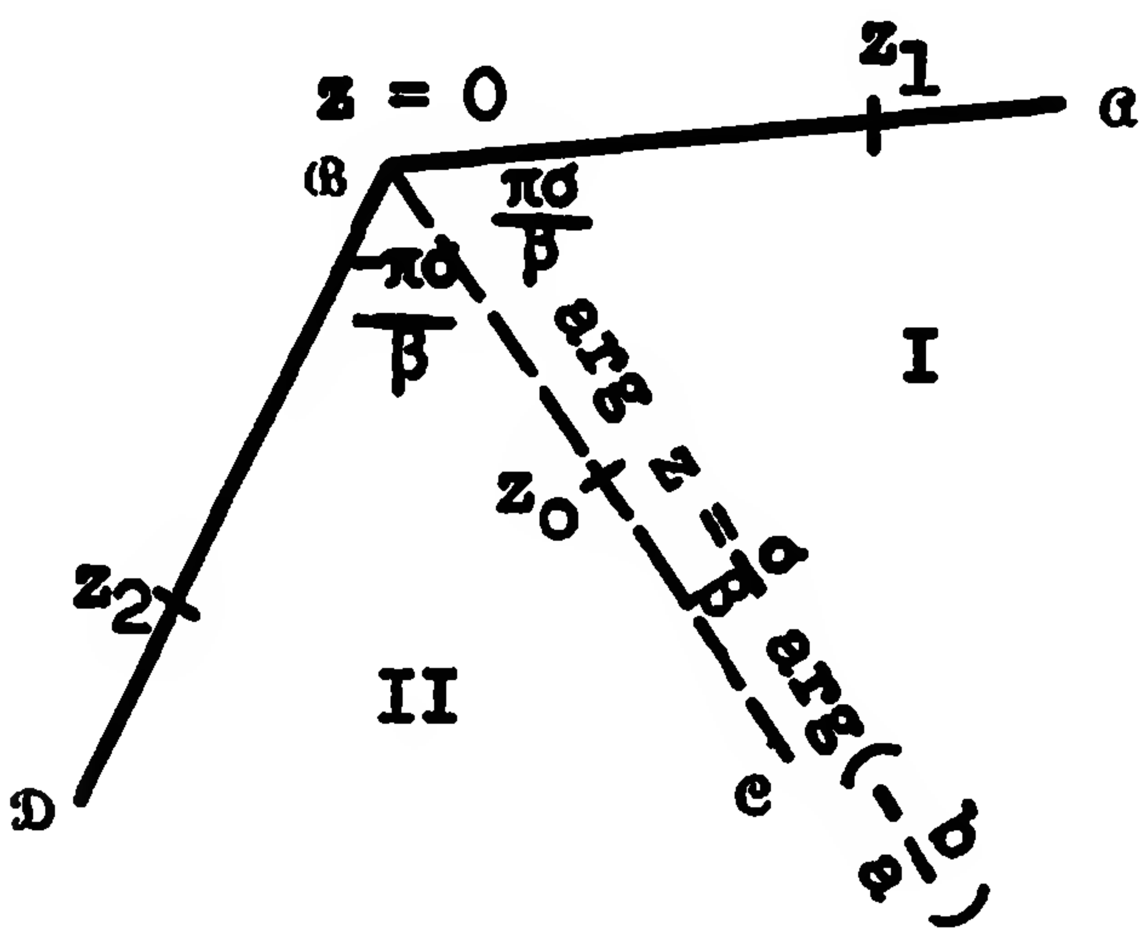
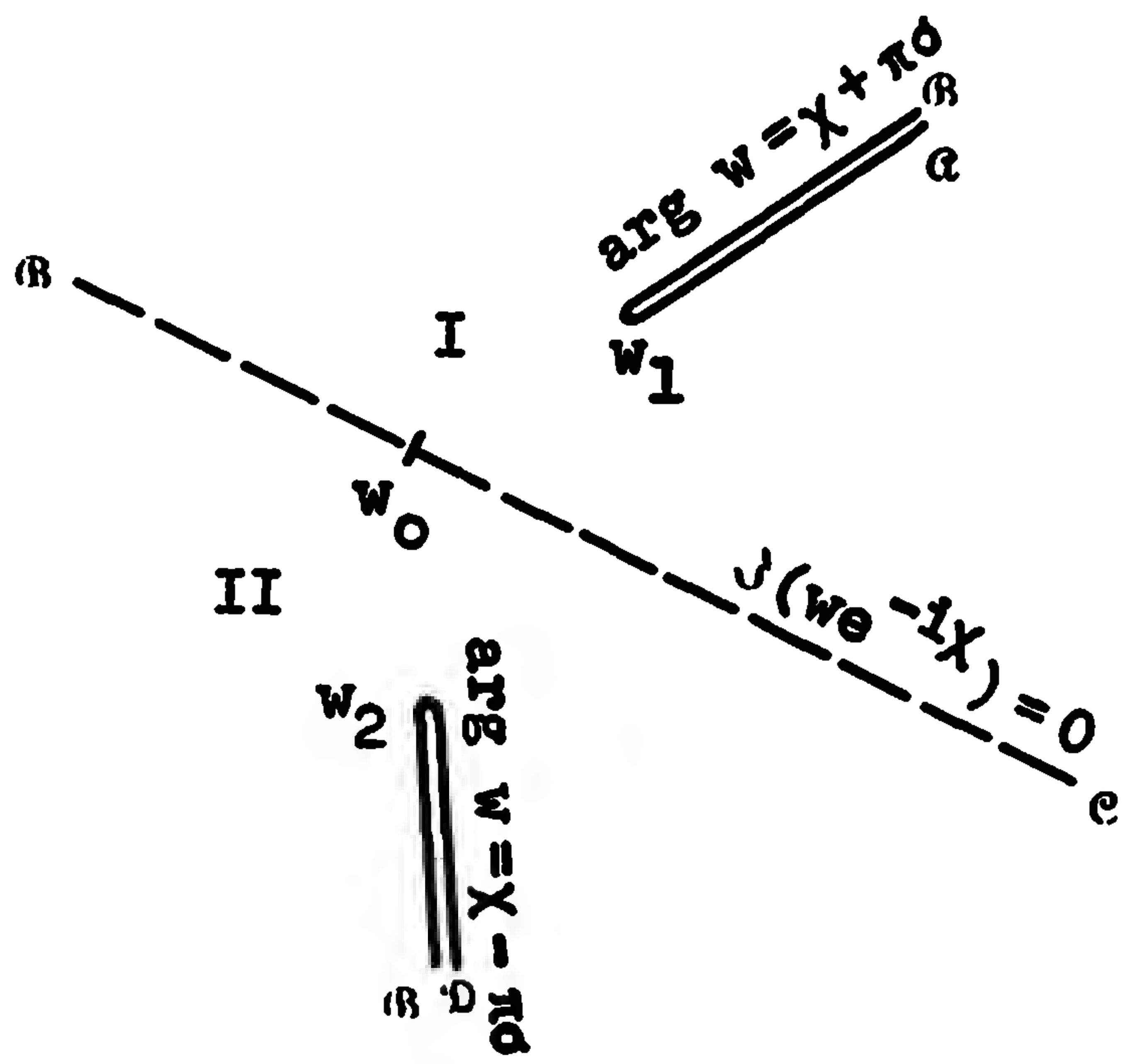
$$w = 2\sqrt{ab}\,\frac{\xi+1}{\xi-1}, \quad z = \zeta\left(\frac{b}{a}\right)^{1/2\alpha} \quad \text{and} \quad \xi = \left(\frac{\zeta^{\alpha}+1}{\zeta^{\alpha}-1}\right)^{2}, \quad \text{see §7.2.}$$

z - plane; $k = 0, \pm 1, \pm 2, \pm 3, \ldots$	w - plane
sector area $\lvert z \rvert < \lvert \frac{b}{a} \rvert^{1/2\alpha}$, $\frac{2k\pi}{\alpha} + \frac{\arg(b/a)}{2\alpha} < \arg z < \frac{\arg(b/a)}{2\alpha} + \frac{\pi(2k+1)}{\alpha}$	one of the two half-planes $\Im\left(\frac{w}{\sqrt{ab}}\right) \gtrless 0$
area $\lvert z \rvert > \lvert \frac{b}{a} \rvert^{1/2\alpha}$, $\frac{2k\pi}{\alpha} + \frac{\arg(b/a)}{2\alpha} < \arg z < \frac{\arg(b/a)}{2\alpha} + \frac{\pi(2k+1)}{\alpha}$	the other of these half-planes.
arc, lying between these two regions, of circle $\lvert z \rvert = \lvert \frac{b}{a} \rvert^{1/2\alpha}$	line segment joining $-2\sqrt{ab}$ to $\sqrt{2ab}$

$\boxed{w = az^{\beta} + bz^{\gamma}}$; $\beta > 0 > \gamma$, $ab \neq 0$. For $\gamma = -\beta$, see also previous case.

$$\sigma = \frac{\beta}{\beta-\gamma}\,; \quad \chi = (1-\sigma)\arg a + \sigma \arg(-b); \quad \psi = \frac{\sigma}{\beta}\arg\left(-\frac{b}{a}\right) \quad [\chi = \psi = 0 \text{ if } a > 0 > b]$$

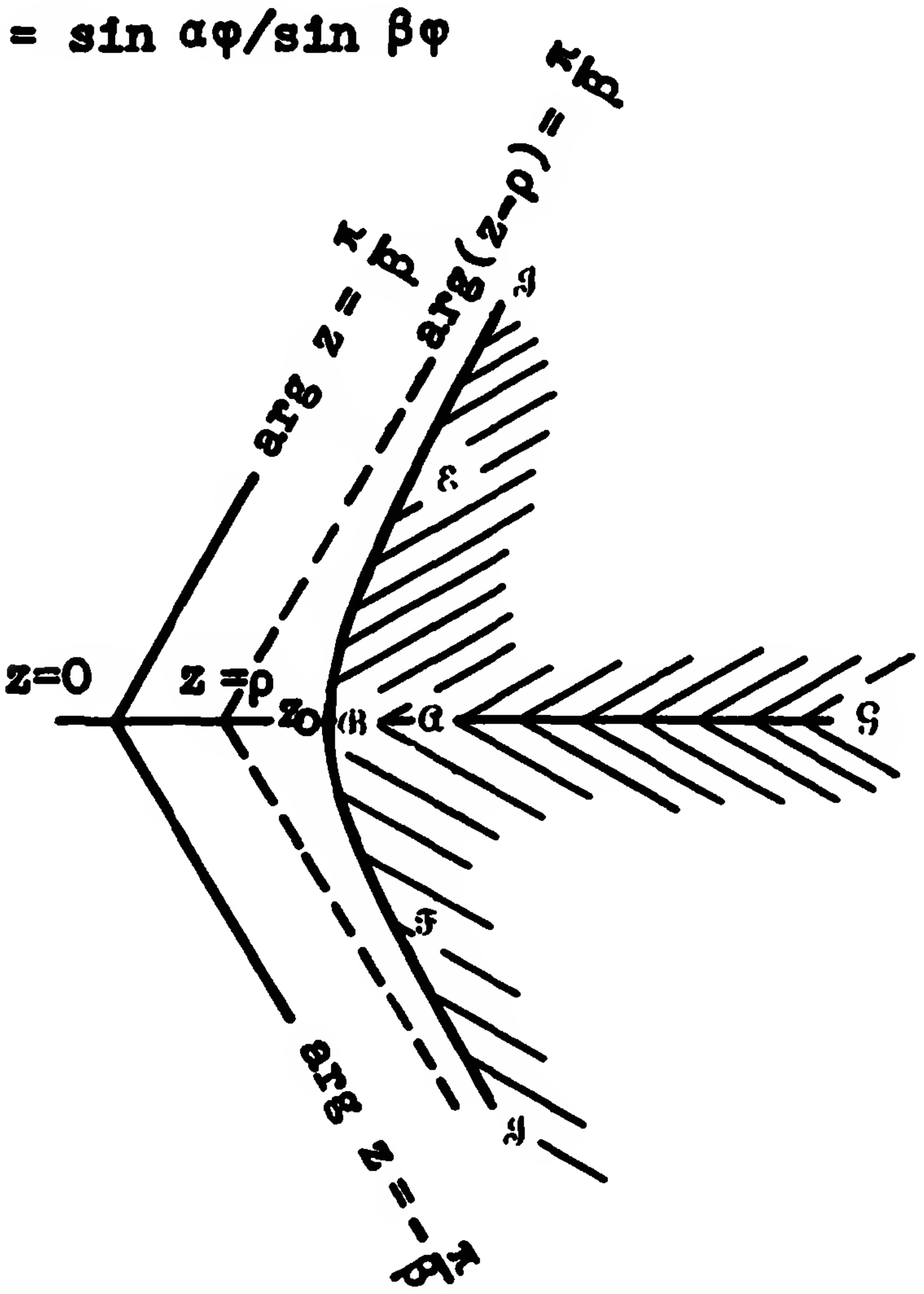
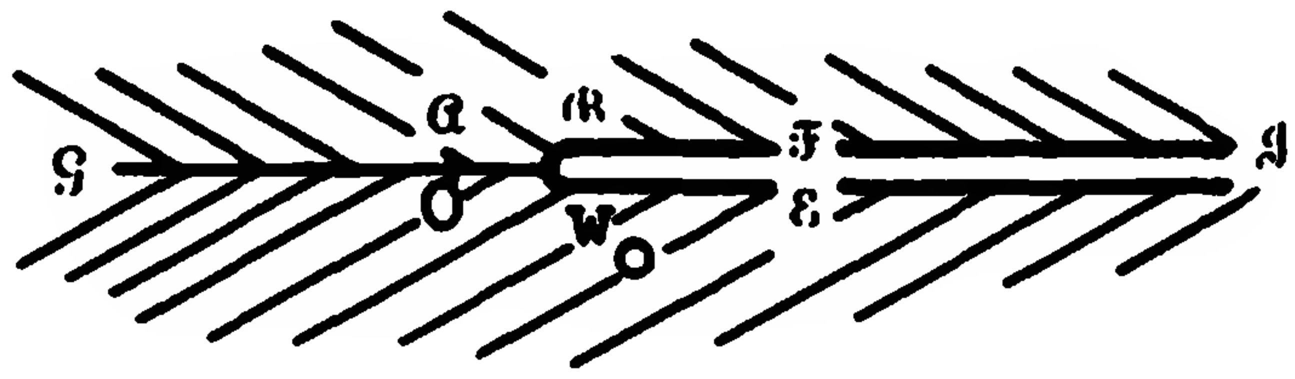
$$c = \left|\frac{b}{a}\,\frac{1-\sigma}{\sigma}\right|^{\sigma/\beta}; \quad d = \left|\left(\frac{a}{1-\sigma}\right)^{1-\sigma}\left(\frac{b}{\sigma}\right)^{\sigma}\right|$$

z - plane	w - plane
	
points $z_o = \left(-\frac{b}{a}\right)^{\sigma/\beta}$; $z_1 = ce^{i\left(\psi + \frac{\pi\sigma}{\beta}\right)}$; $z_2 = ce^{i(\psi - \pi\sigma/\beta)}$	points $w_o = 0$; $w_1 = de^{i(\chi+\pi\sigma)}$; $w_2 = de^{i(\chi-\pi\sigma)}$

6.3 $\boxed{w = z^{\alpha} - z^{\beta}}$ $\quad \beta > \alpha > 0$.

$$\gamma = \beta - \alpha\,; \quad z = re^{i\varphi} \quad (r > 0,\ \varphi \text{ real})$$

z - plane	w - plane
points 0; 1; $z_o = (\alpha/\beta)^{1/\gamma}$	points 0; 0; $w_o = \frac{\gamma}{\beta}\left(\frac{\alpha}{\beta}\right)^{\alpha/\gamma}$
half-line $z_o \leq x < \infty$, $y = 0$	half-line $w_o \geq u > -\infty$, $v = 0$
either of the two halves of $\mathcal{A}\mathcal{E}\mathcal{B}\mathcal{F}\mathcal{A}$	half-line $w_o < u < \infty$, $v = 0$

z - plane	w - plane
$r^{\gamma} = \sin\alpha\varphi/\sin\beta\varphi$ 	
region $r^{\gamma} > \dfrac{\sin\alpha\varphi}{\sin\beta\varphi}$, $0 < \varphi < \dfrac{\pi}{\beta}$	half-plane $v < 0$
region $r^{\gamma} > \dfrac{\sin\alpha\varphi}{\sin\beta\varphi}$, $0 > \varphi > -\dfrac{\pi}{\beta}$	half-plane $v > 0$

Behaviour of the curve $\mathfrak{A}\mathfrak{E}\mathfrak{B}\mathfrak{F}\mathfrak{A}$ at infinity.

$\gamma > 1$: asymptotes $\arg z = \pm\pi/\beta$

$\gamma = 1$: asymptotes $\arg(z-\rho) = \pm\pi/\beta$; $\rho = 1/\beta$; see figure.

When $\alpha = 1(\beta = 2)$ then the curve coincides with its asymptote $x = 1/2$. When $\beta \gtrless 2$, then $\rho \lessgtr z_0$, respectively.

$0 < \gamma < 1$: no asymptotes.

For $\beta = 2\alpha = 4$, curve is one branch of hyperbola $\Re(z^2) = 1/2$.

$\boxed{w = az^{\alpha} + bz^{\beta}}$ $ab \neq 0$, $\beta > \alpha > 0$.

This is a combination of

$$w = a\left(-\frac{a}{b}\right)^{\alpha/\gamma}\xi, \quad z = \left(-\frac{a}{b}\right)^{1/\gamma}\zeta \quad \text{and} \quad \xi = \zeta^{\alpha} - \zeta^{\beta}; \ (\gamma = \beta - \alpha).$$

7. REGIONS BOUNDED BY SEGMENTS OF TWO OR THREE CIRCLES or straight lines on half-plane.

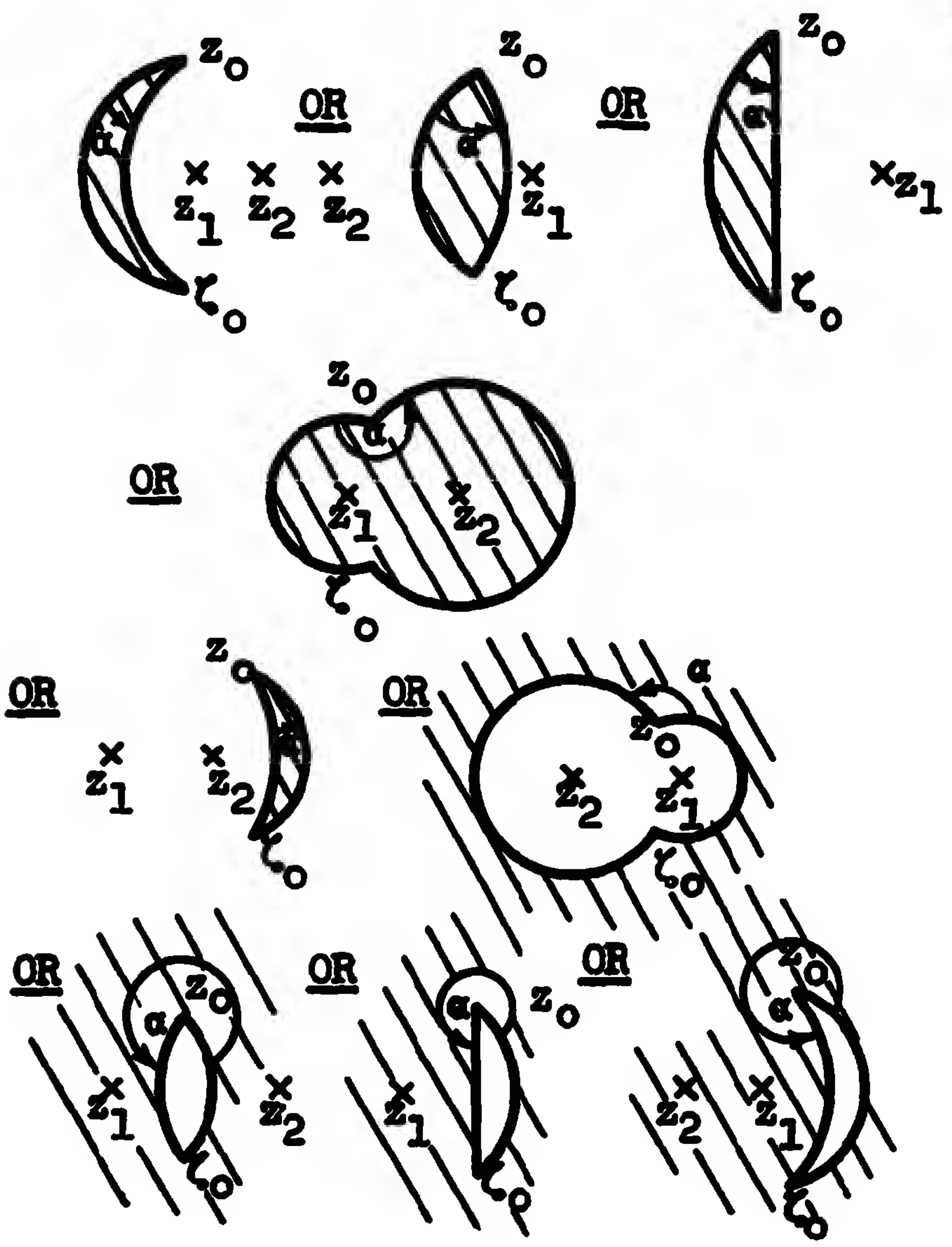
7.1 Region bounded by two circular arcs on half-plane

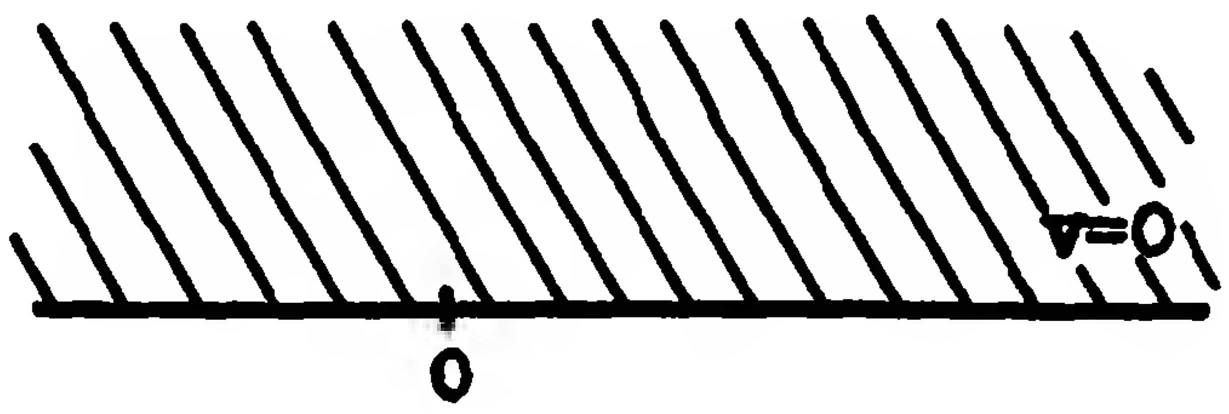
z - plane	w - plane
Transformation: $w = k\left(\frac{z-z_0}{z-\zeta_0}\right)^{\pi/\alpha}$; $k \neq 0$.	
circular crescent, its vertices at $z = z_0$ and $z = \zeta_0$, angle $=\alpha$	some half-plane

Mapping on upper half plane

Transformation: $w = \xi^{\pi/\alpha}$; $\xi = \varepsilon i K \dfrac{(z-z_0)(z_1-\zeta_0)}{(z-\zeta_0)(z_0-\zeta_0)}$ (conf. §5.8);

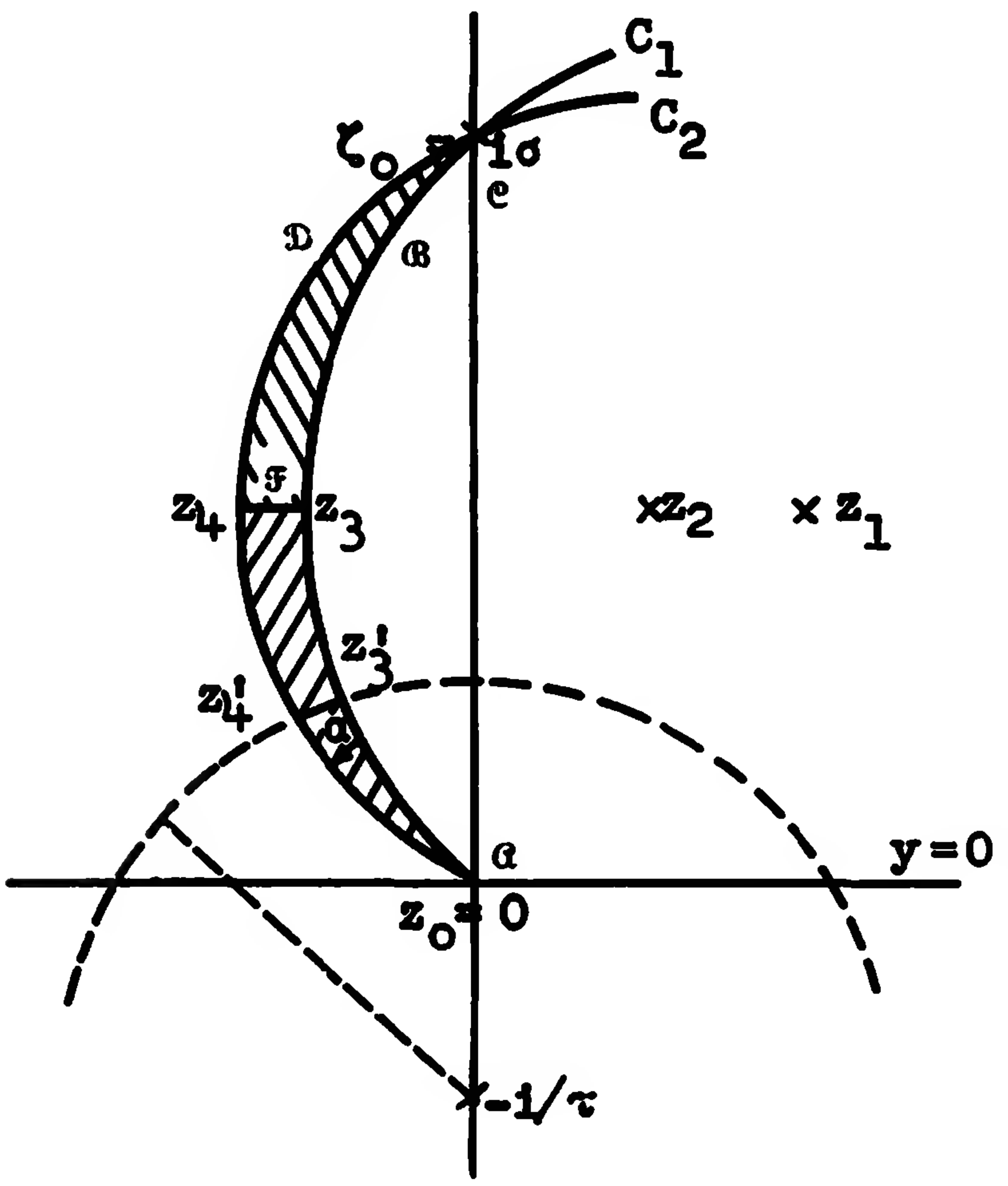
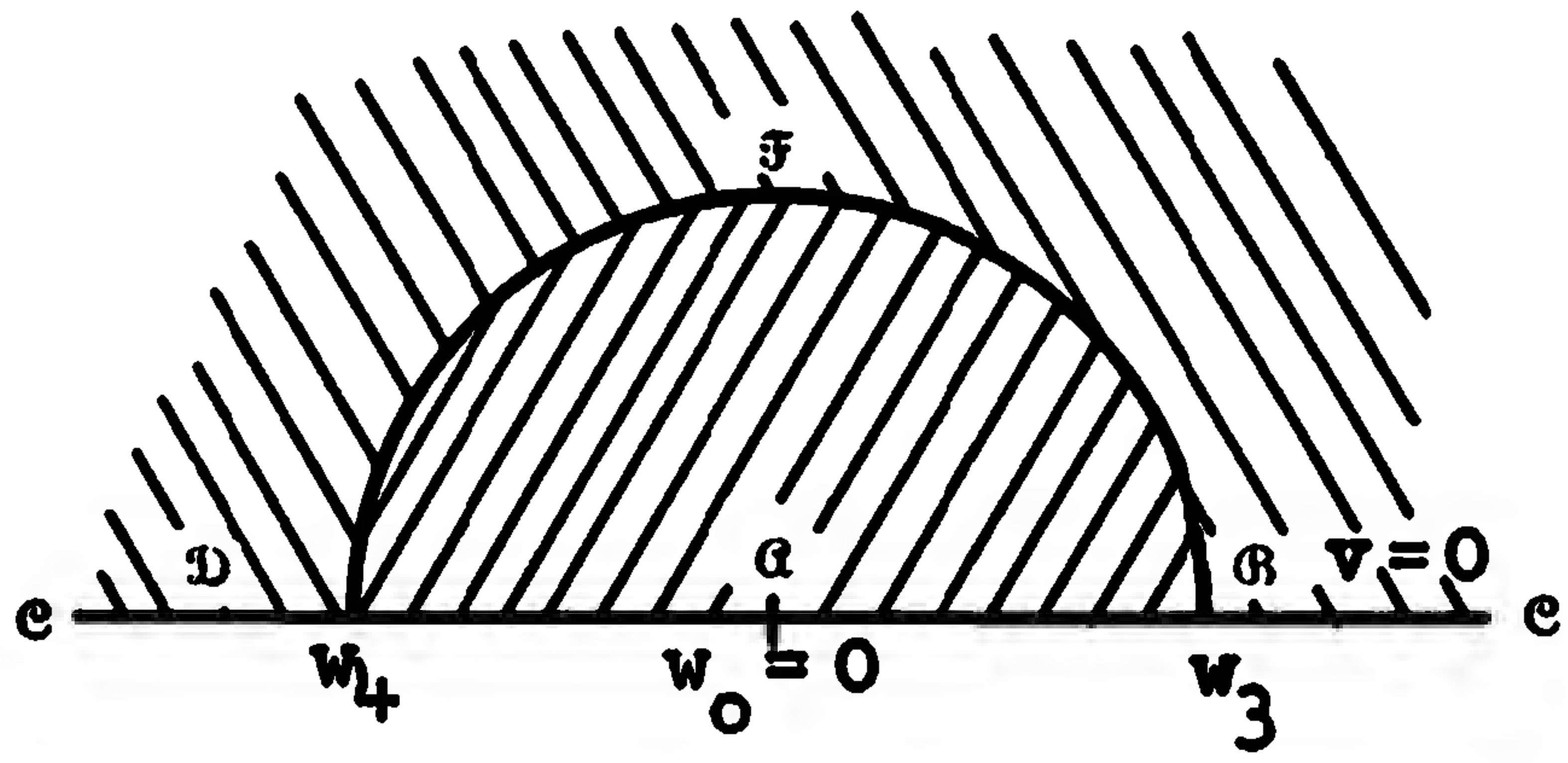
$0 < \alpha < 2\pi$, $K \gtrless 0$, $\varepsilon = \pm 1$; ε is chosen so that ξ maps the arc concerned of the circle (centre z_1) on the positive part of the real axis in the ξ - plane.





z - plane	w - plane
arc (z_o, ζ_o) of circle $\lvert z-z_1\rvert = \lvert z_o-z_1\rvert$	half-line $v = 0, \quad 0 < u < \infty$
arc (z_o, ζ_o) of circle $\lvert z-z_2\rvert = \lvert z_o-z_2\rvert$	half-line $v = 0, \quad 0 > u > -\infty$
points z_o; ζ_o	points 0; ∞

Example (i): $w = c\left(\dfrac{z\bar{z}_1}{i\sigma - z}\right)^{\pi/\alpha}$; $\quad c > 0, \; \sigma > 0, \quad \Im(z_1) = \frac{1}{2}\sigma, \quad 0 < \alpha < \pi - \arg z_1$

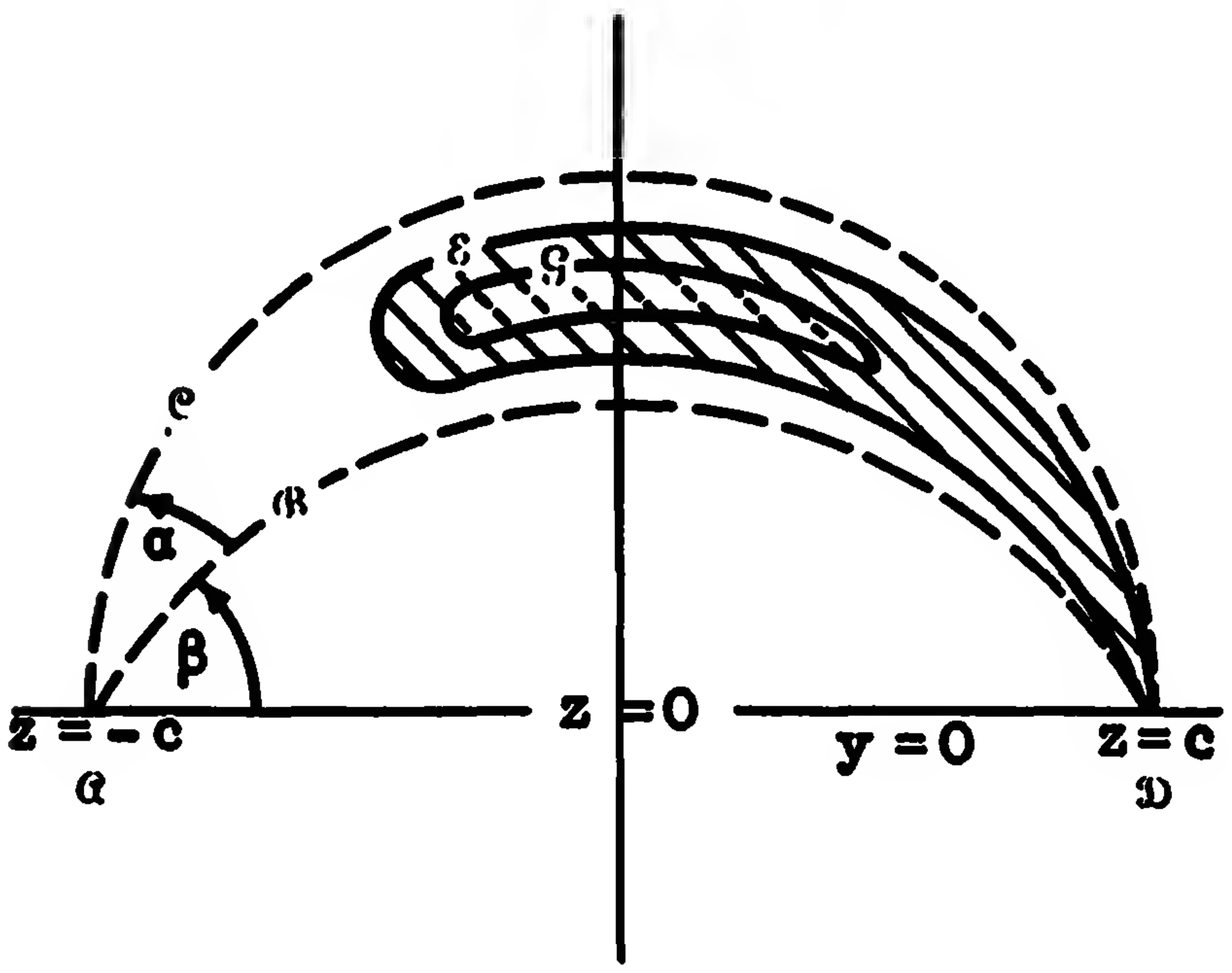
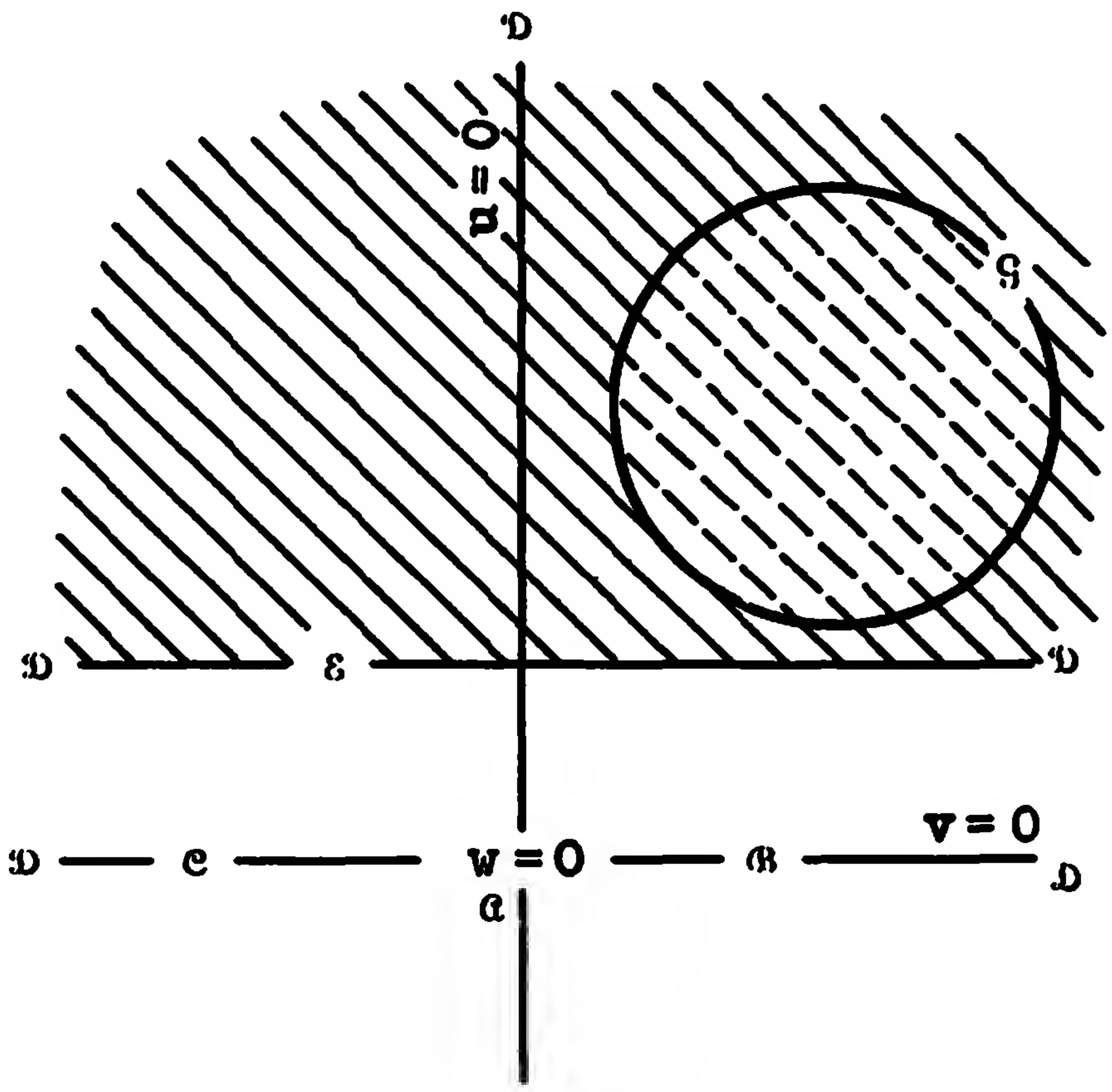
z - plane	w - plane
	
points z_o; $\zeta_o = i\sigma$; $z_3 = z_1 - \lvert z_1\rvert$; $z_4 = z_2 - \lvert z_2\rvert$	points 0; ∞; $w_3 = c\lvert z_1\rvert^{\pi/\alpha}$; $w_4 = -c\lvert z_1\rvert^{\pi/\alpha}$
arc$(z_o z_3 \zeta_o)$ of the circle C_1, with centre z_1	half-line $v = 0, \; 0 \le u \le \infty$
arc$(z_o z_4 \zeta_o)$ of the circle C_2, with centre z_2	half-line $v = 0, \quad 0 \ge u \ge -\infty$

z - plane	w - plane
arc of a circle forming $\angle\beta$ $(0<\beta<\alpha)$ with C_1 at z_0, and with end-points z_0, ζ_0	half-line $w=\rho e^{i\pi\beta/\alpha}$, $0\le\rho\le\infty$
Set of coaxal circles with limiting points z_0, ζ_0.	
arc, lying in the crescent, of circle $\lvert\tau z+i\rvert=\sqrt{1+\sigma\tau}$, $\sigma\tau>-1$ (centre $-i/\tau$)	semi-circle $\lvert w\rvert=c\left\lvert\dfrac{z_1}{\sqrt{1+\sigma\tau}}\right\rvert^{\pi/\alpha}$, $v>0$

<u>Example (ii)</u>: <u>Interior of aerofoil on half-plane</u>‡ (Exterior see §8.5)

$$\boxed{w=ce^{-i\beta\pi/\alpha}\left(\frac{c+z}{c-z}\right)^{\pi/\alpha}};\quad c>0,\ \beta>0,\ 0<\alpha<\alpha+\beta<\pi.$$

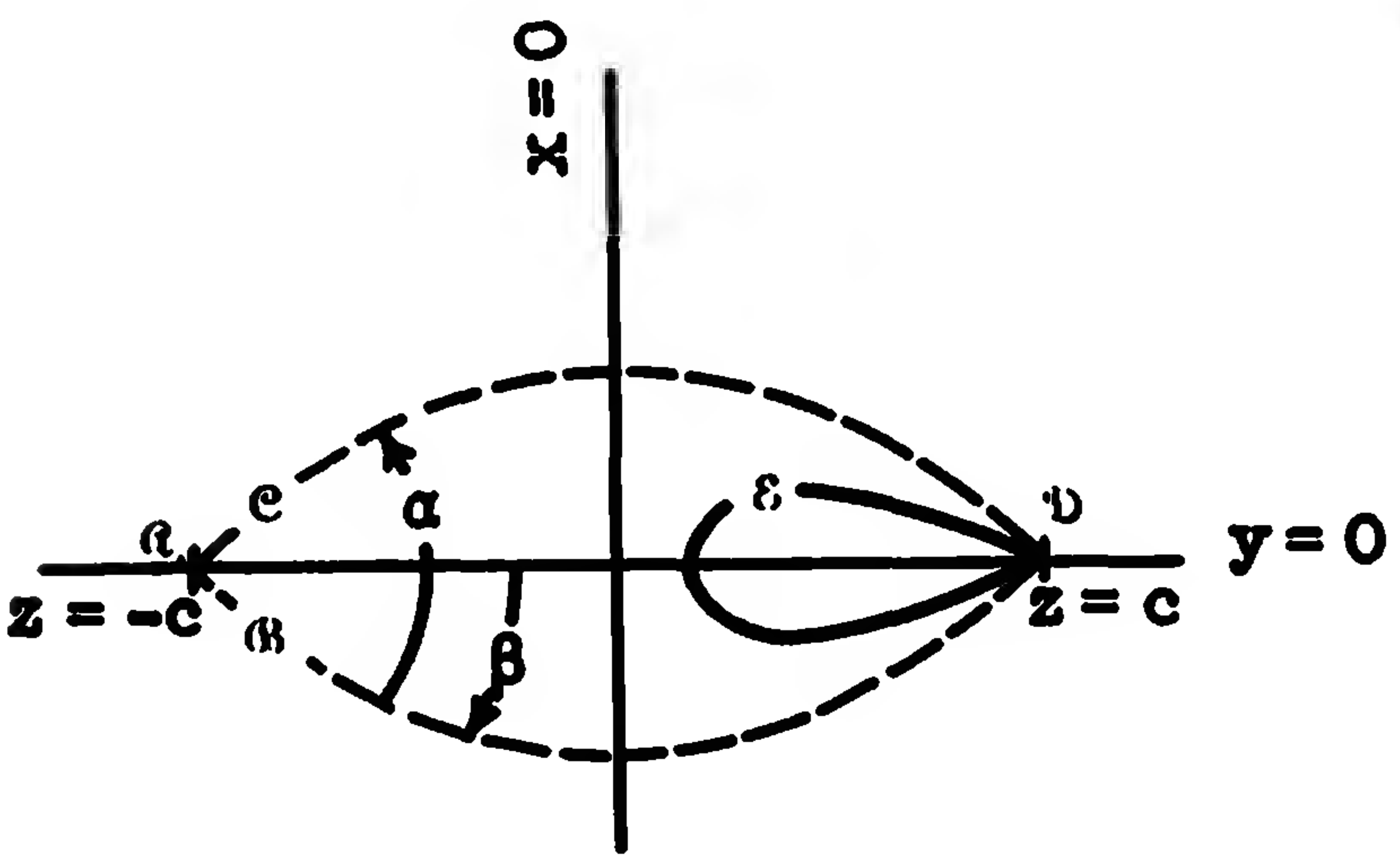
$$\boxed{z=c\,\frac{w^{\alpha/\pi}-c^{\alpha/\pi}e^{-i\beta}}{w^{\alpha/\pi}+c^{\alpha/\pi}e^{-i\beta}}}$$

z - plane	w - plane
	

‡ For further particulars see R. A. Frazer, Conformal representation of the internal area of ovals and aerofoils, and its applications, 1932, Aeronautical Research Committee. The above diagrams are copied from this report.

z - plane	w - plane
points $z = -c;\ c$	points $w = 0;\ \infty$
arc $\mathfrak{A}\mathfrak{B}\mathfrak{D}$	half-line $v = 0,\ 0 \le u \le \infty$
arc $\mathfrak{A}\mathfrak{E}\mathfrak{D}$	half-line $v = 0,\ 0 \ge u \ge -\infty$
region bounded by these arcs	half-plane $v > 0$
aerofoil $\mathfrak{D}\mathcal{E}\mathfrak{D}$, with angle α at $\mathfrak{D}$	line parallel to $v = 0$ in upper half-plane
curve $\mathfrak{G}$	circle $\mathfrak{G}$

When $\beta < 0$ and $\beta = -\frac{\alpha}{2}$, then the aerofoil is symmetrical with respect to $y = 0$.

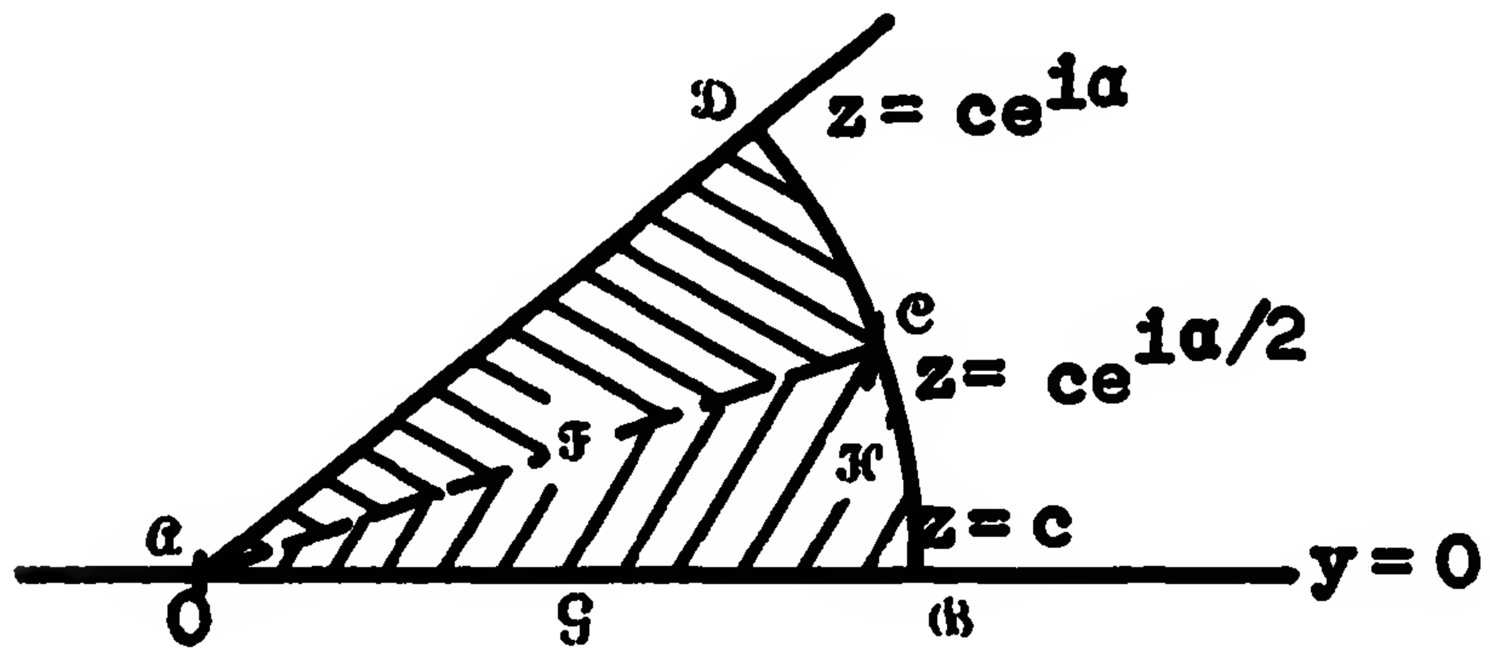


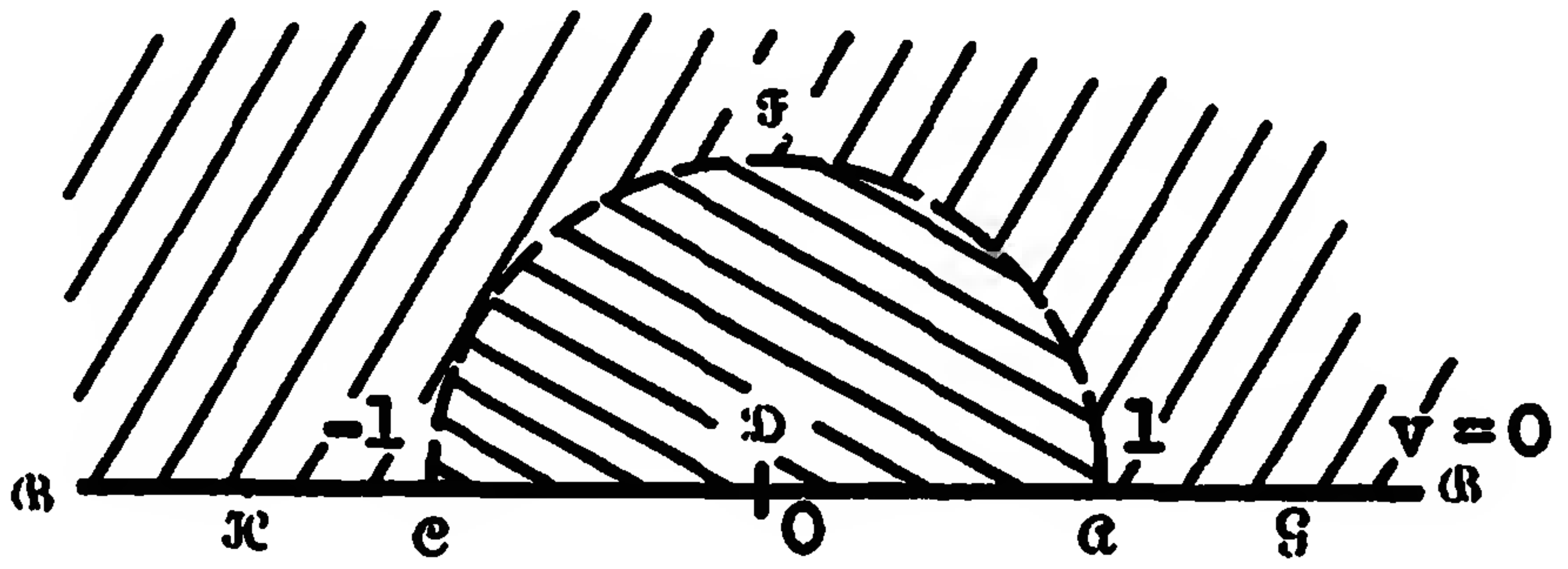
7.2 Sector on upper half-plane.

$$w = \left\{\frac{(z/c)^{\pi/\alpha} + 1}{(z/c)^{\pi/\alpha} - 1}\right\}^2 ;\quad c > 0;\quad 0 < \alpha \le 2\pi;\quad z = c\left(\frac{\sqrt{w}+1}{\sqrt{w}-1}\right)^{\alpha/\pi}$$

Critical points for the sector: $z = 0$, $z = c$, $z = ce^{i\alpha}$.

z - plane	w - plane
points $z = 0;\ ce^{i\varphi}\ (0 < \varphi < \alpha);\ c;\ ce^{i\alpha/2};\ ce^{i\alpha}$	points $1;\ -\cot^2\frac{\pi\varphi}{2\alpha};\ \infty;\ -1;\ 0$



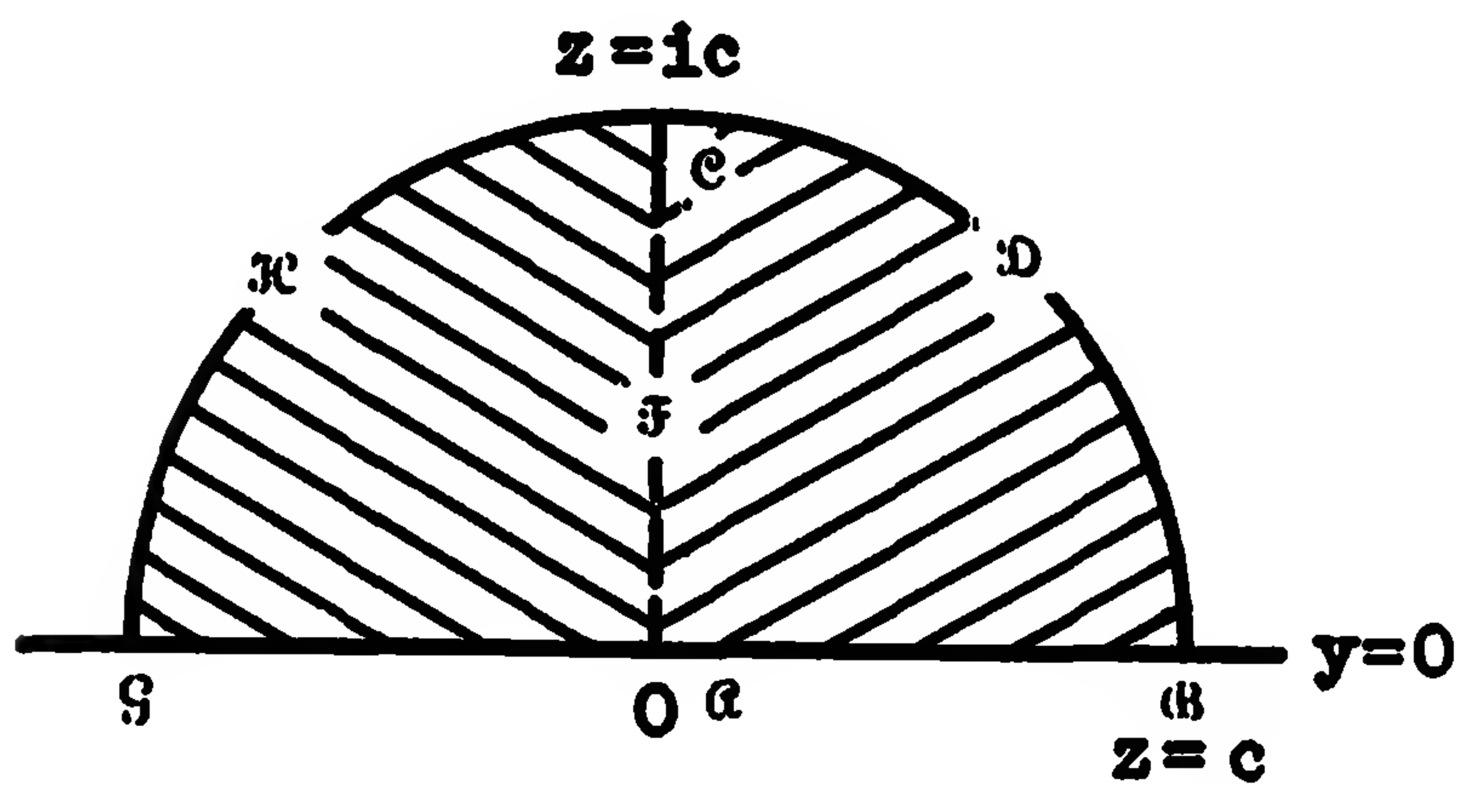


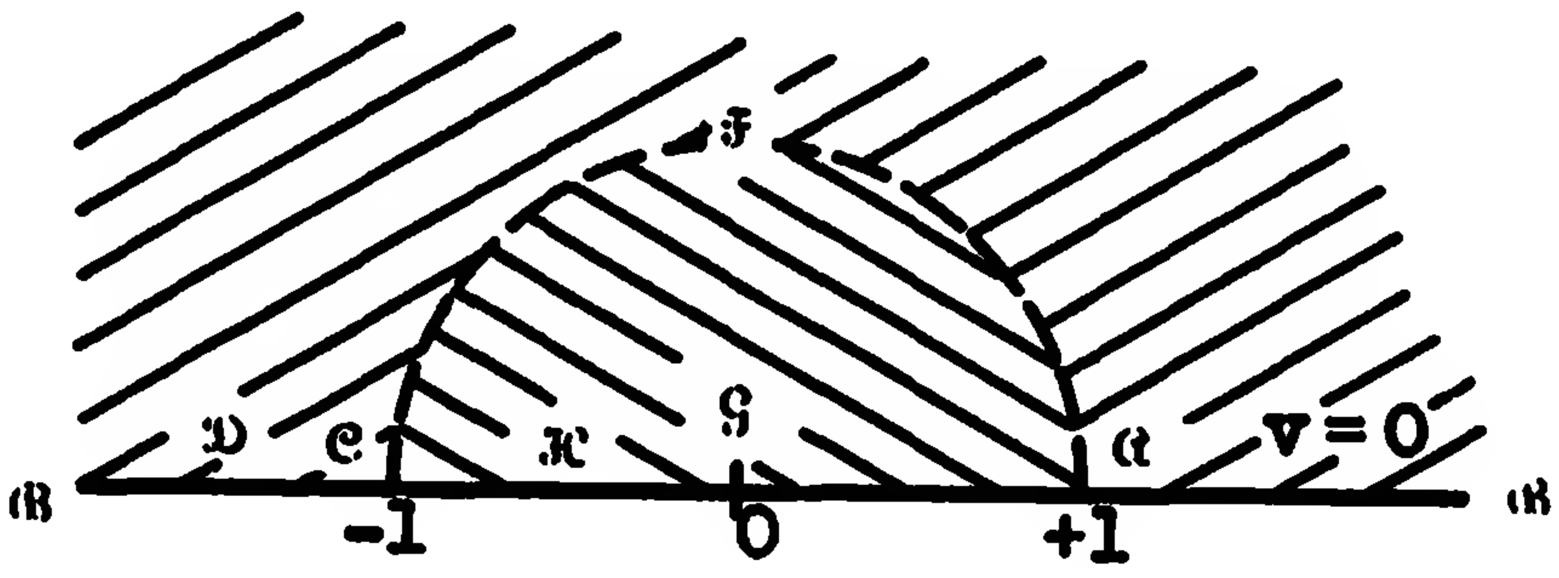
z - plane	w - plane
line segment $y = 0,\ 0 \leq x < c$	half-line $v = 0,\ 1 \leq u < \infty$
arc $z = ce^{i\varphi},\ 0 < \varphi \leq \frac{\alpha}{2}$	half-line $v = 0,\ -\infty < u \leq -1$
arc $z = ce^{i\varphi},\ \frac{\alpha}{2} \leq \varphi \leq \alpha$	line segment $v = 0,\ -1 \leq u \leq 0$
line segment $z = re^{i\alpha},\ c \geq r \geq 0$	line segment $v = 0,\ 0 \leq u \leq 1$
line segment $z = re^{i\alpha/2},\ 0 \leq r \leq c$	semicircle $\lvert w \rvert = 1,\ y \geq 0$

<u>Example (i)</u> $\boxed{w = \left(\frac{z+c}{z-c}\right)^2}$; $z = c\,\frac{\sqrt{w}+1}{\sqrt{w}-1}$.

Area of semicircle on upper half-plane

z - plane	w - plane
point $z = ce^{i\varphi},\ 0 < \varphi < \alpha = \pi$	point $w = -\cot^2 \frac{\varphi}{2}$





<u>Example (ii)</u> $\boxed{w = \frac{z^2-c^2+2cz}{z^2-c^2-2cz}}$;

combination of $\zeta = iz$; $\xi = \left(\frac{\zeta+c}{\zeta-c}\right)^2$, $w = i\,\frac{\xi-i}{\xi+i}$;

or of $w = \frac{\xi+1}{\xi-1}$, $\xi = (2c)^{-1}z + (-\frac{c}{2})/z$ (see §8).

z - plane	w - plane
area of semi-circle $\lvert z \rvert < c,\ x > 0$	$\lvert w \rvert < 1$

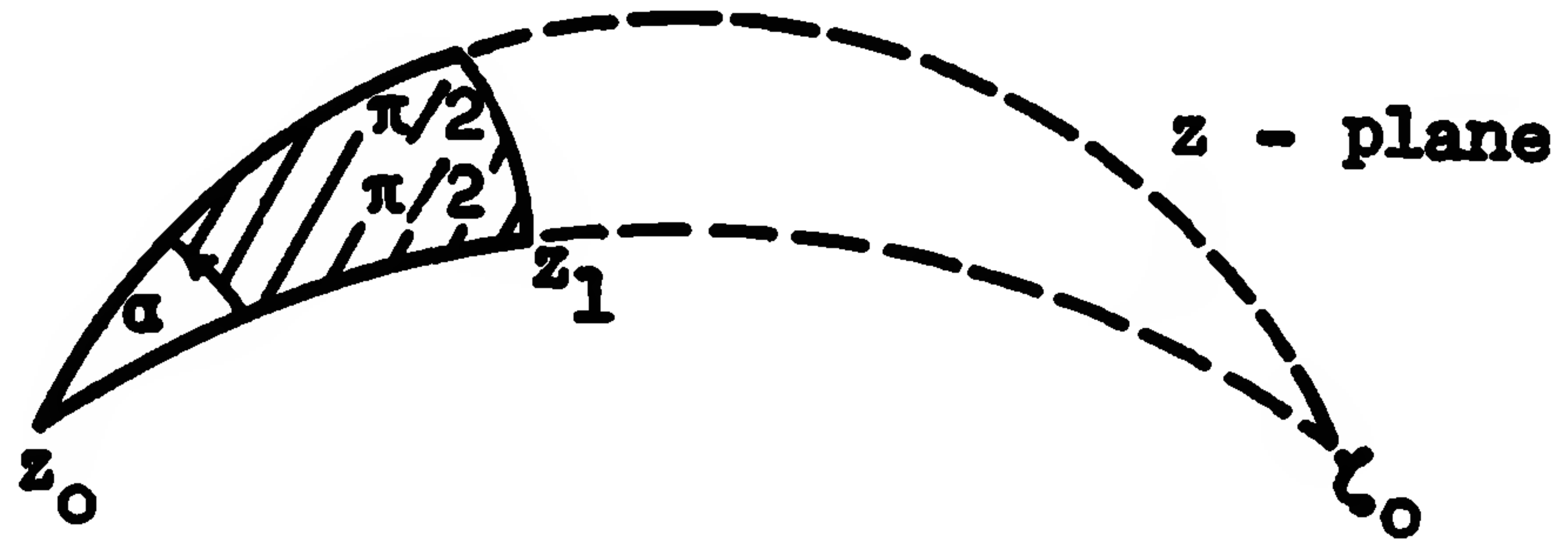
Example (iii) $\boxed{w = \dfrac{(1+z^3)^2 - i(1-z^3)^2}{(1+z^3)^2 + i(1-z^3)^2}}$;

combination of $w = \frac{\xi-i}{\xi+i}$, $\xi = \left(\frac{z^3+1}{z^3-1}\right)^2$.

z - plane	w - plane
points $z = 0;\ 1;\ e^{i\pi/3};\ e^{i\pi/6}$	points $w = -i;\ 1;\ -1;\ i$
sector area $\lvert z\rvert <1,\ \ 0 < \arg z < \frac{\pi}{3}$	$\lvert w\rvert <1$

7.3 **Curvilinear triangle with two right angles on half-plane.**

Third angle $\alpha \neq \pi$; for $\alpha = \pi$ cf. §7.1



Given z_o, z_1, $\angle\alpha$, ζ_o; z_o, ζ_o finite; $0 < \alpha < 2\pi$.

$$w = \left(\frac{\zeta^{\pi/\alpha}+1}{\zeta^{\pi/\alpha}-1}\right)^2 \quad \text{where} \quad \zeta = \frac{z_1-\zeta_o}{z_1-z_o}\,\frac{z-z_o}{z-\zeta_o}\ .$$

Example (i) Cf. figure of §7.1, example (i).

$$w = \left\{ \frac{\left(\frac{\bar{z}_1 z}{i\sigma - z}\right)^{\pi/\alpha} + \left(\frac{|z_1|}{\sqrt{1+\sigma\tau}}\right)^{\pi/\alpha}}{\left(\frac{\bar{z}_1 z}{i\sigma - z}\right)^{\pi/\alpha} - \left(\frac{|z_1|}{\sqrt{1+\sigma\tau}}\right)^{\pi/\alpha}} \right\}^2 \ ; \qquad \Im(z_1) = \tfrac{1}{2}\sigma > 0, \quad 0 < \alpha < \pi - \arg z_1$$

z - plane	w - plane
region ($z_o = 0$, z_3', z_4') bounded by C_1, C_2, and circle $\lvert\tau z+i\rvert = \sqrt{1+\sigma\tau}$ ($\sigma > 0$, $\tau > 0$)	half-plane $y > 0$
arc of circle passing through $z_o = 0$ and $\zeta_o = i\sigma$, bisecting $\angle\alpha$	semi-circle $\lvert w\rvert = 1$, $v \geq 0$
end-point of this arc, on circle $\lvert\tau z+i\rvert = \sqrt{1+\sigma\tau}$	point $w = -1$
points $z_o = 0$, z_3', z_4'	points 1, ∞, 0

Note: if $\tau = 0$, the region is ($z_o = 0$, z_3, z_4) and is bounded by C_1, C_2 and $\Im(z-z_1) = 0$.

If π/α is an integer n, then the z-plane can be divided up into $4\lvert n\rvert$ triangular regions, mapped in turn on $v > 0$ and $v < 0$; the transformation is a rational function, cf. §7.4 and 7.5.

Example (ii): $z_o = \frac{1}{2} + \frac{i}{2}\sqrt{3}$, $\zeta_o = \frac{1}{2} - \frac{i}{2}\sqrt{3}$, $z_1 = 0$, $\angle\alpha = -\pi/3$, $n = -3$.

Curvilinear triangle with angles $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{3}$.

$$1 - w = \frac{4}{27}\,\frac{(z^2-z+1)^3}{(z^2-z)^2}.$$

Critical points: $z = 0$; 1, ∞; 2; $\frac{1}{2}$; -1; z_o; ζ_o.

z - plane	w - plane
points z; z^{-1}; $1-z$; $(1-z)^{-1}$; $z(z-1)^{-1}$; $1-z^{-1}$	point $w = w(z)$
points 2; $\frac{1}{2}$; -1; 0; 1; ∞	points 0; 0; 0; ∞; ∞; ∞
points z_o; ζ_o; $\pm i$; $1\pm i$; $\frac{1}{2}(1\pm i)$	points 1; 1; $\frac{25}{27}$; $\frac{25}{27}$; $\frac{25}{27}$

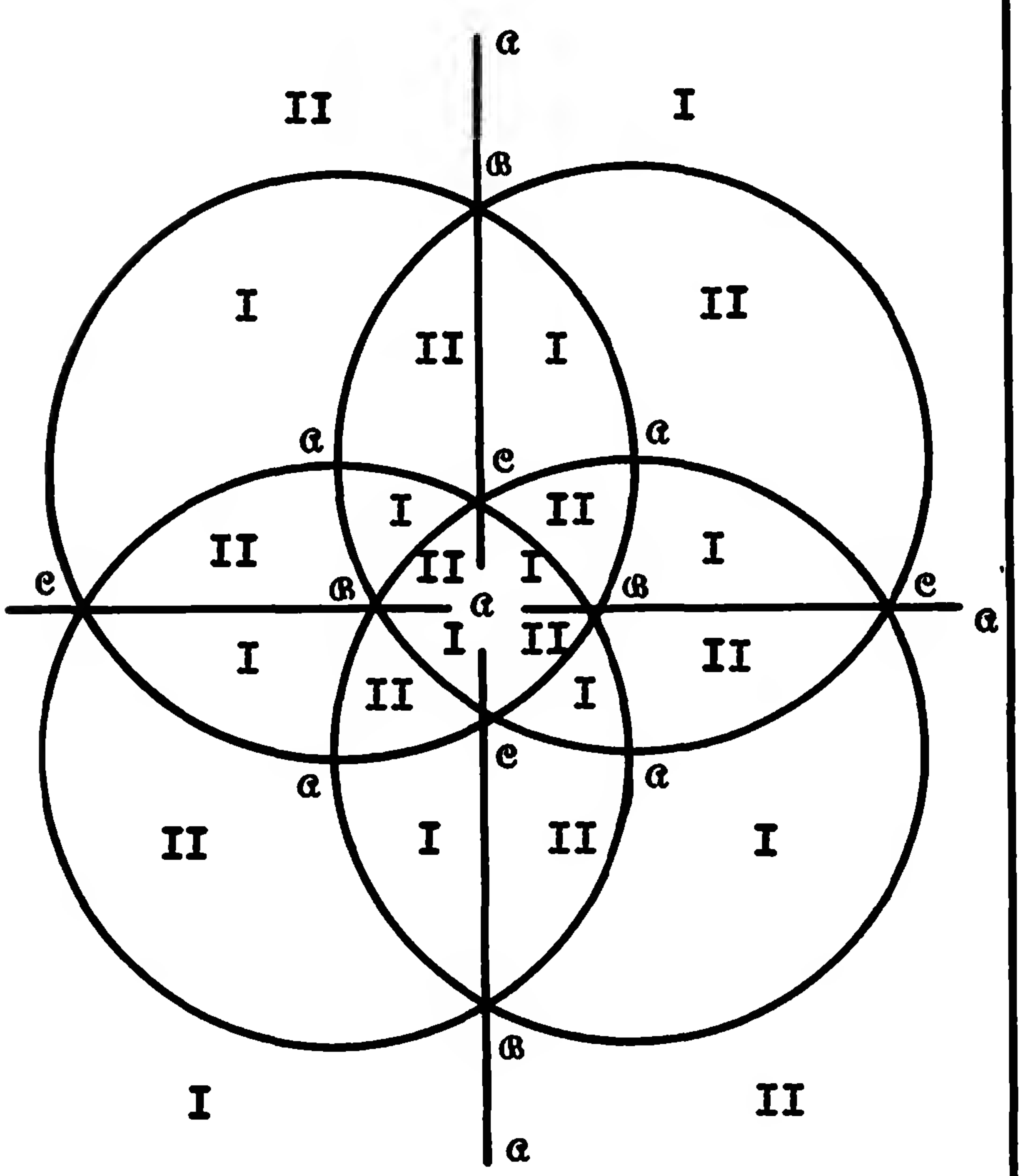
z - plane	w - plane
each of the 12 triangular regions, formed by $x = \frac{1}{2}$; $y = 0, \lvert z \rvert = 1$ and $\lvert z-1 \rvert = 1$ Diagram of $\tilde{w} = 1-w$: see p. 199.	half-plane $v > 0$ or half-plane $v < 0$

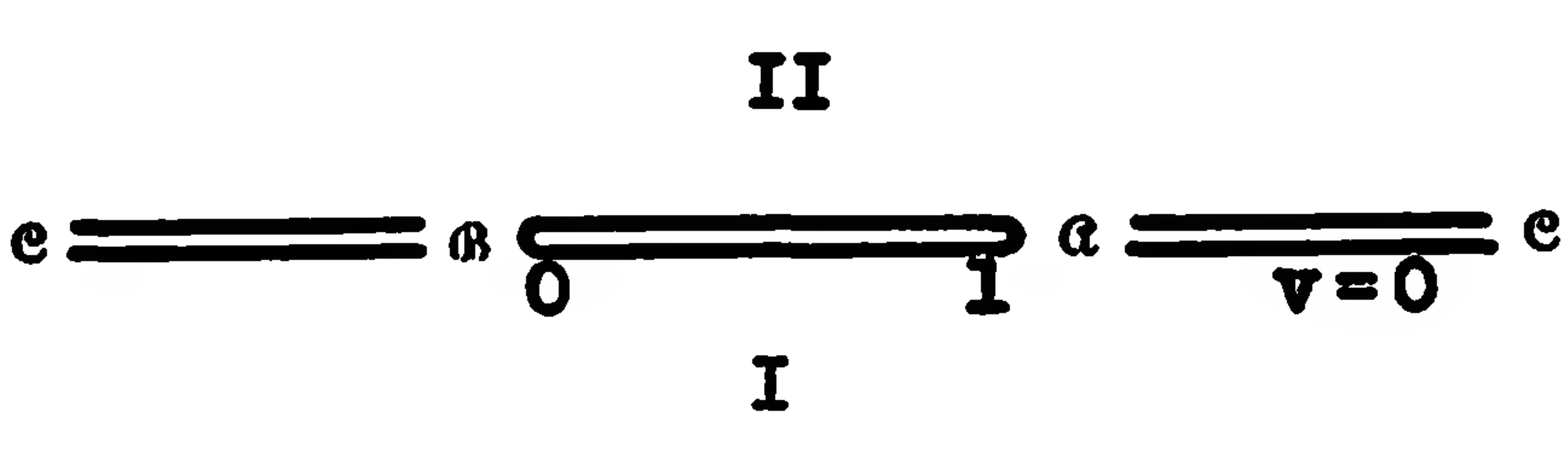
7.4 <u>Curvilinear triangle with angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{3}\pi$ on half-plane.</u>

$$w = \left(\frac{z^4 + 2z^2\sqrt{3} - 1}{z^4 - 2z^2\sqrt{3} - 1}\right)^3 .$$

Critical points: $z = 0$; $\frac{\eta}{2}(\sqrt{6} - \sqrt{2})$; $\frac{\eta}{2}(\sqrt{6} + \sqrt{2})$; $\eta e^{i\pi/4}$; ∞; where $\eta = 1, i, -1, -i$.

z - plane	w - plane
a: 0; ∞; $e^{i\pi/4}$; $e^{-i\pi/4}$; $-e^{i\pi/4}$; $-e^{-i\pi/4}$.	point 1
b: $\pm\frac{1}{2}(\sqrt{6} - \sqrt{2})$; $\pm\frac{1}{2}(\sqrt{6} + \sqrt{2})$;	points 0; 0
c: $\pm\frac{1}{2}(\sqrt{6} + \sqrt{2})$; $\pm\frac{1}{2}(\sqrt{6} - \sqrt{2})$	points ∞; ∞





z - plane	w - plane
parts of "circles" $x = 0$; $y = 0$; $\lvert z \pm e^{i\pi/4}\rvert = \sqrt{2}$; $\lvert z \pm e^{-i\pi/4}\rvert = \sqrt{2}$	parts of $v = 0$; see figure.
each of the 12 regions II	half-plane $v > 0$
each of the 12 regions I	half-plane $v < 0$

Each of the angles at α is $\pi/2$; at β and at γ: $\pi/3$.

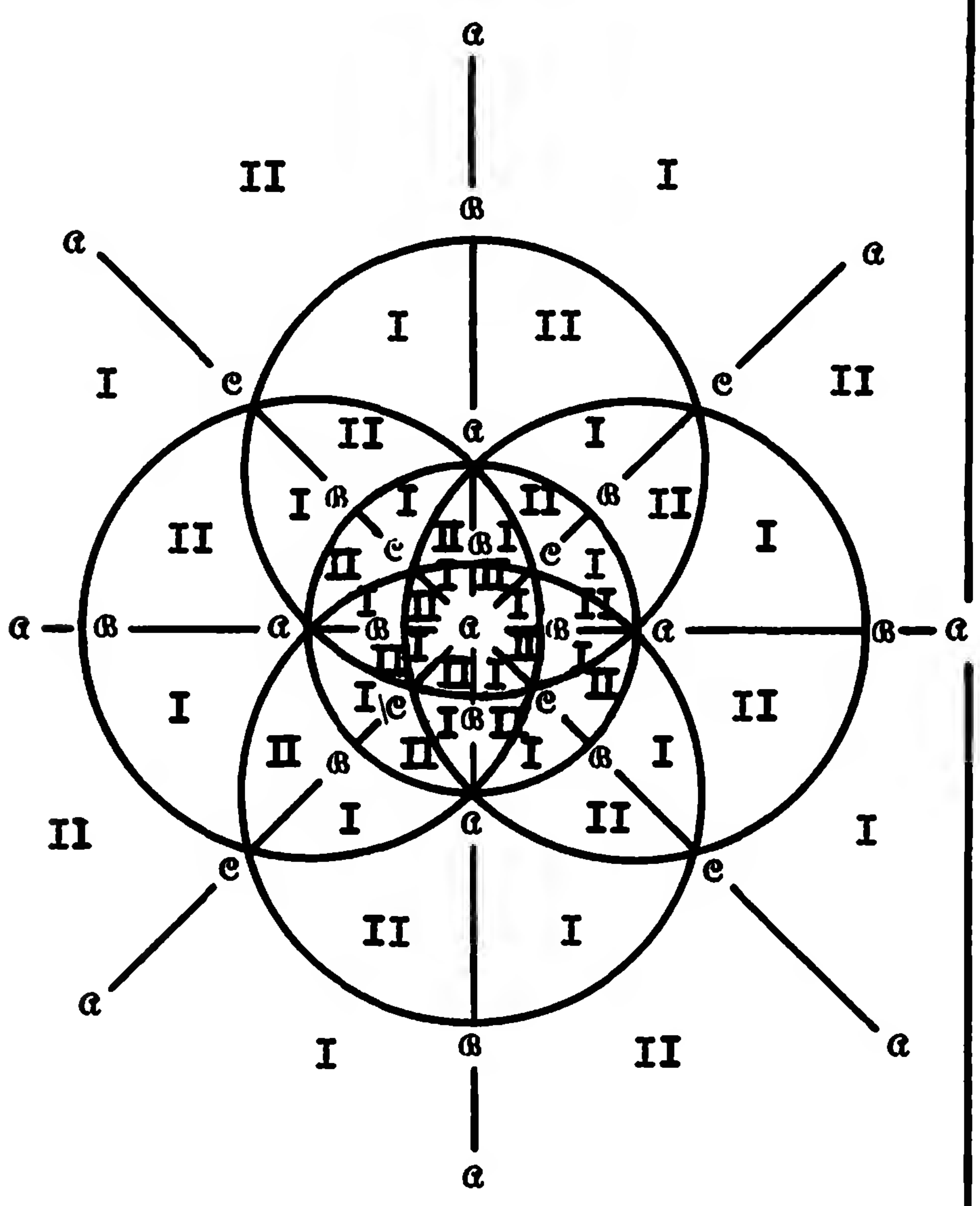
7.5 **Curvilinear triangle with angles $\frac{1}{2}\pi$, $\frac{1}{3}\pi$, $\frac{1}{4}\pi$ on half-plane.**

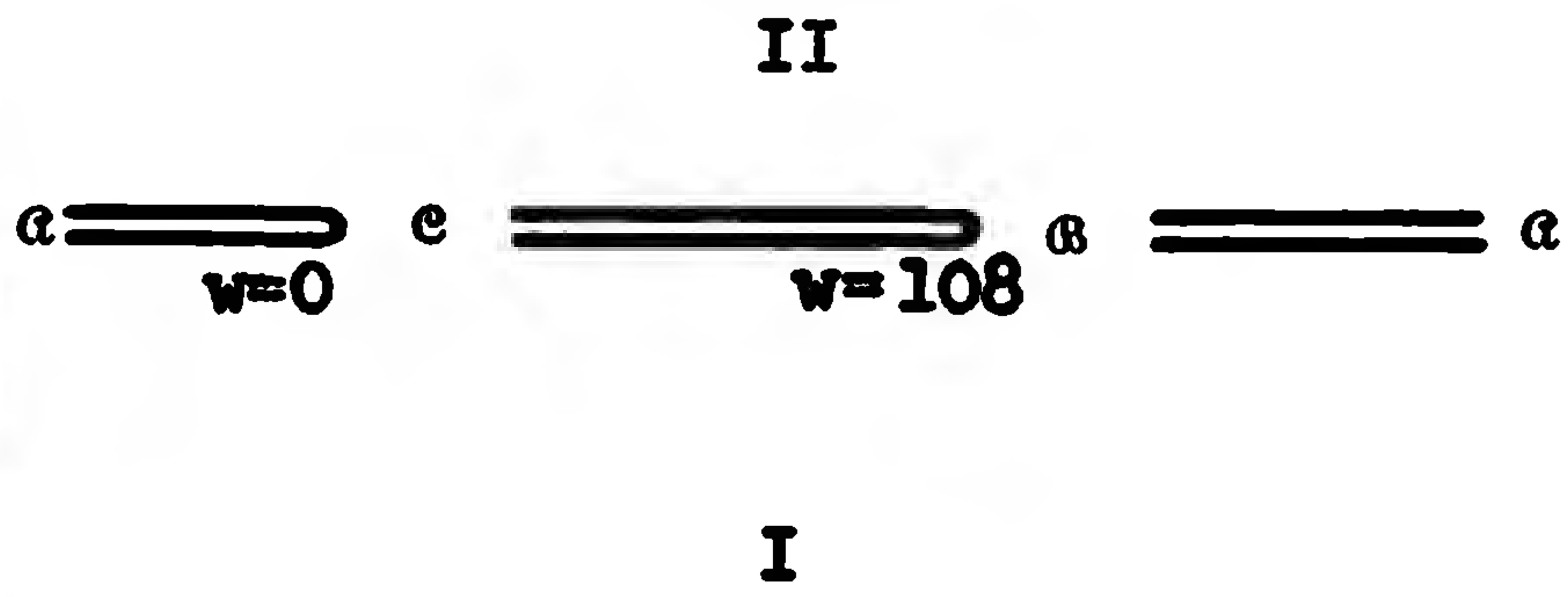
$$w = \frac{(z^4 + z^{-4} + 14)^3}{(z^4 + z^{-4} - 2)^2}.$$

Critical points: $z = 0$; $\eta(\sqrt{2}-1)$; η; $\eta e^{i\pi/4}$; $\eta(\sqrt{2}+1)$; $\frac{\eta}{2}(\sqrt{6}-\sqrt{2})e^{i\pi/4}$; $\frac{\eta}{2}(\sqrt{6}+\sqrt{2})e^{i\pi/4}$; ∞; where $\eta = 1, i, -1, -i$.

z - plane	w - plane
ηz; $\frac{\eta}{z}$; $\eta\frac{z+1}{z-1}$; $\eta\frac{z-1}{z+1}$; $\eta\frac{z+i}{z-i}$; $\eta\frac{z-i}{z+i}$	point $w = w(z)$
points α: 0; η; ∞, see figure.	point ∞
points β: $\eta(\sqrt{2}\pm 1)$; $\eta e^{i\pi/4}$	point 108
points γ: $\frac{\eta}{2}(\sqrt{6}\pm\sqrt{2})e^{i\pi/4}$	point 0
parts of $x = 0$; $y = 0$; $\arg z = e^{\pm i\pi/4}$, $e^{\pm 3i\pi/4}$; $\lvert z\rvert = 1$; $\lvert z\pm 1\rvert = \sqrt{2}$; $\lvert z \pm i\rvert = \sqrt{2}$.	parts of $v = 0$, see figure.

z - plane	w - plane





each of the 24 regions II	half-plane $v > 0$
each of the 24 regions I	half-plane $v < 0$

Each of the small angles at a is $\pi/4$; at b: $\pi/2$; at c: $\pi/3$.

8. $\boxed{w = \alpha z + \frac{\beta}{z} \quad \text{or} \quad \frac{w+2k}{w-2k} = \left(\frac{z+k/\alpha}{z-k/\alpha}\right)^2}$ ‡

$k = \sqrt{\alpha\beta} \neq 0, \quad -\frac{\pi}{2} \leq \arg k < \frac{\pi}{2}. \qquad 2\alpha z = w + \sqrt{w^2-4k^2}.$

Critical points: $z = 0, \quad z = \infty, \quad z = \pm\sqrt{\beta/\alpha} = \pm k/\alpha.$

z - plane	w - plane
points $z;\ \frac{\beta}{\alpha z}$	point $w = \alpha z + \frac{\beta}{z}$
points $0;\ \infty$	point ∞
points $\frac{ik}{\alpha};\ -\frac{ik}{\alpha}$	point 0
points $\frac{k}{\alpha}e^{i\varphi};\ \frac{k}{\alpha}e^{-i\varphi}$	point $2k\cos\varphi$
points $i\frac{k}{\alpha}\tan\varphi;\ -i\frac{k}{\alpha}\cot\varphi$	point $-2ik\cot 2\varphi$

‡ A drawing representing the curves u = const., v = const. for $\alpha = \beta = 1$,

$u = 1.0,\ 1.1,\ 1.5,\ 2.0,\ 3.0;$

$\pm v = \ 0,\ 0.5,\ 1.0,\ 2.0,\ 3.0,\ 4.0;$

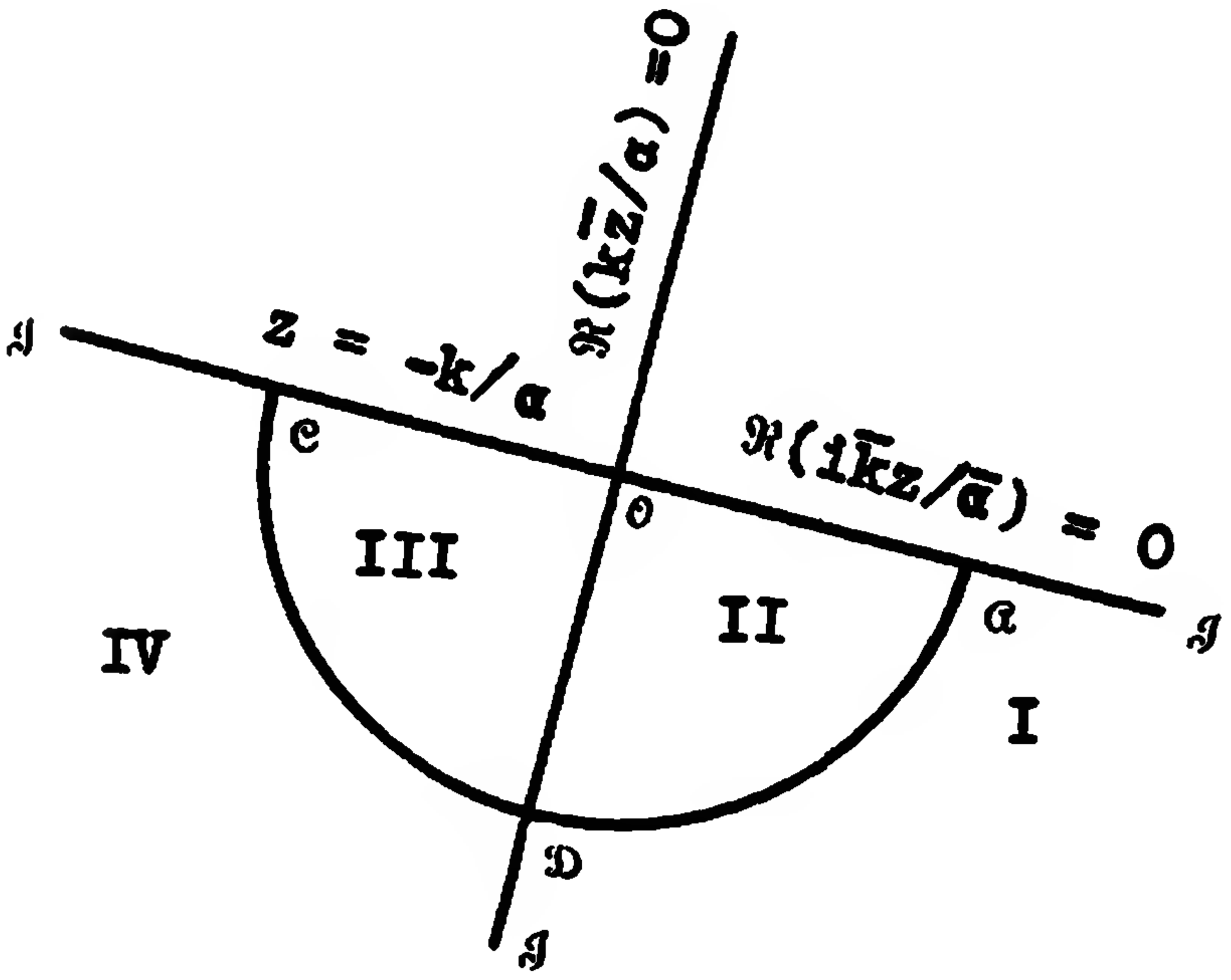
is contained in

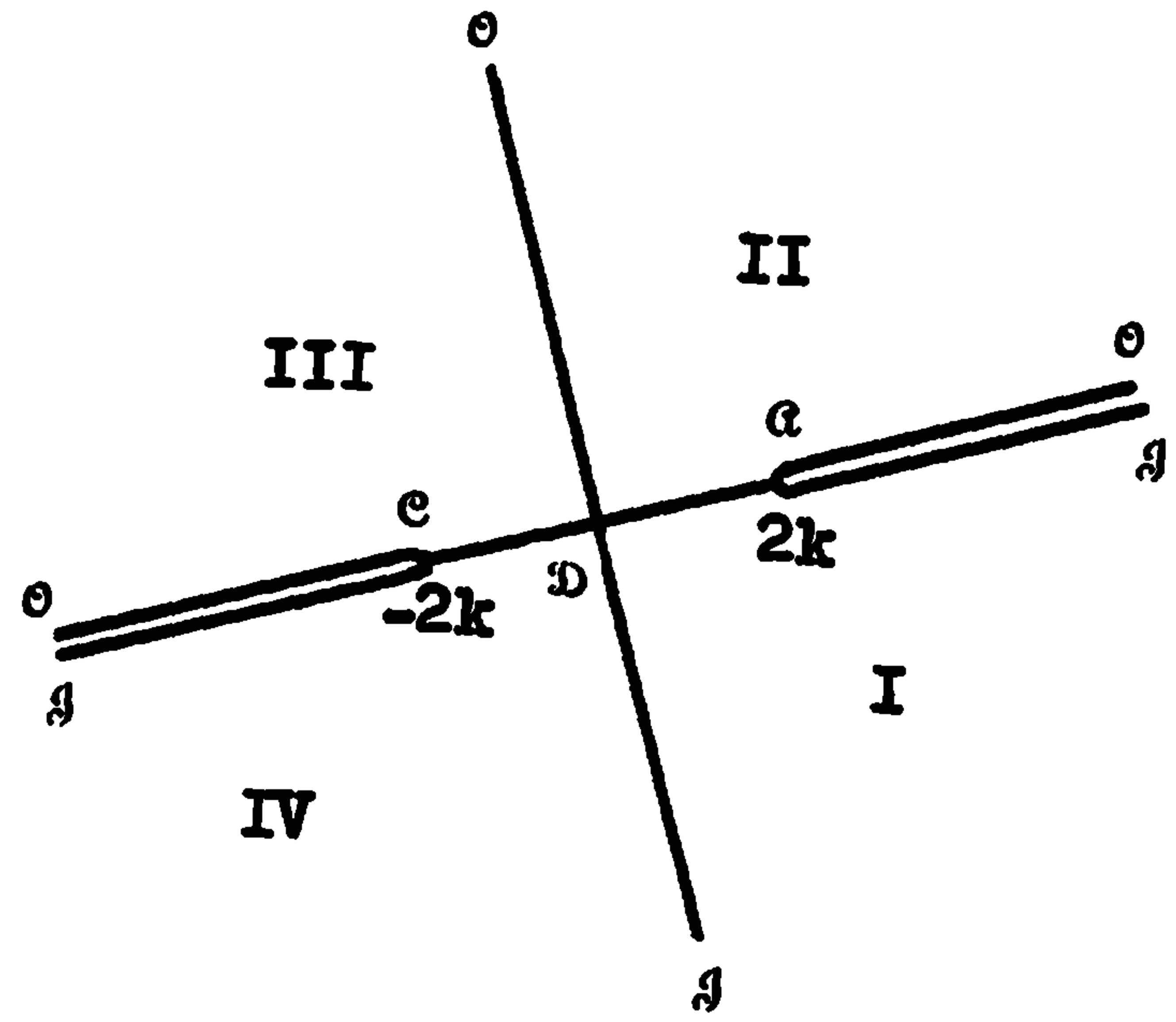
Stress distribution near a surface crack in a bar under tension, by N. Rosen. (A. 45/770).

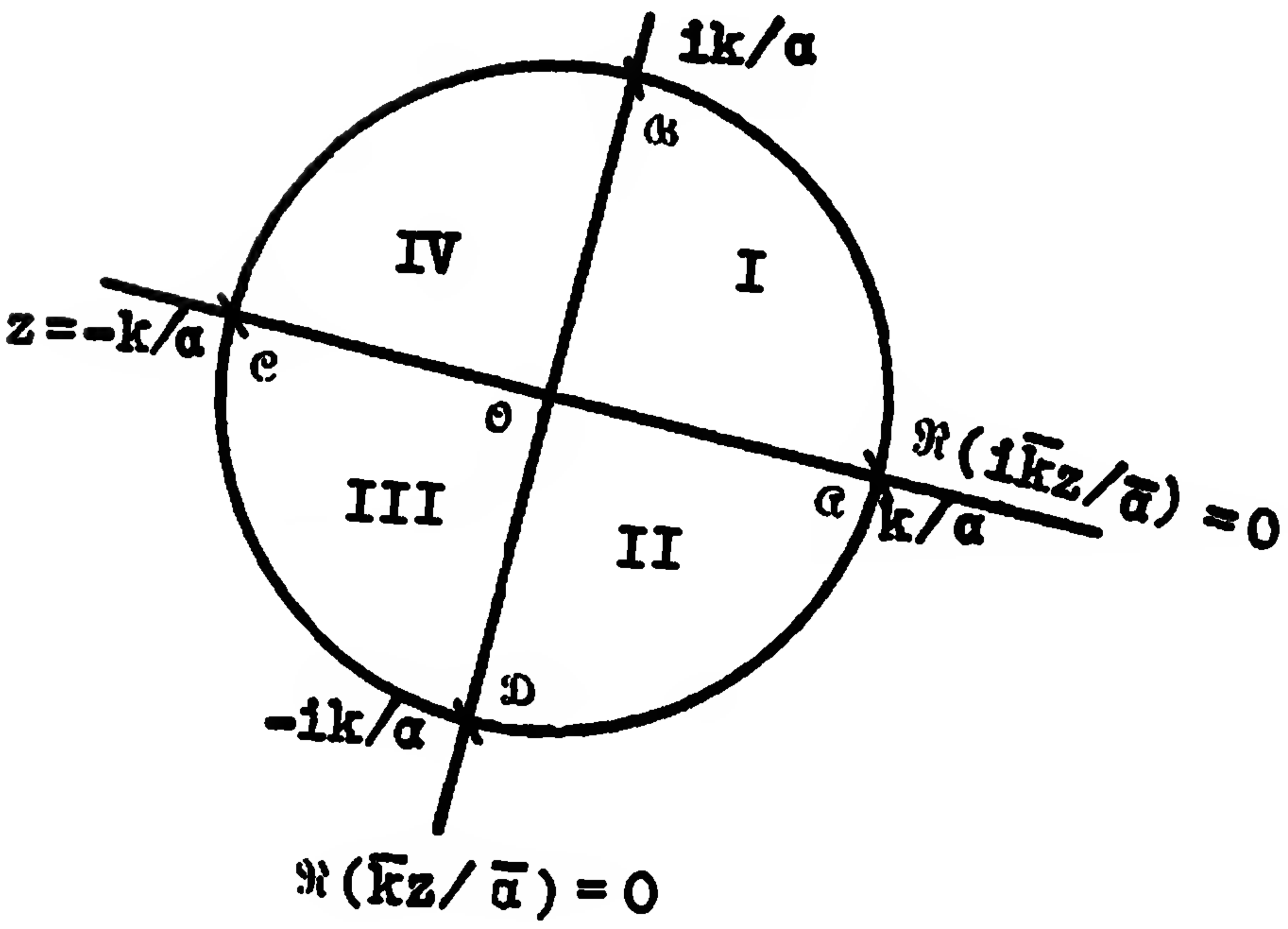
8.1

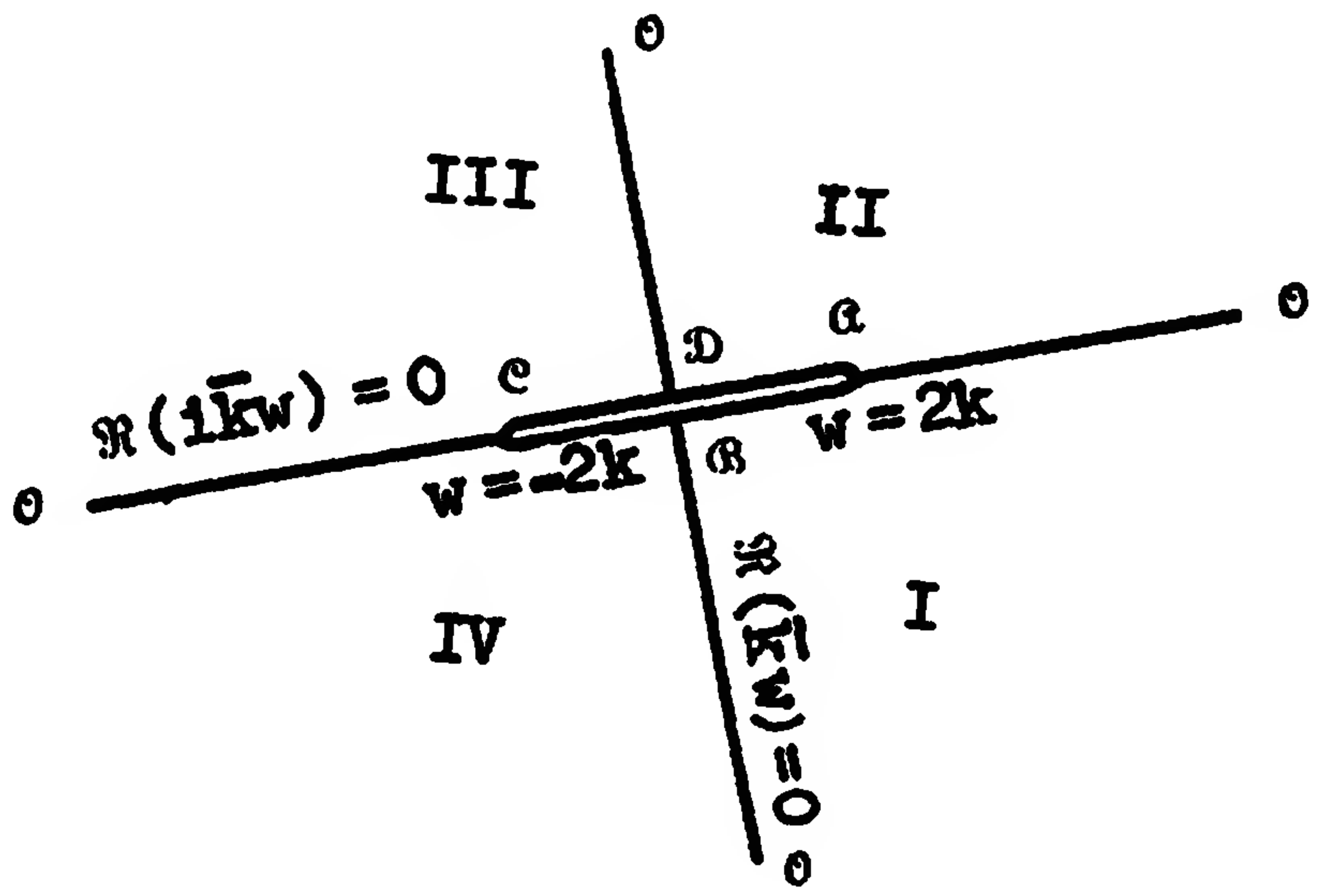
Lower Upper	half-plane on cut plane;
Interior Exterior	of circle on cut plane.

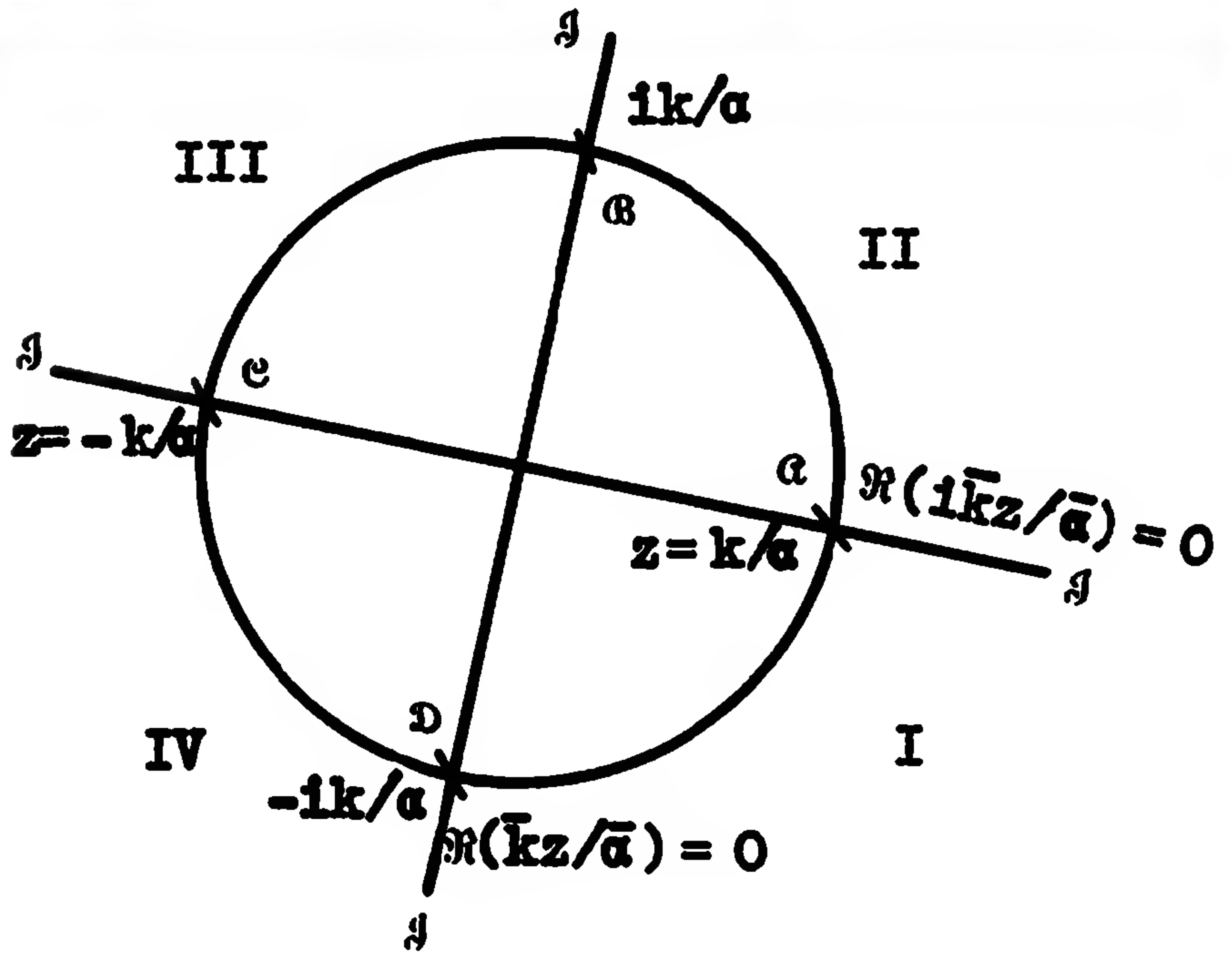
z - plane	w - plane

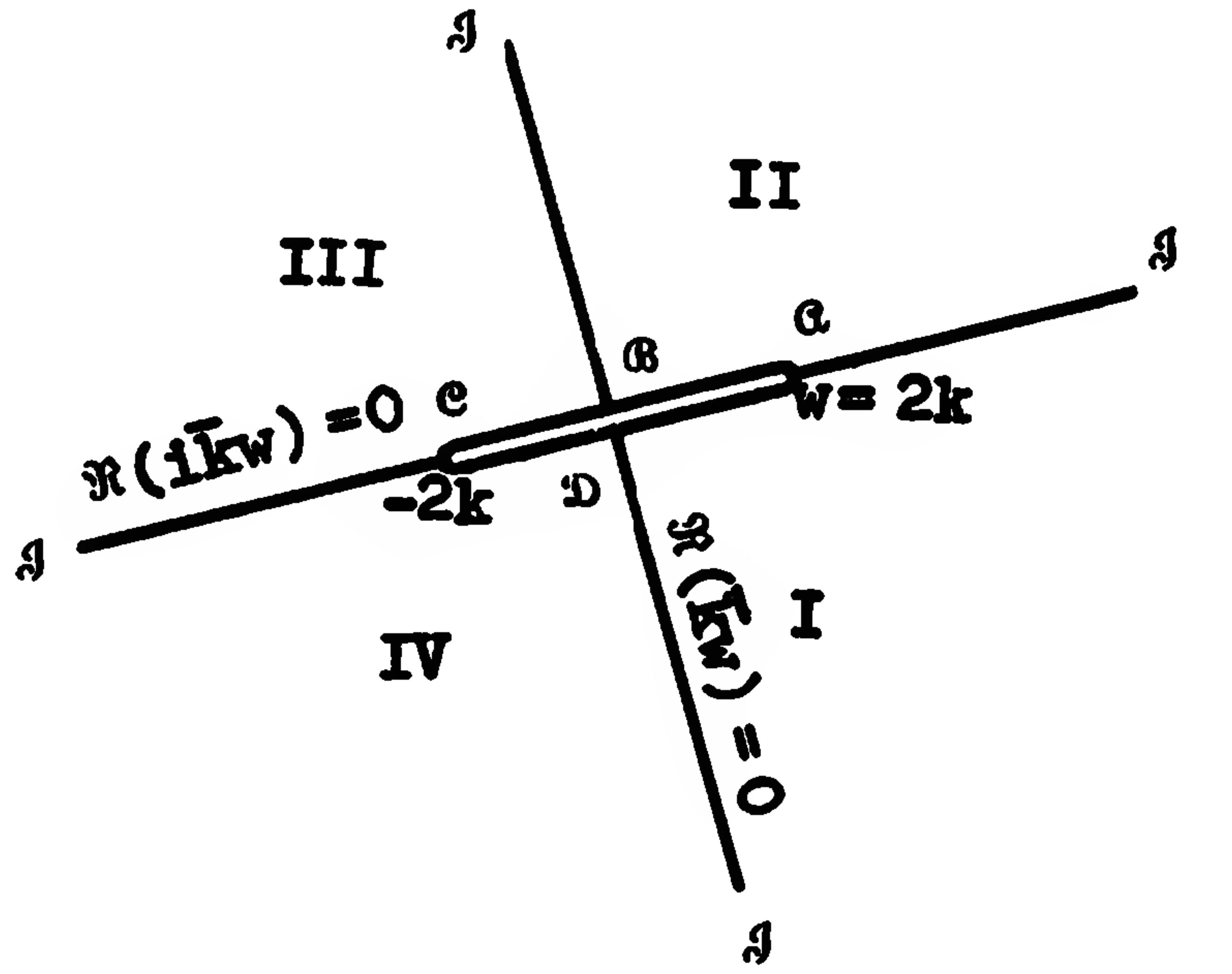








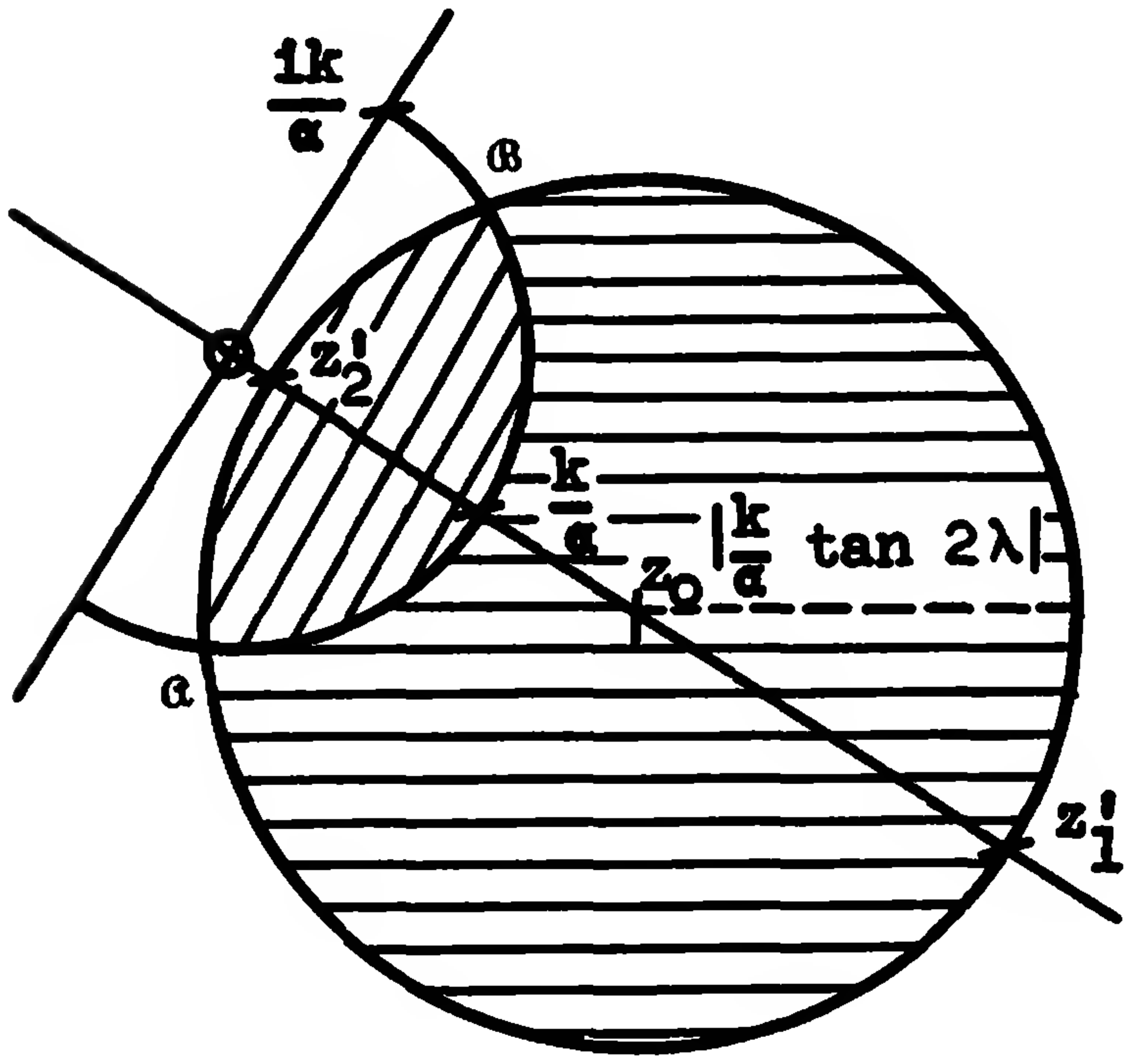
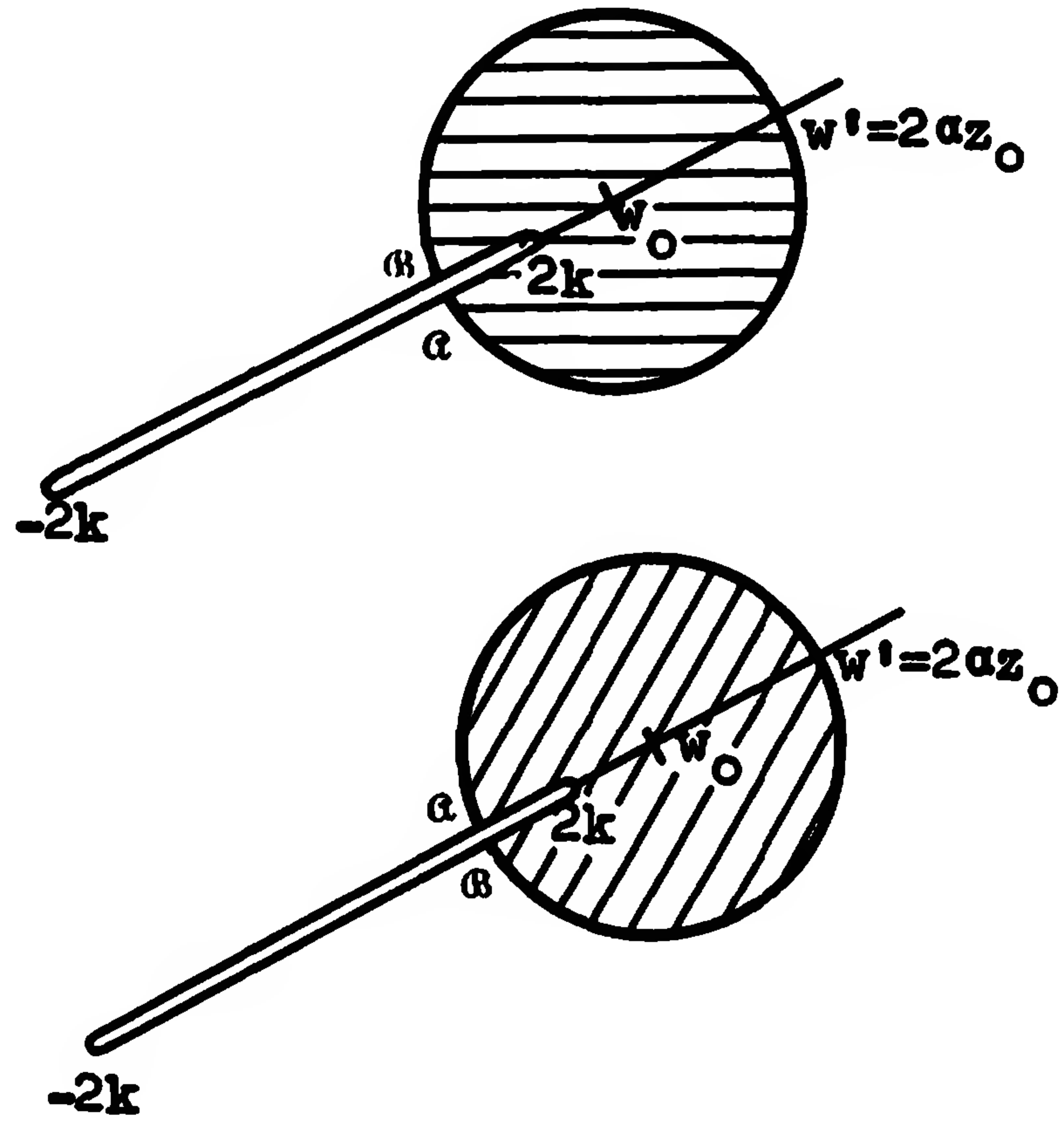


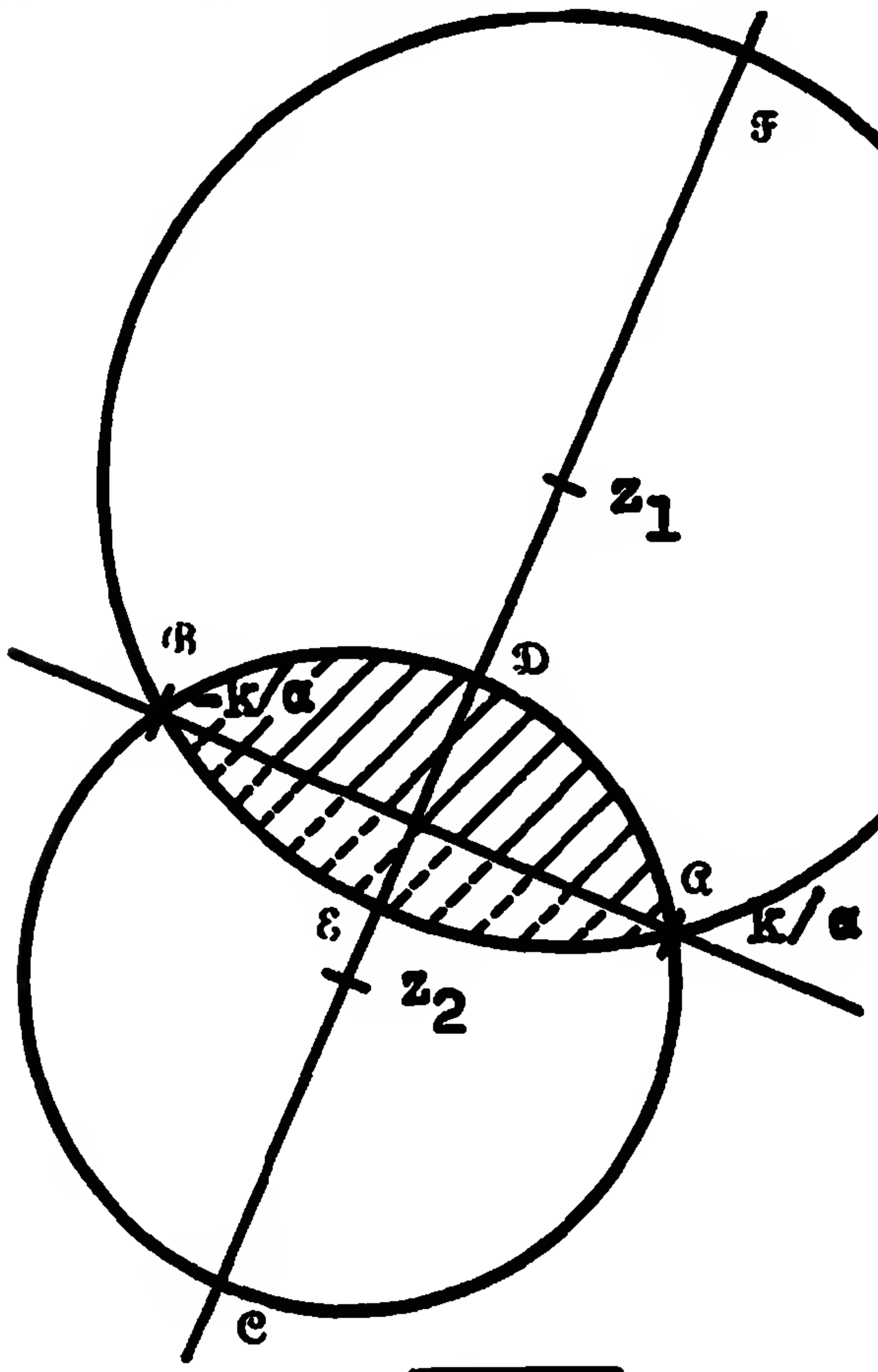


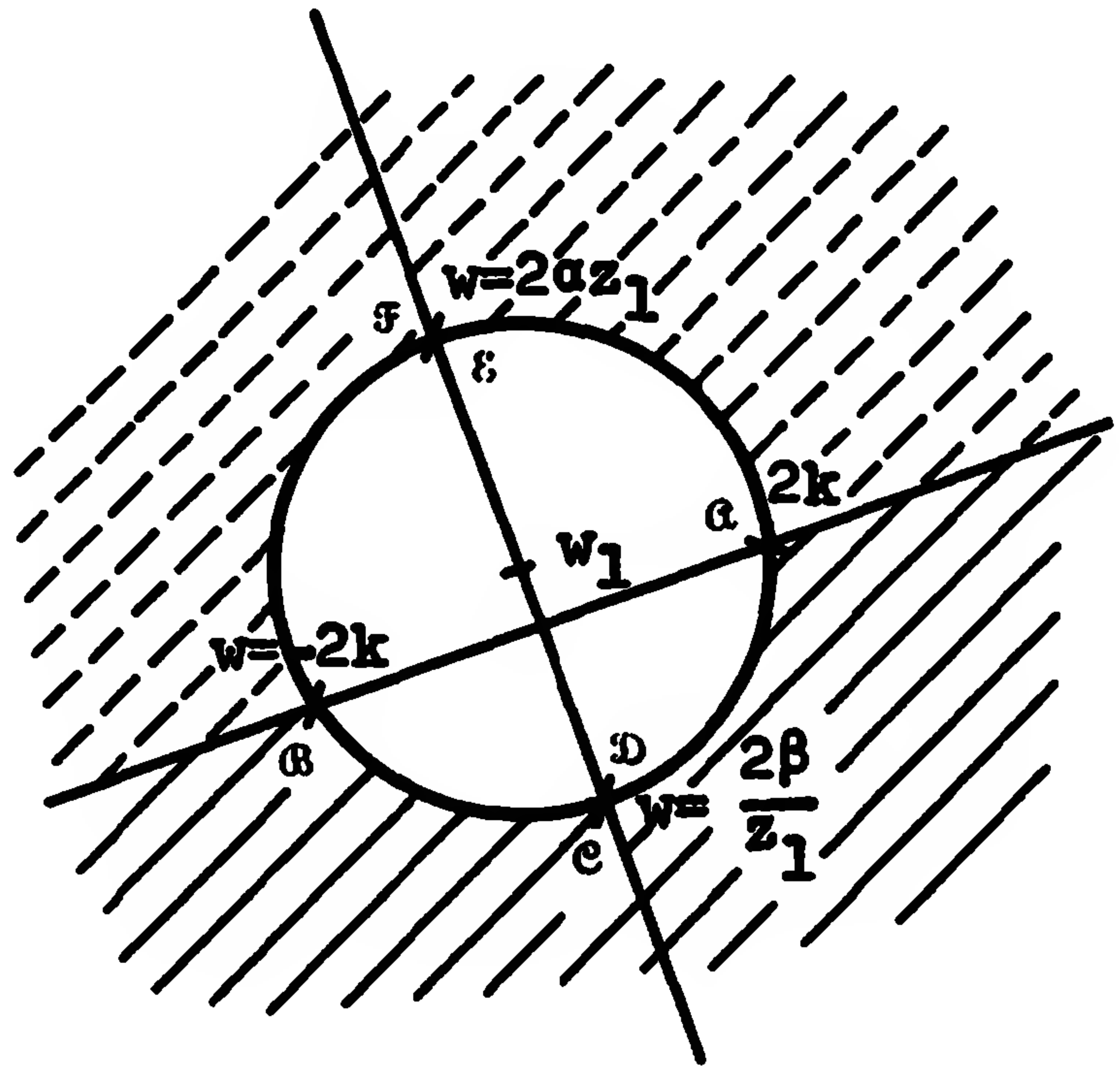
z - plane	w - plane
circle $\lvert z \rvert = \lvert k/\alpha \rvert$	segment $(-2k, 2k)$ of line $\Re(i\bar{k}w) = 0$, counted twice
line $\Re(iz\bar{k}/\bar{\alpha}) = 0$	line $\Re(i\bar{k}w) = 0$, excluding this segment, counted twice
line $\Re(z\bar{k}/\bar{\alpha}) = 0$	line $\Re(\bar{k}w) = 0$, counted twice
both the region $\lvert z \rvert < \lvert \frac{k}{\alpha} \rvert$ and $\lvert z \rvert > \lvert \frac{k}{\alpha} \rvert$	whole plane, cut from $-2k$ to $2k$
half-plane $\Re(i\bar{k}z/\bar{\alpha}) > 0$ } half-plane $\Re(i\bar{k}z/\bar{\alpha}) < 0$	whole plane, cut from $2k$ to ∞ and from $-2k$ to $-\infty$

8.2 <u>Circles on circles; centre on centre</u>.

z - plane	w - plane
$z_0 = \frac{k}{\alpha} \sec 2\lambda$; $0 < \lambda < \frac{\pi}{2}$, $\lambda \neq \frac{\pi}{4}$	$w_0 = 2k \sec 2\mu$; $\tan \mu = \tan^2 \lambda$, $0 < \mu < \frac{\pi}{2}$
circle $\lvert z - z_0 \rvert = \lvert \frac{k}{\alpha} \tan 2\lambda \rvert$	circle $\lvert w - w_0 \rvert = 2 \lvert k \tan 2\mu \rvert$, counted twice
set of coaxal circles, with limiting points $\pm k/\alpha$.	set of coaxal circles, with limiting points $\pm 2k$, counted twice.

z - plane	w - plane
	
z_1'; z_2'	$w' = 2\alpha z_0$
Each of the two regions bounded by $\lvert z-z_0\rvert = \lvert\sqrt{z_0^2-\beta/\alpha}\rvert$ and $\lvert z\rvert = \lvert k/\alpha\rvert$	$\lvert w-w_0\rvert < \lvert\sqrt{w_0^2-4\alpha\beta}\rvert$, slit along the segment from -2k to 2k
$\left.\begin{array}{l} z_1 = i\frac{k}{\alpha}\tan\nu \\ z_2 = -\frac{ik}{\alpha}\cot\nu = \frac{\beta}{\alpha z_1} \end{array}\right\} 0<\nu<\frac{\pi}{2}$	$w_1 = -2ik\cot 2\nu$
$\left.\begin{array}{l} \text{circle } \lvert z-z_1\rvert = \lvert\frac{k}{\alpha}\sec\nu\rvert \\ \text{circle } \lvert z-z_2\rvert = \lvert\frac{k}{\alpha}\operatorname{cosec}\nu\rvert \end{array}\right\}$	arc of $\lvert w-w_1\rvert = 2\lvert k\operatorname{cosec}2\nu\rvert$, counted twice (cf. figure)
set of coaxal circles passing through $\pm k/\alpha$	set of coaxal circles passing through $\pm 2k$, counted twice.

<table>
<tr><th>z - plane</th><th>w - plane</th></tr>
<tr><td>


</td><td>


</td></tr>
<tr><td>points $z_1 \pm\sqrt{z_1^2-\beta/\alpha}$ (i.e. $\mathfrak{E}$, $\mathfrak{F}$)</td><td>point $2\alpha z_1$</td></tr>
<tr><td>points $z_2 \pm\sqrt{z_2^2-\beta/\alpha}$ (i.e. $\mathfrak{C}$, $\mathfrak{D}$)</td><td>point $2\alpha z_2 = 2\beta/z_1$</td></tr>
<tr><td>circular crescent $\mathfrak{A}\,\mathfrak{D}\,\mathfrak{B}\,\mathfrak{E}\,\mathfrak{A}$
region exterior to both circles</td><td>exterior of circle $\mathfrak{A}\,\mathfrak{D}\,\mathfrak{B}\,\mathfrak{E}\,\mathfrak{A}$</td></tr>
</table>

8.3 **Ellipses and hyperbolae.**

t, t', τ fixed, $0 < t < t' < 1$; $\ell = t + 1/t$, $\ell' = t' + 1/t'$;

$\sigma = \arg \frac{k}{\alpha}$; $0 < \tau < \pi/2$.

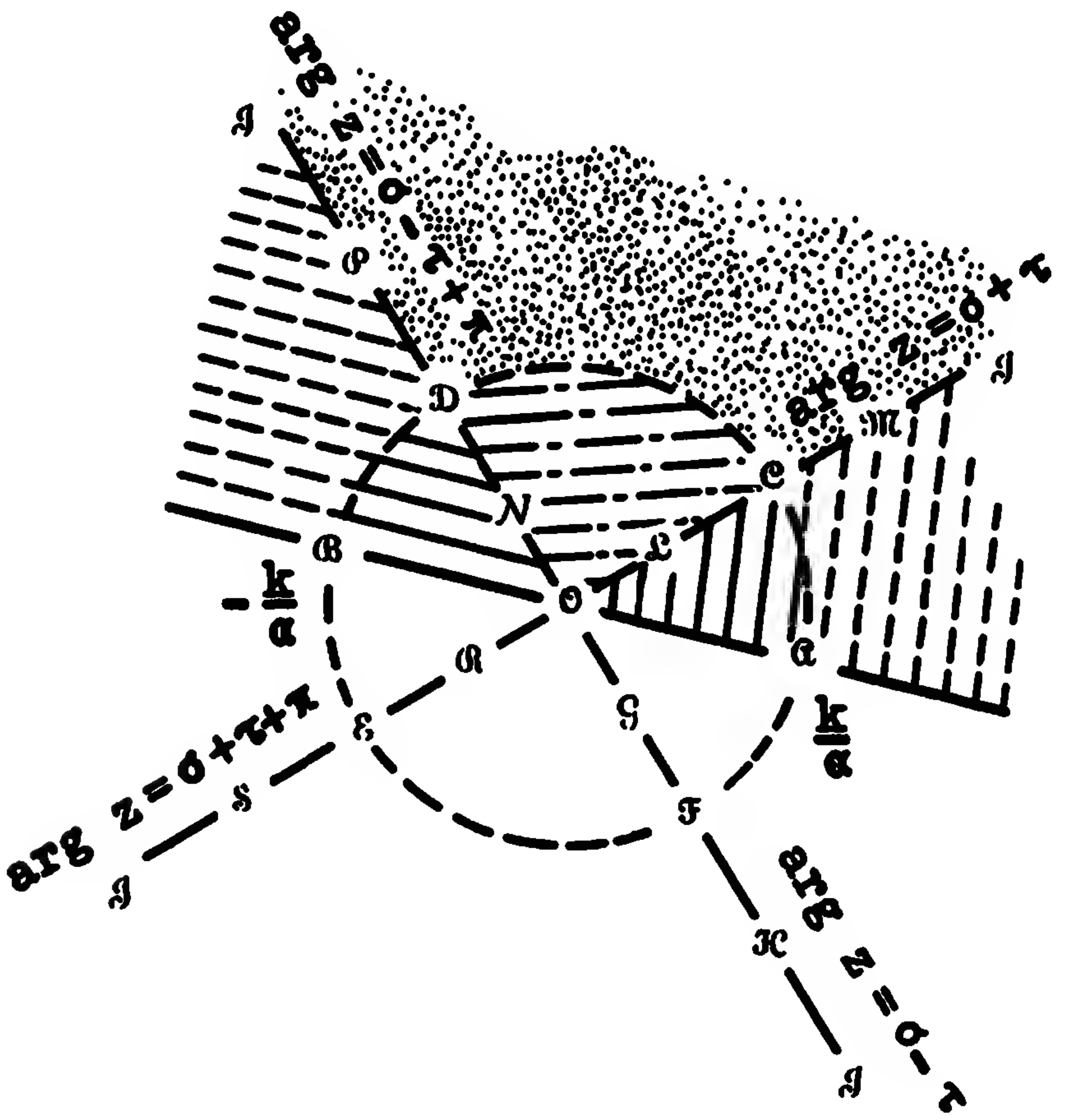
Ellipses

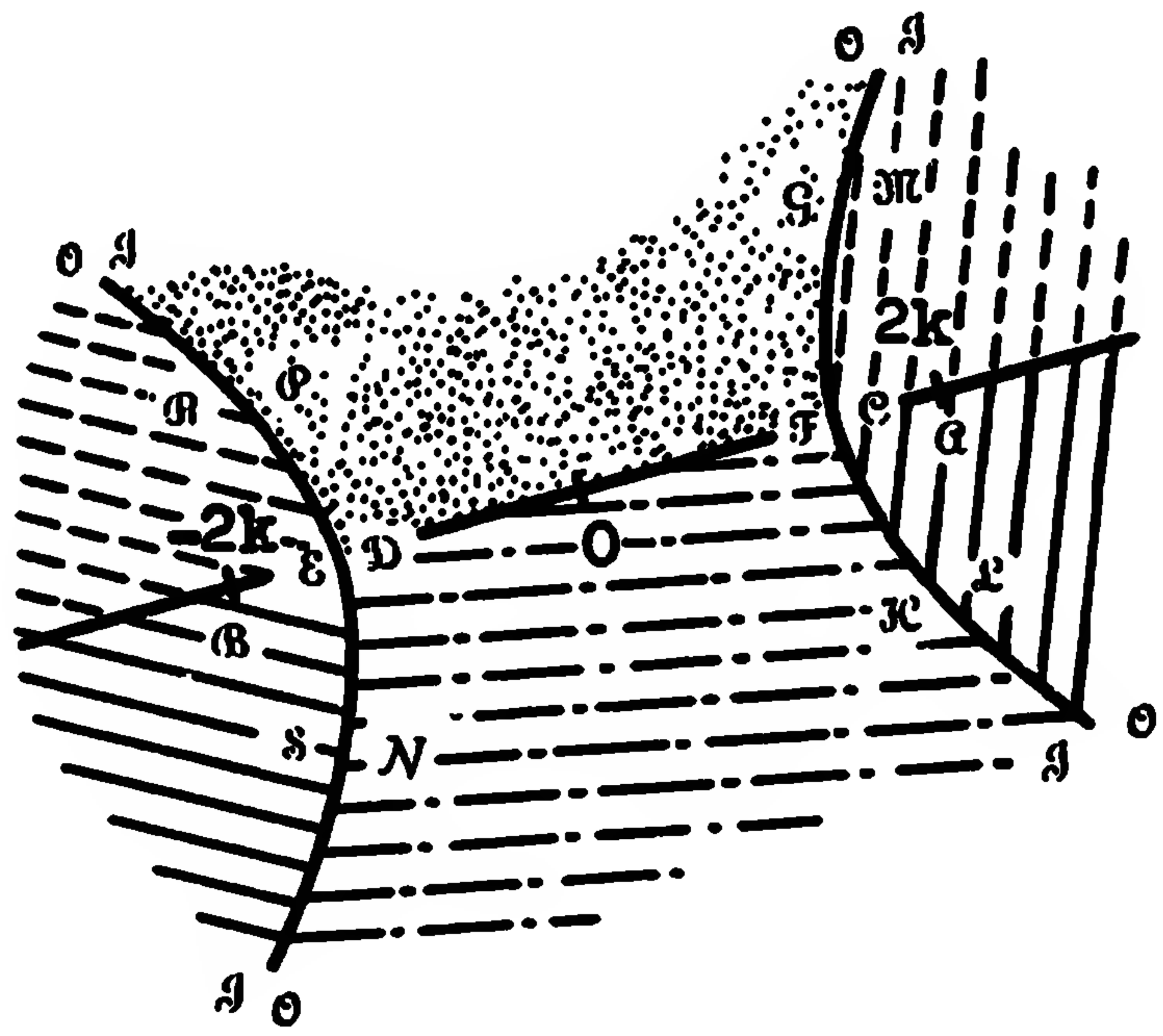
<table>
<tr><th>z - plane</th><th>w - plane</th></tr>
<tr><td>region $|z| < |k/\alpha|t$
region $|z| > |k/\alpha|t^{-1}$</td><td>$|w+2k|+|w-2k| > 2\ell|k|$,
i.e. exterior of an ellipse</td></tr>
<tr><td>annular region $t|k/\alpha|<|z|<|k/\alpha|$
annular region $t^{-1}|k/\alpha|>|z|>|k/\alpha|$</td><td>$2\ell|k|>|w+2k|+|w-2k|>4|k|$,
i.e. interior of the same ellipse, excluding the segment joining the foci</td></tr>
</table>

z - plane	w - plane
annular region $t\|k/\alpha\|<\|z\|<t'\|k/\alpha\|$ annular region $t^{-1}\|k/\alpha\|>\|z\|>(1/t')\|k/\alpha\|$	$2\ell\|k\|>\|w+2k\|+\|w-2k\|>2\ell'\|k\|$, i.e. annular region bounded by two confocal ellipses.

<u>Hyperbolae</u>

z - plane	w - plane





z - plane	w - plane
half-line $\arg z = \sigma + \tau$ half-line $\arg z = \sigma - \tau$	$\|w+2k\|-\|w-2k\| = 4\|k\|\cos\tau$; i.e. one branch of a hyperbola, with asymptotes $\arg w = \arg k \pm \tau$.
half-line $\arg z = \sigma + \tau + \pi$ half-line $\arg z = \sigma - \tau + \pi$	$\|w+2k\|-\|w-2k\| = -4\|k\|\cos\tau$, i.e. the other branch of the same hyperbola
half-line $\arg z = \sigma + \frac{1}{2}\pi$ half-line $\arg z = \sigma - \frac{1}{2}\pi$	line $\Re(\bar{k}w) = 0$, see 8.1

z - plane	w - plane
half-line $\arg z = \sigma$	half-line $\|w\| > 2\|k\|$, $\arg \frac{w}{k} = 0$, counted twice
half-line $\arg z = \sigma + \pi$	half-line $\|w\| > 2\|k\|$, $\arg \frac{w}{k} = \pi$, counted twice

8.4 <u>Examples</u>

(i) $w = \frac{1}{2}z + \frac{\beta}{z}$, $\beta \neq 0$.

*Fixed points: $F_1 = 2k = \frac{k}{\alpha} = \sqrt{2\beta} = -F_2$.

z - plane	w - plane
exterior of $\|z\| = \|\sqrt{2\beta}\|$ interior of $\|z\| = \|\sqrt{2\beta}\|$	whole plane, cut from $-\sqrt{2\beta}$ to $\sqrt{2\beta}$
set D, as a whole (see §1)	set D, as a whole, counted twice
set E, as a whole (see §1)	set E, as a whole, counted twice

(ii) $w = z + \frac{i}{z}$

z - plane	w - plane
exterior of $\|w\| = 1$ interior of $\|w\| = 1$	whole plane cut from $-(1+i)\sqrt{2}$ to $(1+i)\sqrt{2}$

(iii) $w = \dfrac{az^2+bz+c}{z+d}$; $a \neq 0$; $ad^2 - bd + c = \beta \neq 0$.

Combination of $w = \xi + b - 2ad$, $z = \zeta - d$, $\xi = a\zeta + \beta/\zeta$.

(iv) $w = \dfrac{az^2+bz+c}{z^2+dz+f}$; $a \neq 0$, $w = w(z)$ not reducing to a linear or bilinear function; equivalent to

$$\frac{w}{a-w} = \frac{az^2+bz+c}{(ad-b)z+(af-c)} \qquad \{\text{cf. (iii)}\}$$

(v) $w = \dfrac{i\{b(1+z)^2+2(1-z)\}}{2(1+z)}$ ‡

Combination of $w = \xi - i$; $z = \dfrac{2i}{\zeta} - 1$; $\xi = \zeta - \dfrac{b}{\zeta}$.

1) b real, $0 < b \leq 1$.

ζ - plane	z - plane	w - plane
line $\Im(\zeta) = 1$ circle $\lvert\zeta - \frac{ib}{2}\rvert = \frac{b}{2}$	circle $\lvert z\rvert = 1$ line $\Re(z) = \frac{2}{b} - 1$	curve $\mathfrak{O\,A\,G\,B\,O}$ (fig. 1), symmetric with respect to $u = 0$; cusp at $\mathfrak{G}$ for $b = 1$, as in the first figure of 8.5

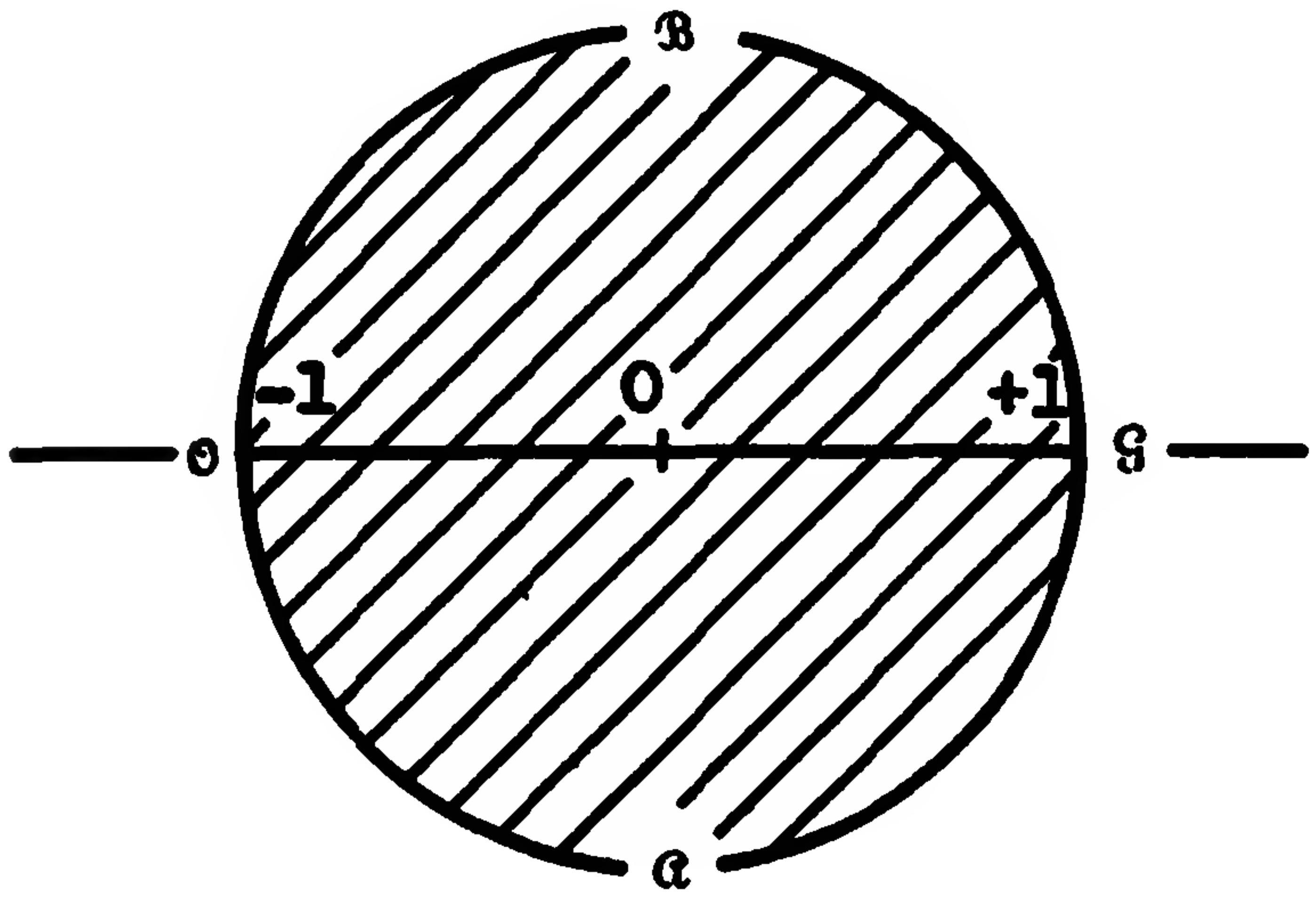
2) $b = -2i\beta$, β real, $0 < \beta < 1$.

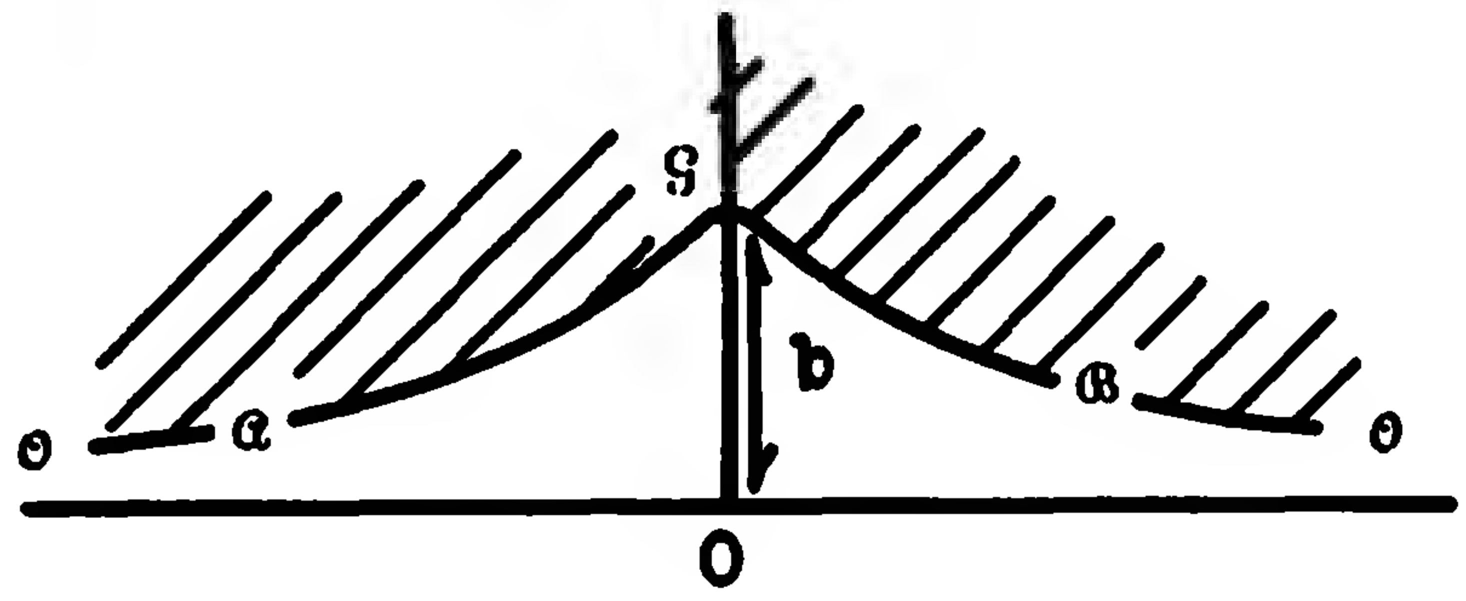
ζ - plane	z - plane	w - plane
line $\Im(\zeta) = 1$ circle $\lvert\zeta - \beta\rvert = \beta$	circle $\lvert z\rvert = 1$ line $\Im(z) = \frac{1}{\beta}$	curve $\mathfrak{O\,A\,G\,B\,O}$ (fig. 2), case $\beta = 1$; cusp at $\mathfrak{G}$, asymptote $v = 0$.

‡ W. R. Dean. The diagrams are copied from his paper, with the author's permission.

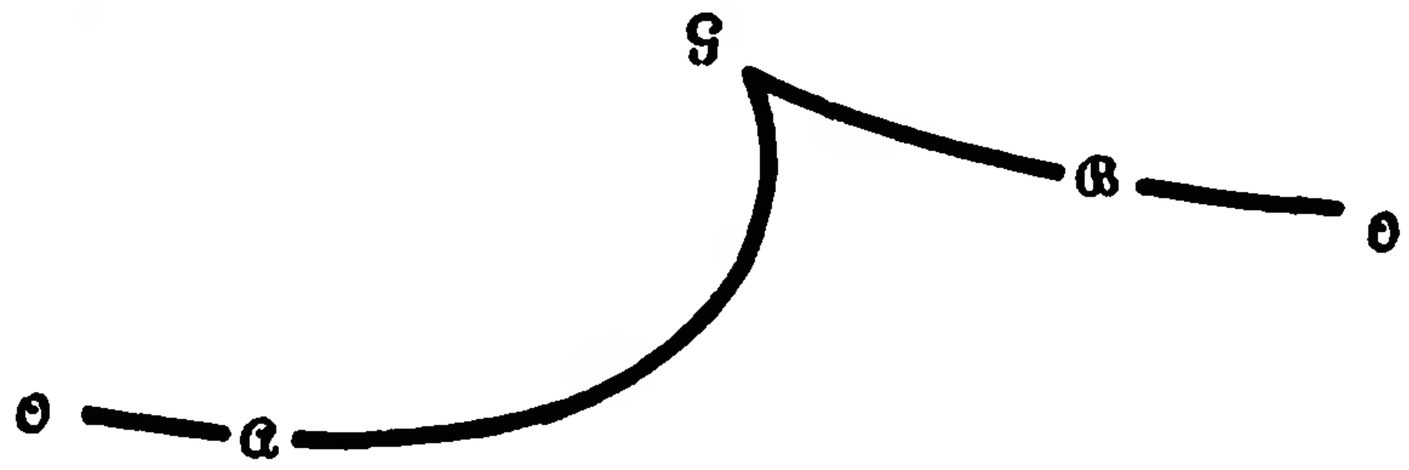
z - plane	w - plane

1.) Case $b = \frac{3}{4}$





2.) Case $b = -2i$ $(\beta = 1)$



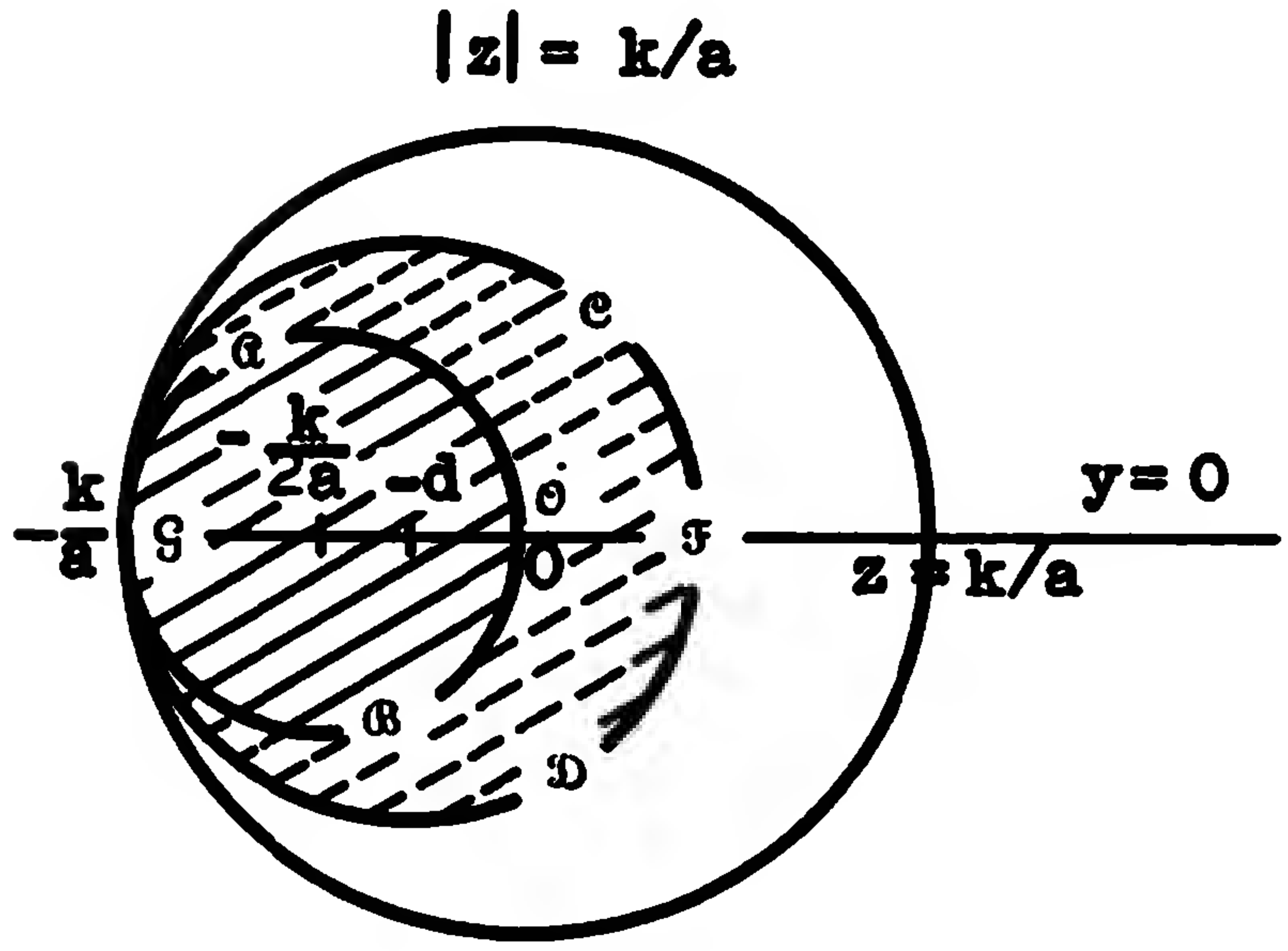
8.5 Joukowski's Aerofoil; its exterior on the interior or exterior of a circle.

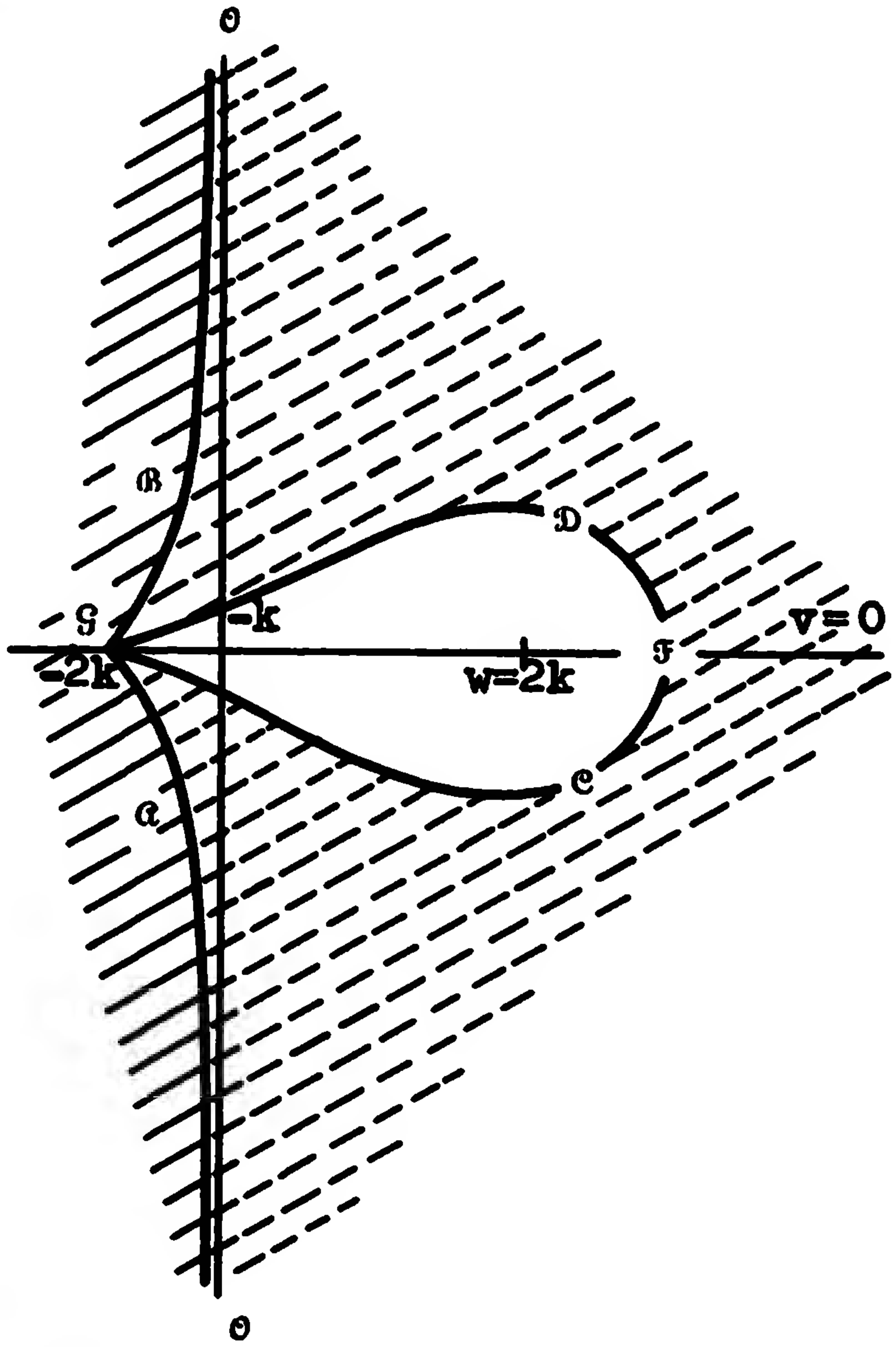
$$w = az + \frac{b}{z}; \quad a > 0, \quad b > 0; \quad k = \sqrt{ab} > 0.$$

(i) Symmetric aerofoil: d real.

z - plane	w - plane
circle $\lvert z \rvert = k/a$	segment $-2k \leq u \leq 2k$ of $v = 0$, counted twice
circle $\lvert z + \frac{k}{2a} \rvert = \frac{k}{2a}$ through θ line $x = -\frac{k}{a}$	curve $\theta\,\alpha\,\mathfrak{g}\,\mathfrak{B}\,0$, cusp at $\mathfrak{g}$, with asymptote $u = -k$

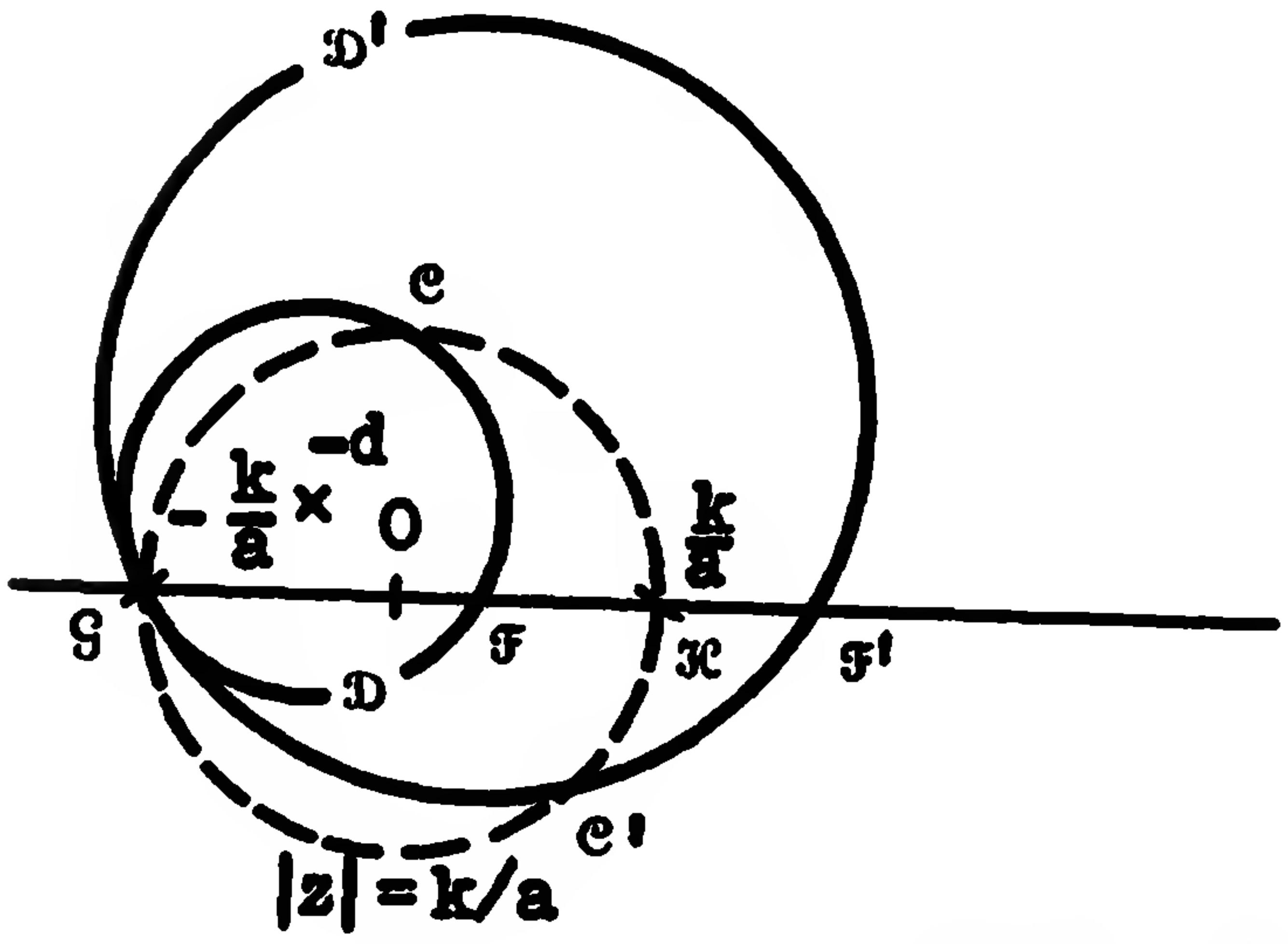
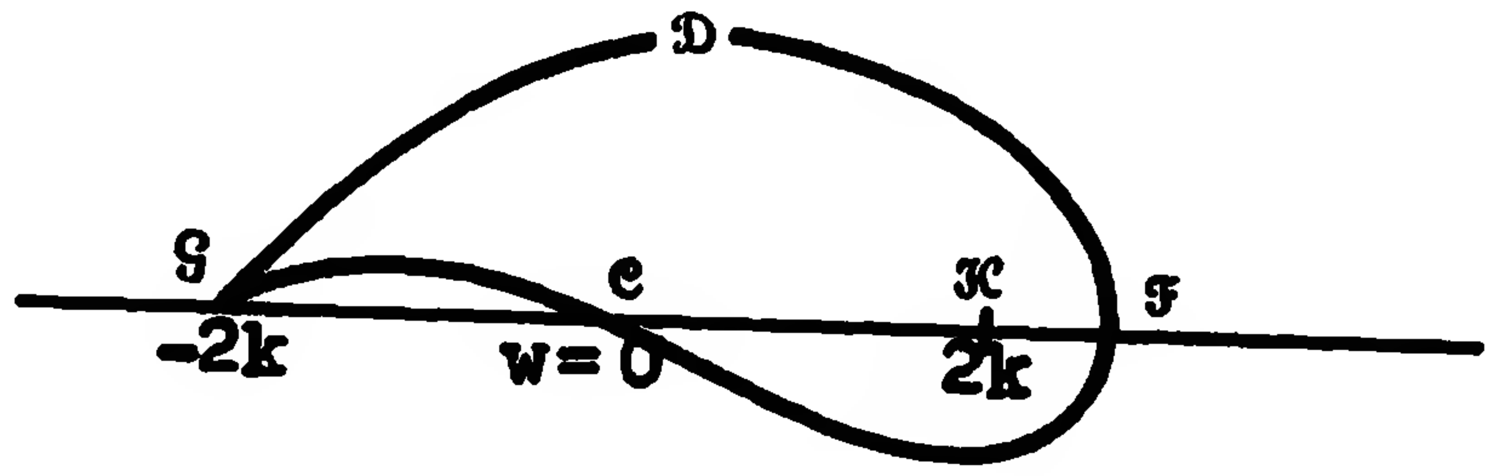
z - plane	w - plane
circle $\lvert z + d\rvert = \frac{k}{a} - d,\quad 0 < d < \frac{k}{2a}$ circle $\left\lvert z - \frac{kd}{k-2ad}\right\rvert = \frac{b - kd}{k-2ad}$	symmetric aerofoil $\mathfrak{G}\,\mathfrak{D}\,\mathfrak{F}\,\mathfrak{C}$; angle at $\mathfrak{G}$ is zero
interior of $\lvert z + d\rvert = \frac{k}{a} - d$ exterior of $\left\lvert z - \frac{kd}{k-2ad}\right\rvert = \frac{b-kd}{k-2ad}$	exterior of aerofoil





(ii) <u>Unsymmetric aerofoil; d not real.</u>

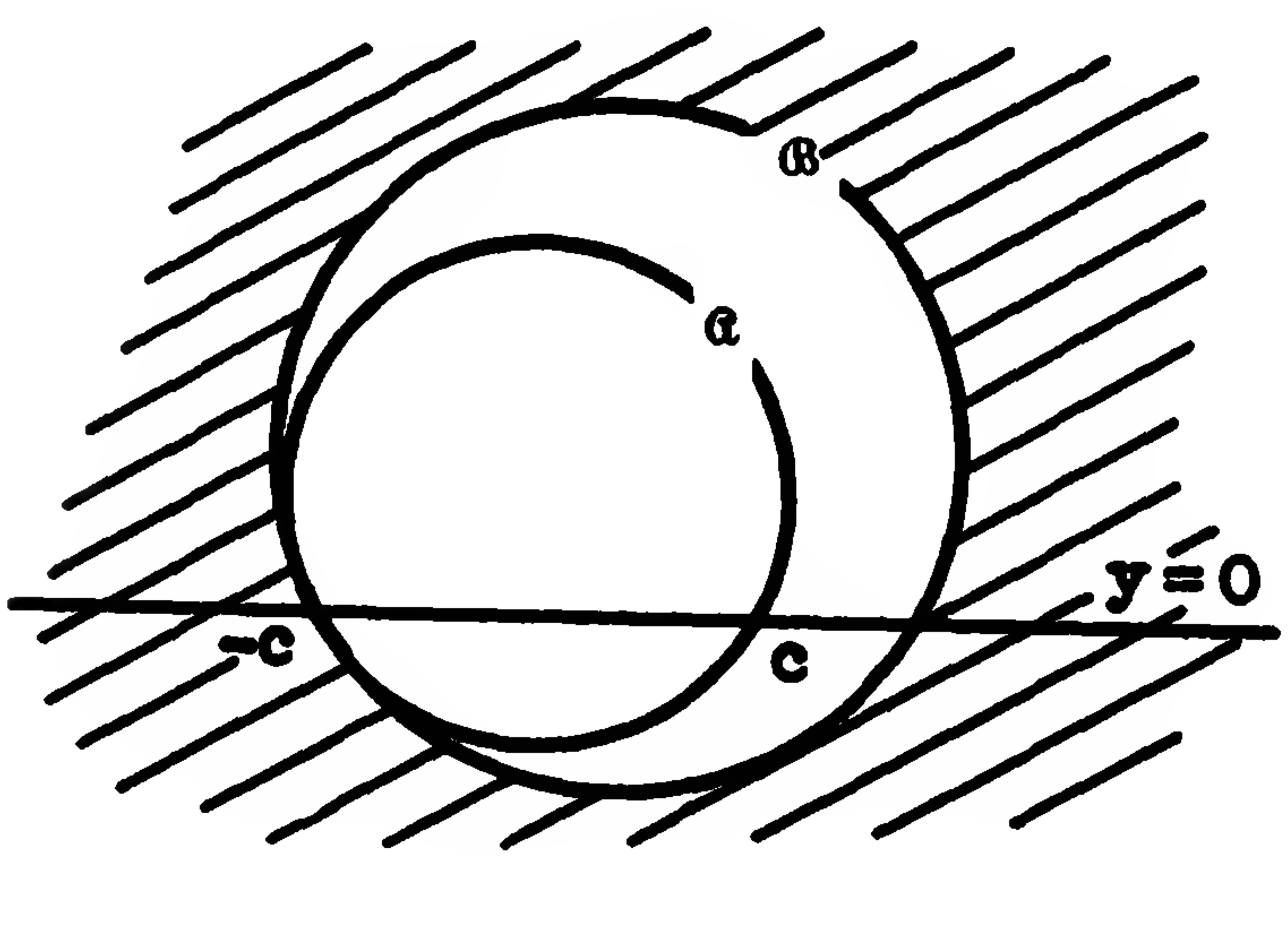
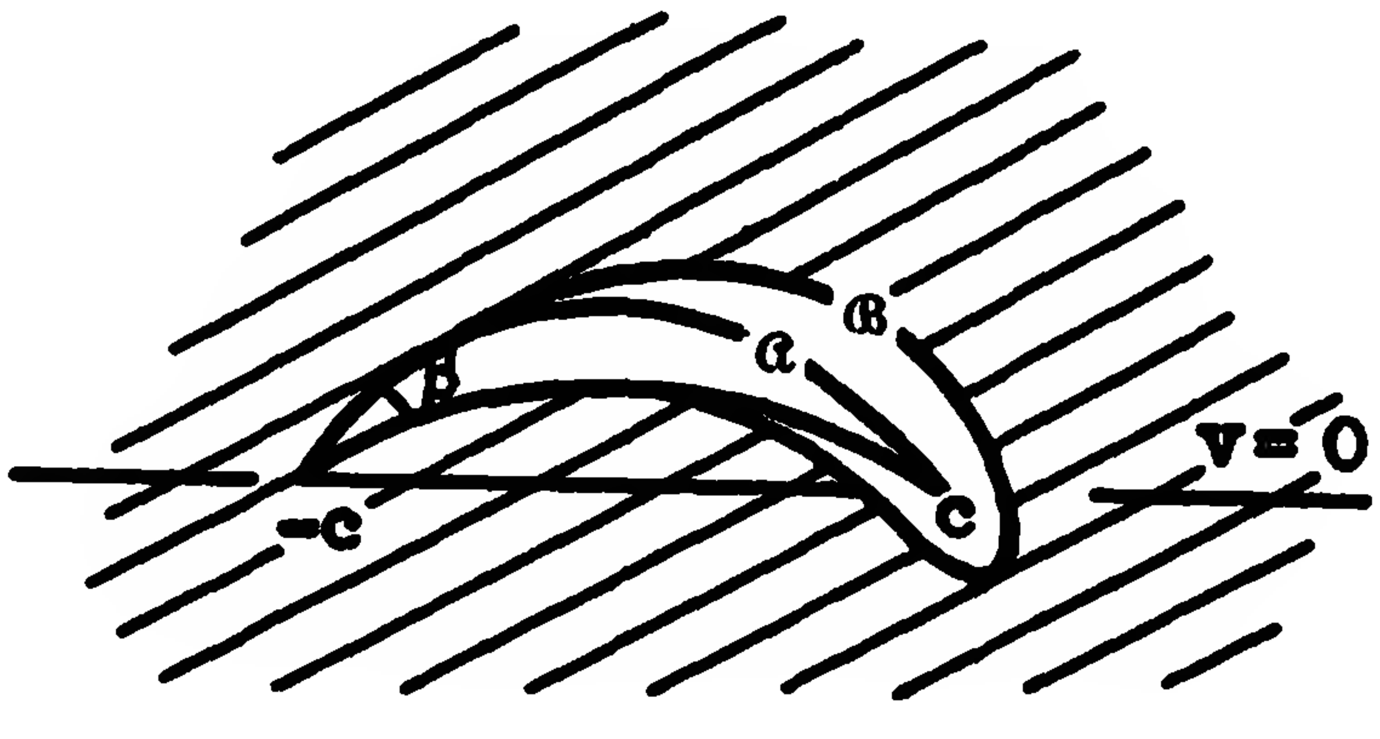
z - plane	w - plane
circle $\mathfrak{G}\,\mathfrak{D}\,\mathfrak{F}\,\mathfrak{C}$: $\lvert z+d\rvert = \left\lvert \frac{k}{a} - d\right\rvert$ circle $\mathfrak{G}\,\mathfrak{D}'\mathfrak{F}'\mathfrak{C}'$: $\left\lvert z - \frac{k^2\bar{d}}{s}\right\rvert = \lvert k-ad\rvert\,\frac{b}{s}$ where $\lvert d\rvert < \left\lvert\frac{k}{a} - d\right\rvert < \left\lvert\frac{k}{a} + d\right\rvert$, d not real, $s = \lvert k-ad\rvert^2 - \lvert ad\rvert^2$	unsymmetric aerofoil $\mathfrak{G}\,\mathfrak{D}\,\mathfrak{F}\,\mathfrak{C}$, with $\angle\mathfrak{G} = 0^{\circ}$

z - plane	w - plane
line $\Re\left\{\left(\frac{k}{a} - d\right)z\right\} = \frac{k}{a}\Re(d) - \frac{k^2}{a^2}$, touching both circles at $z = -\frac{k}{a}$, i.e., at $\mathfrak{G}$	curve with cusp at $\mathfrak{G}$, with asymptote, $\Re\left\{w\left(\frac{k}{a} - d\right)\right\} = k\Re(d) - \frac{k^2}{a}$
$\left.\begin{array}{l}\text{interior of } \mathfrak{G}\,\mathfrak{D}\,\mathfrak{F}\,\mathfrak{C} \\ \text{exterior of } \mathfrak{G}\,\mathfrak{D}'\,\mathfrak{F}'\,\mathfrak{C}'\end{array}\right\}$	exterior of aerofoil
	

Generalised Joukowski Aerofoils

$$\boxed{\frac{w+c}{w-c} = \left(\frac{z+c}{z-c}\right)^n}\;; \qquad c > 0, \quad 0 < n < 2$$

Combination of $\frac{z+c}{z-c} = \zeta$ and $\zeta = \left(\frac{w+c}{w-c}\right)^{\pi/\alpha}$, c.f. §7.1; $\angle\alpha = n\pi$.

z - plane	w - plane; $\angle\beta = \pi(2-n)$
exterior of circle $\mathfrak{A}$	exterior of circular crescent
	

z - plane	w - plane
exterior of the circle $\mathcal{B}$ touching $\mathcal{A}$ at $z = -c$, point $+c$ lying inside $\mathcal{B}$	exterior of aerofoil $\mathcal{B}$

Other transformations mapping the exterior of a circle on the exterior of an aerofoil:

$$\boxed{\frac{w+nc}{w-nc} = \left(\frac{z+c}{z-c}\right)^n}\ ; \quad c > 0, \quad 0 < n < 2.$$

For further transformations, see publications in the list following.

cf. $w = e^{i\beta} \log \frac{m-z}{m+z} + e^{-i\beta} \log \frac{1-mz}{1+mz}$, part III, §11.12, pp. 131-135.

<u>Selection of Original Memoirs on the Theory of Aerofoils</u>.

Some abbreviations:

C.R.:	Comptes Rendus
P.R.S:	Proceedings of the Royal Society
Q.J.M.:	Quarterly Journal of Mathematics, Oxford
R. & M.:	Reports and Memoranda of the Aeronautical Research Committee
Z.A.M.M.:	Zeitschrift für angewandte Mathematik und Mechanik
Z.F.M.	Zeitschrift für Mathematik und Physik

<u>Bickley, W. G.</u>: The effect of Rotation upon the Lift and Moment of a Joukowski Aerofoil. P.R.S. A 127 (1930), p. 186.

<u>Carafoli, E.</u>: (1) Méthode générale pour le tracé des profils d'aviation. C.R. 185 (1927), p. 842.

(2) Sur les profils aérodynamiques de forme générale. C.R. 185 (1928), p. 1014.

(3) Tracé général des profils avec dièdre à la pointe. C.R. 185 (1928), p. 1189.

(4) Sur le mouvement général autour d'un contour. C.R. 186 (1928), p. 1196.

Daymond, S. D. and Hodgkinson, J.

A type of aerofoil. Q.J.M. 10 (1939), p. 136.

Girault, M. (1) Sur une construction très générale des profils d'ailes par transformation conforme d'un circle. C.R. 183 (1926), p. 1331.

(2) Construction géométrique de profils d'ailes par représentation conforme d'un circle. C.R. 185 (1928), p. 1584.

Glauert, H. (1) A method of calculating the characteristics of a tapered wing. R. & M. 824 (1922).

(2) A theory of thin aerofoils. R. & M. 910 (1924)

(3) The force and moment on an oscillating aerofoil. R. & M. 1242 (1929).

(4) Aerofoil and airscrew theory. (Cambridge, 1929).

Höhndorf, F. Verfahren zur Berechnung des Auftriebes gegebener Tragflächen-Profile. Z.A.M.M. 6 (1926), p. 265.

Jeffreys, H. On aerofoils of small thickness. P.R.S. A 121 (1928).

Joukowski, N. Über die Konturen der Drachenflieger. Z.F.M. (1910).

Kármán, T. v., and Trefftz, E.

Potentialströmungen um gegebene Tragflächen-Querschnitte. Z.F.M. (1918).

Kawada, S. A contribution to the theory of the latticed wing. Proc. 3rd. Int. Congr. Appl. Mechs. (1930), p. 393.

König, E. Potentialströmung durch Gitter. Z.A.M.M. 2 (1922), p. 422.

Kutta, W. M. Über ebene Zirkulationsströmungen. Ber. Bayer. Akad. d. Wiss. (1911).

Lagally, M. Die reibungslose Strömung im Aussengebiet zweier Kreise. Z.A.M.M. 9 (1929), p. 299.

Müller, W. (1) Zur Konstruction von Tragflächen-Profilen. Z.A.M.M. 4 (1924), p. 213.

(2) Stromlinien und Kraftlinien in der konformen Abbildung. Z.A.M.M. 6 (1926), p. 284.

Müller, W. (3) Zylinder in einer unstetigen Potentialströmung. Z.A.M.M. 7 (1927), p. 13.

(4) Abbildungstheoretische Grundlagen für das Problem des Tragflügels in Erdbodennähe. Z.A.M.M. 11 (1931), p. 231.

Perring, W. G. A. The theoretical pressure distribution around Joukowski aerofoils. R. & M. 1106 (1927).

Rosenhead, L. (1) The lift on a flat plate between parallel walls. P.R.S. A 132 (1931), p. 127.

(2) Aerofoil in a wind tunnel. P.R.S. A 140 (1933), p. 583.

Rosenhead, L. and Daymond, S. D.

The lateral force on a keel and rudder. Proc. Camb. Phil. Soc. 33 (1937), p. 62.

Rosenhead, L. and Davison, B.

Wind tunnel correction for a circular open jet tunnel with a reflection plate. P.R.S. A 177 (1941), p. 366.

Schmieden, C. Die Berechnung kavitationssicherer Tragflügel-profile. Z.A.M.M. 12 (1932), p. 228.

Trefftz, E. Graphische Konstruction Joukowskischer Tragflächen. Z.F.M. (1913).

Prandtlsche Tragflächen- und Propeller Theorie. Z.A.M.M. 1 (1921), p. 206.

Wenig, F. Der Hodograph der Gitterströmung als Weg zur Ermittlung günstiger Turbinen- und Propellerprofile. Proc. 3rd. Int. Congr. Appl. Mechs. (1930), p. 437.

Witoszynski, C. Modification du principle de circulation. Proc. 1st. Int. Congr. Appl. Mechs. (1924), p. 418.

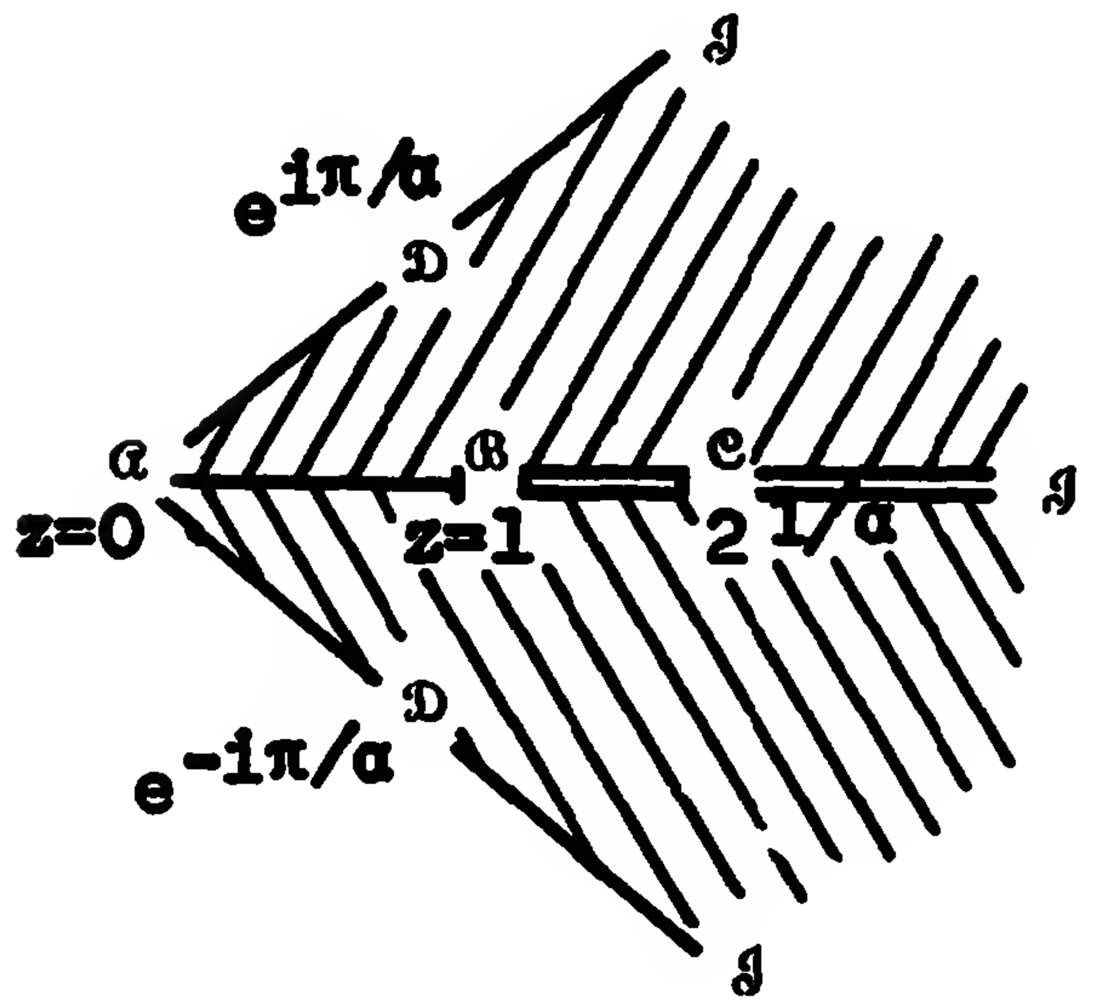
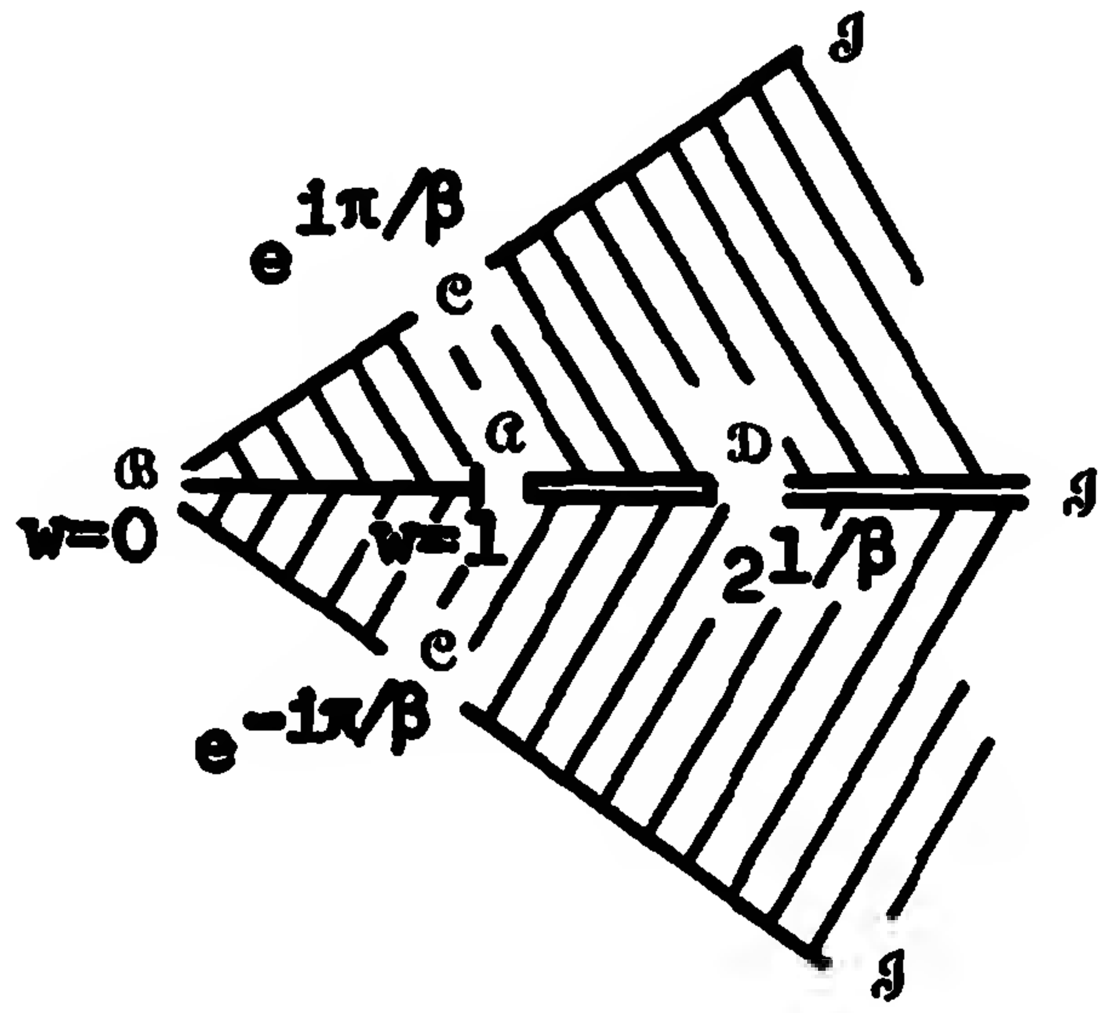
Yokota, S. Discontinuous flow past an aerofoil. Phil. Mag. VII 3 (1927), p. 216.

9. FURTHER TRANSFORMATIONS.

9.1 $w^{\beta} = 1 - z^{\alpha}$; $\alpha > 0$, $\beta > 0$, $\alpha \neq 1$, $\beta \neq 1$; $|\arg z| \leq \frac{\pi}{\alpha}$, $|\arg w| \leq \frac{\pi}{\beta}$.

Critical points: $0, 1, \infty$.

φ, ψ real, $\sigma = \frac{\alpha\varphi}{\beta} - \frac{\pi}{2\beta}$; $|w| = R$.

<table>
<tr><th>z - plane</th><th>w - plane; $w = Re^{i\theta}$</th></tr>
<tr><td>points 0; 1; ∞; $2^{1/\alpha}$; $e^{\pm i\pi\alpha}$;
$e^{i\varphi}$ $(0 \leq \varphi \leq \frac{\pi}{\alpha})$;
$e^{i\varphi}$ $(-\frac{\pi}{\alpha} \leq \varphi \leq 0)$;
$(2 \sin \frac{1}{2}\beta\psi)^{1/\alpha} e^{i(\beta\psi-\pi)/2\alpha}$;
$(2 \sin \frac{\beta|\psi|}{2})^{1/\alpha} e^{i(\beta\psi+\pi)/2\alpha}$</td><td>points 1; 0; ∞; $e^{\pm i\pi/\beta}$; $2^{1/\beta}$;
$(2 \sin \frac{\alpha\varphi}{2})^{1/\beta} e^{i(\alpha\varphi-\pi)/2\beta}$;
$(2 \sin \frac{\alpha|\varphi|}{2})^{1/\beta} e^{i(\alpha\varphi+\pi)/2\beta}$;
$e^{i\psi}$ $(0 \leq \psi \leq \frac{\pi}{\beta})$;
$e^{i\psi}$ $(-\frac{\pi}{\beta} \leq \psi \leq 0)$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>segment $0 < x < 1$, $y = 0$</td><td>segment $1 > u > 0$, $v = 0$</td></tr>
<tr><td>half-line $1 < x < \infty$, $y = 0$</td><td>half-line $\arg w = \pi/\beta$
half-line $\arg w = -\pi/\beta$</td></tr>
<tr><td>half-line $\arg z = \pi/\alpha$
half-line $\arg z = -\pi/\alpha$</td><td>half-line $1 < u < \infty$, $v = 0$</td></tr>
</table>

z - plane	w - plane
half-line $\arg z = \varphi$; φ fixed, $0 < \varphi < \frac{\pi}{\alpha}$	part $0 > \theta > \sigma - \pi/(2\beta)$ of the curve $R^\beta \cos\beta(\theta-\sigma) = \sin\alpha\varphi$, with asymptotes $\theta - \sigma = \pm\frac{\pi}{2\beta}$, axis of symmetry $\theta = \sigma$. Compare the figure in 9.2
half-line $\arg z = \varphi - \frac{\pi}{\alpha}$	part $0 < \theta < \sigma + \frac{\pi}{2\beta}$ of the same curve
$\lvert z\rvert = 1$, $-\pi/\alpha \le \arg z \le \pi/\alpha$	curve $R^\beta = 2\cos\theta\beta$, $\lvert\theta\rvert \le \pi/(2\beta)$

When $\beta = 2$, the latter curve is one half of the lemniscate

$$|w-1|\,|w+1| = 1,$$

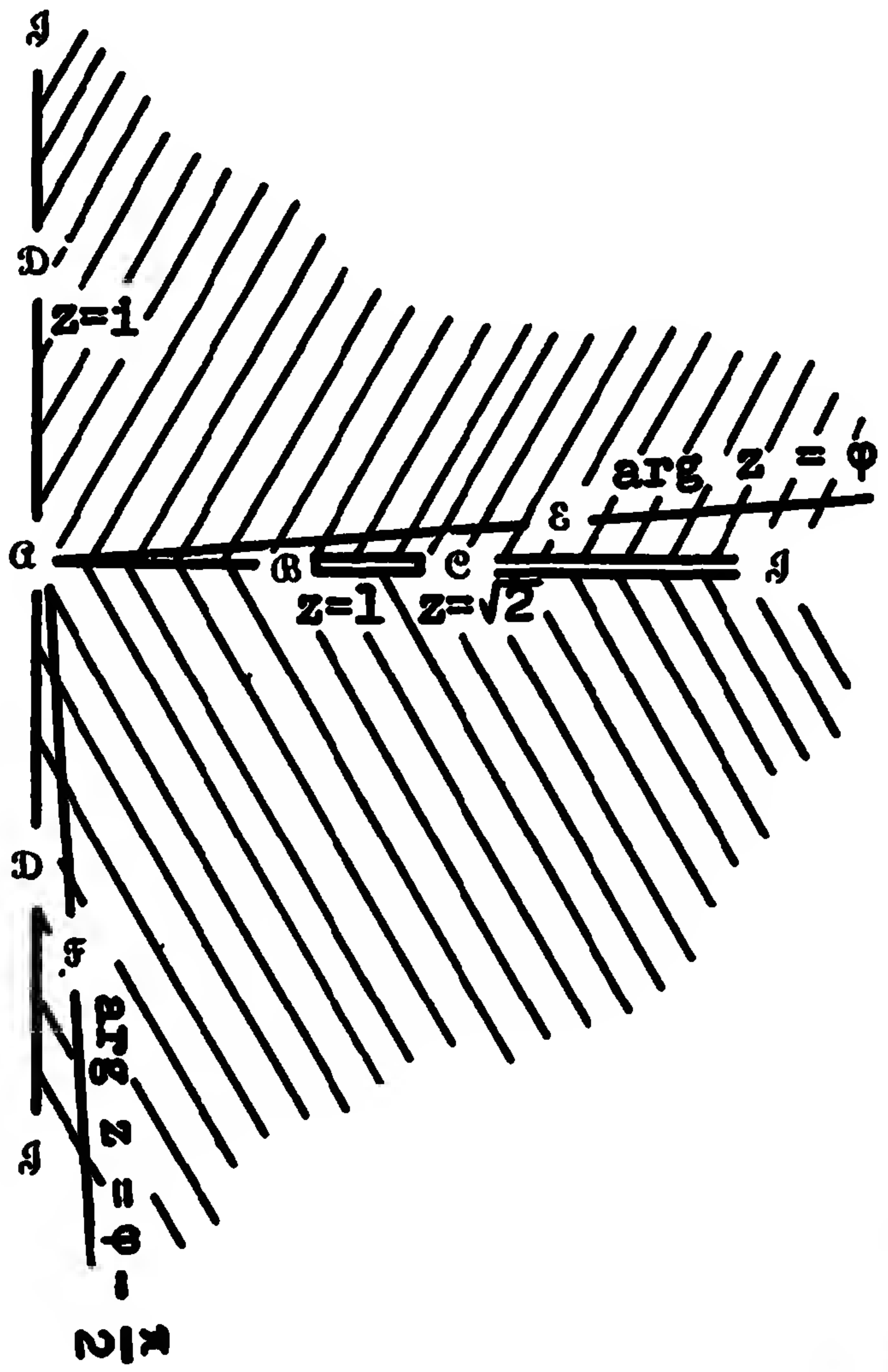
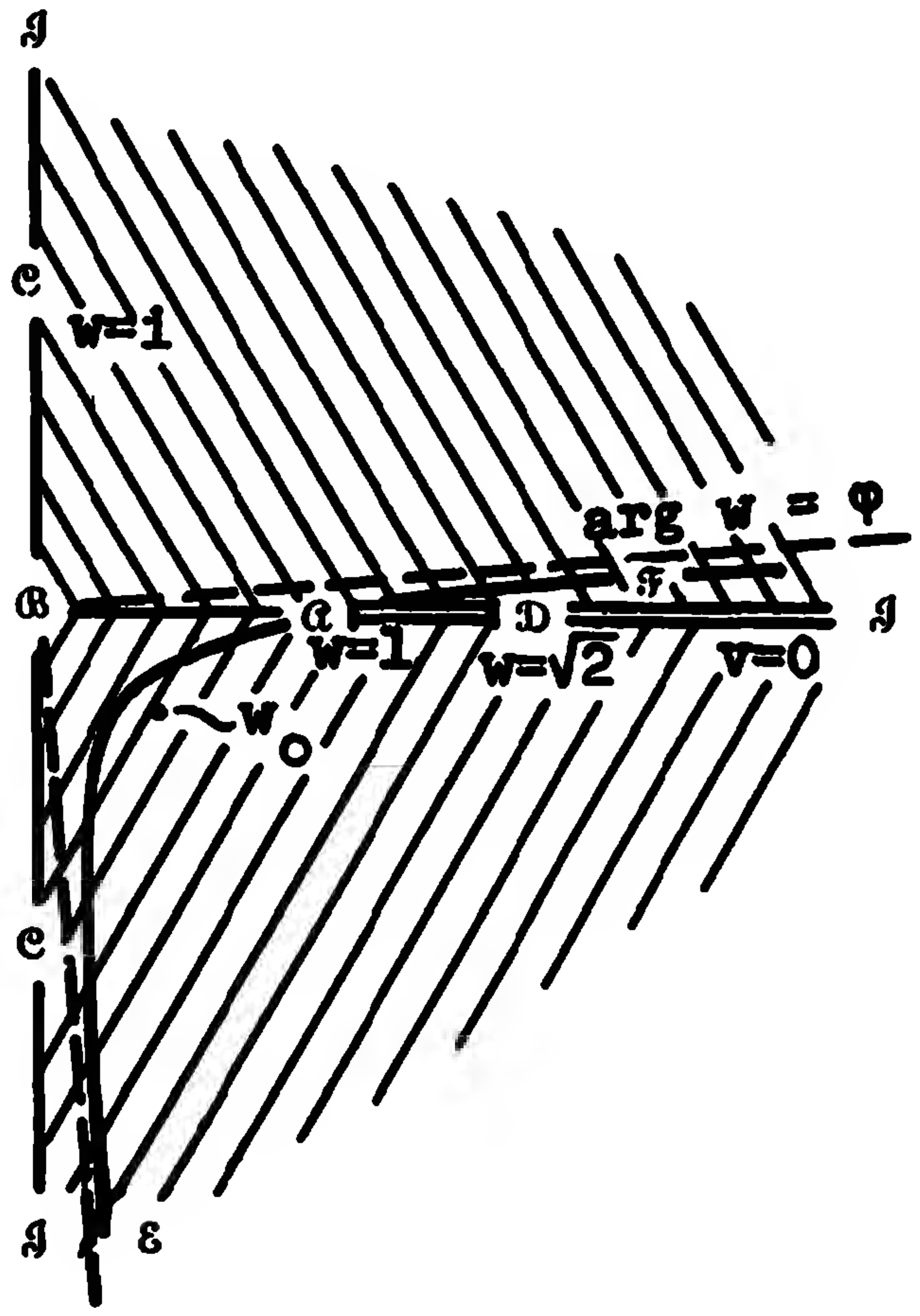
and the curve $R^\beta \cos\beta\theta = \sin\alpha\varphi$ is then one branch of the rectangular hyperbola

$$\Re\left(\frac{w^2}{1-e^{2i\alpha\varphi}}\right) = \frac{1}{2}\,.$$

9.2 Example: $\boxed{w = \sqrt{1-z^2}}$

The transformation is involutory.

z - plane, $\Re(z) \ge 0$	w - plane, $\Re(w) \ge 0$
points 0; 1; ∞; $\pm i$; $\sqrt{2}$	points 1; 0; ∞; $\sqrt{2}$; $\pm i$
quadrant $x > 0$, $y > 0$	quadrant $u > 0$, $v < 0$
quadrant $x > 0$, $y < 0$	quadrant $u > 0$, $v > 0$
half-line $\arg z = \varphi$; φ fixed, $0 < \varphi < \frac{\pi}{2}$	the part in the fourth quadrant of the rectangular hyperbola $\Re\left(\frac{w^2}{1-e^{4i\varphi}}\right) = \frac{1}{2}$ through a, with asymptotes $\arg w = \varphi - \frac{\pi}{2}$, φ

z - plane, $\Re(z) \geq 0$	w - plane, $\Re(w) \geq 0$
	
half-line $\arg z = \varphi - \frac{\pi}{2}$	the part in the first quadrant of the same hyperbola
region $\varphi - \pi/2 < \arg z < \varphi$, cut along $\mathfrak{B}\mathfrak{C}\mathfrak{J}$	exterior of the hyperbola, bounded by $u = 0$
half-plane $x > 0$, cut along $\mathfrak{B}\mathfrak{C}\mathfrak{J}$	half-plane $u > 0$, cut along $\mathfrak{A}\mathfrak{D}\mathfrak{J}$
either of quadrants $\left.\begin{matrix} x < 0,\ y > 0 \\ x > 0,\ y < 0 \end{matrix}\right\}$	either of quadrants $\left.\begin{matrix} u > 0,\ v > 0 \\ u < 0,\ v < 0 \end{matrix}\right\}$
either of quadrants $\left.\begin{matrix} x < 0,\ y < 0 \\ x > 0,\ y > 0 \end{matrix}\right\}$	either of quadrants $\left.\begin{matrix} u > 0,\ v < 0 \\ u < 0,\ v > 0 \end{matrix}\right\}$

<u>Hyperbolae, Ellipses, Cassinians</u>

One-one correspondences can be obtained by considering the above quadrants. Otherwise each curve is to be counted twice.

z - plane	w - plane
$\Re(z^2/z_o^2) = \frac{1}{2}$; $z_o \neq 0$, $z_o^2 \neq 1+e^{4i\varphi_o}$ where $\varphi_o = \arg z_o$; i.e. a rectangular hyperbola with asymptotes $\arg z = \varphi_o \pm n\pi/4$; $n = 1,5$	$\Re(w^2/w_o'^2) = \frac{1}{2}$; $w_o'^2 + z_o^2 = 1 + e^{4i\varphi_o}$; i.e. a rectangular hyperbola with asymptotes $\arg w = \varphi_o \pm n\pi/4$; $n = 1,5$ Note that $(w_o'/z_o)^2$ is real
$\Re\left(\frac{z^2}{1+e^{4i\varphi_o}}\right) = \frac{1}{2}$; $\lvert\varphi_o - k\pi\rvert < \frac{\pi}{4}$, $k = 0,1$	lines $\Re(e^{i(\pm\pi/4-\varphi_o)}w) = 0$
hyperbola $\lvert z+1\rvert - \lvert z-1\rvert = \pm 2\cos v$; v fixed; with asymptotes $\arg z = \pm(v+k\pi)$, $k = 0,1$	hyperbola $\lvert w+1\rvert - \lvert w-1\rvert = \pm 2\sin v$, with asymptotes $\arg w = \pm(\frac{\pi}{2} - v + k\pi)$, $k = 0,1$
ellipse $\lvert z+1\rvert + \lvert z-1\rvert = h$, $h > 2$	ellipse $\lvert w+1\rvert + \lvert w-1\rvert = h$
Cassinian $\lvert z-z_o\rvert\lvert z+z_o\rvert = c$; $z_o \neq 0, \neq \pm 1$; $c > 0$	Cassinian $\lvert w-w_o\rvert\lvert w+w_o\rvert = c$; $w_o^2 = 1 - z_o^2$
circle $\lvert z\rvert = \sqrt{c}$, $\sqrt{c} > 0$	Cassinian $\lvert w-1\rvert\lvert w+1\rvert = c$
Cassinian $\lvert z-1\rvert\lvert z+1\rvert = c$, $c > 0$	circle $\lvert w\rvert = \sqrt{c}$

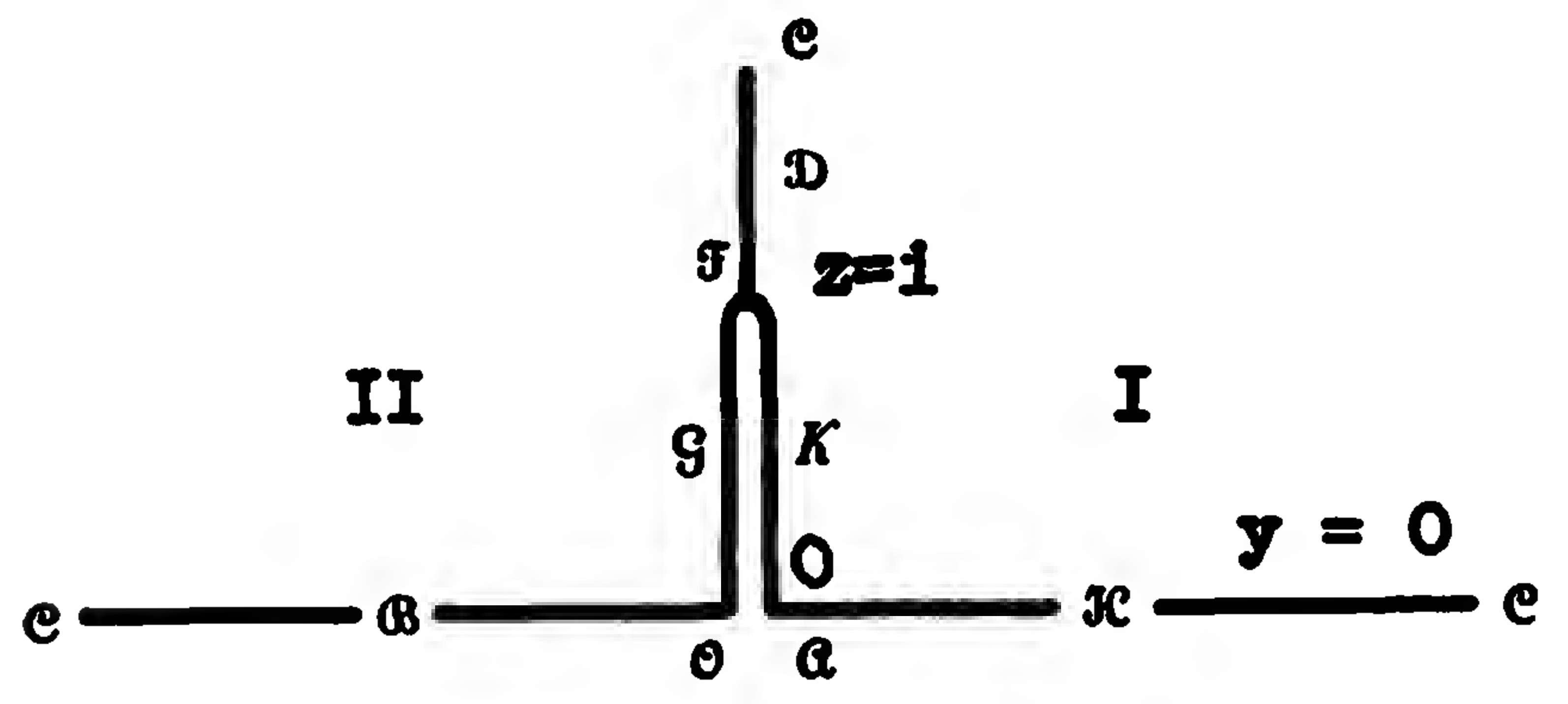
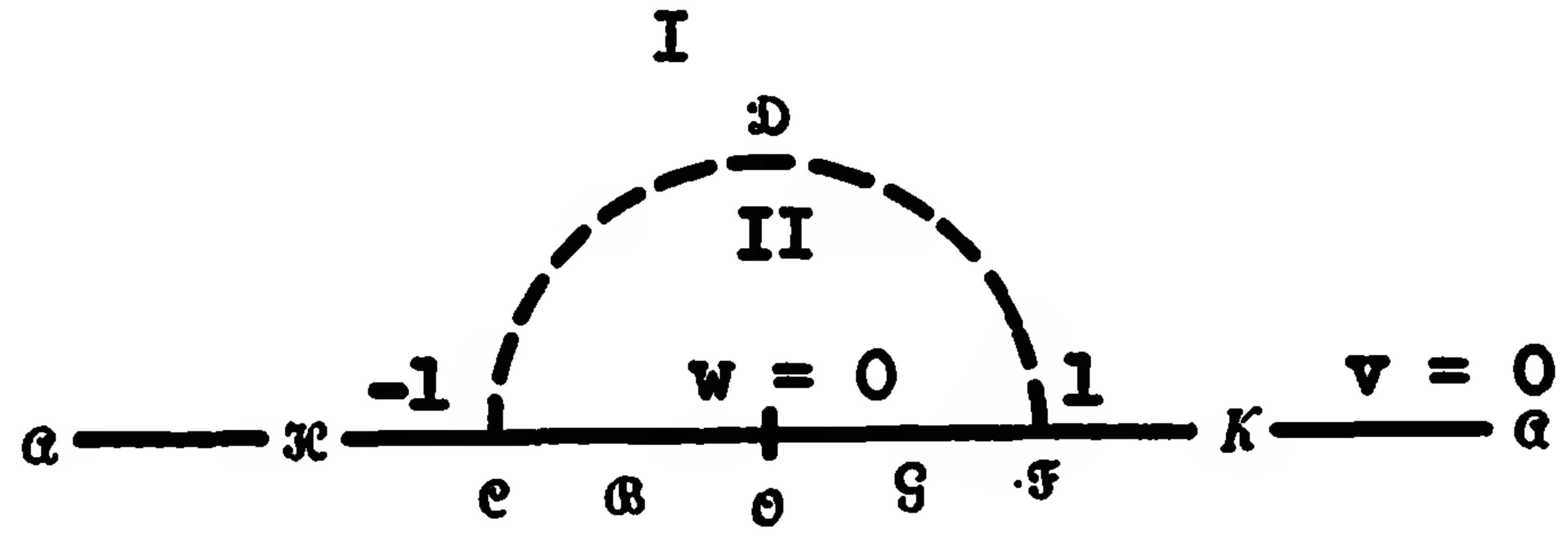
9.3 Cut half-plane, cut circle, cut plane on half-plane.

$$w = -\left(\frac{1+\sqrt{1+z^2}}{z}\right)^2; \qquad z = \frac{2i\sqrt{w}}{w+1}$$

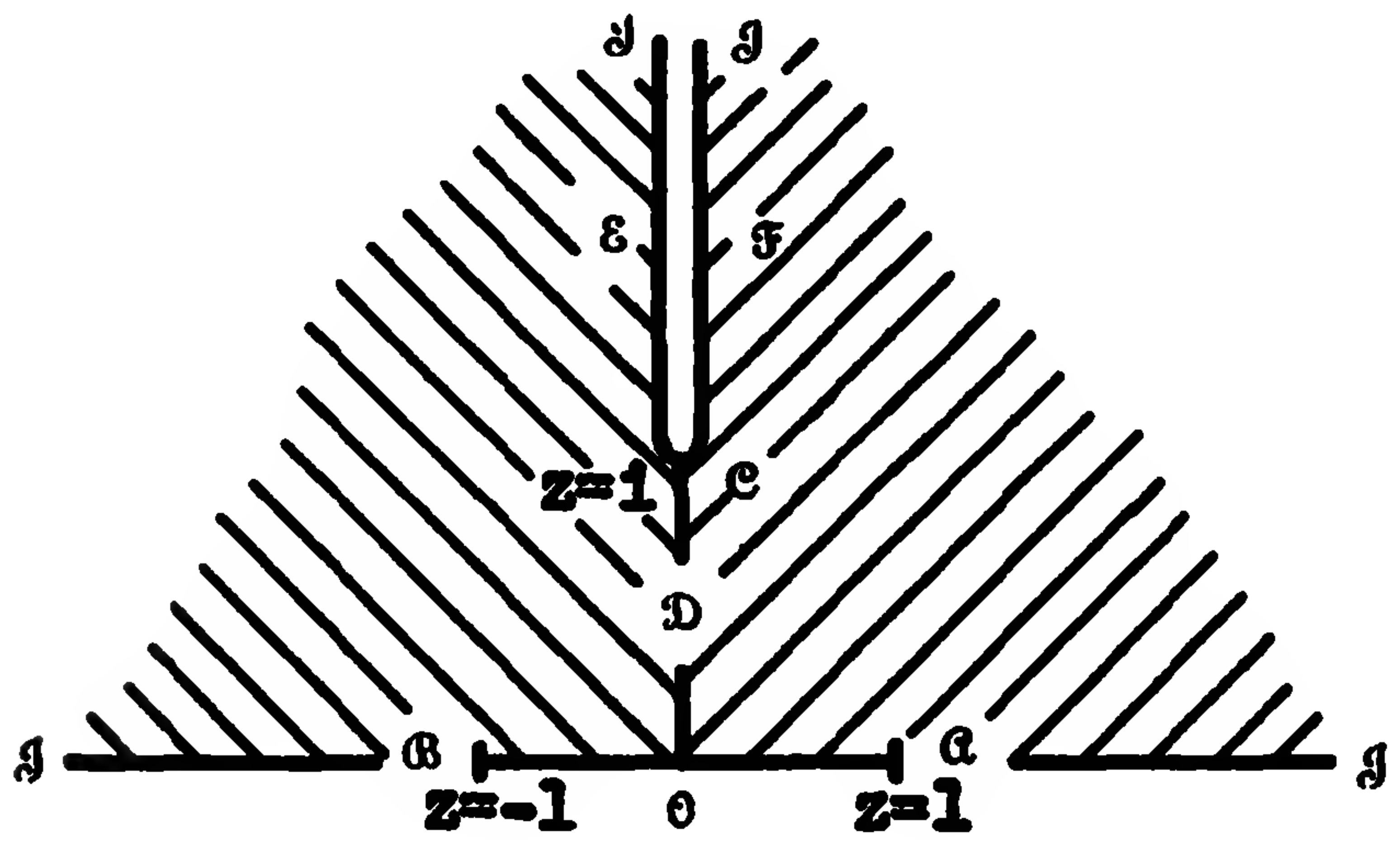
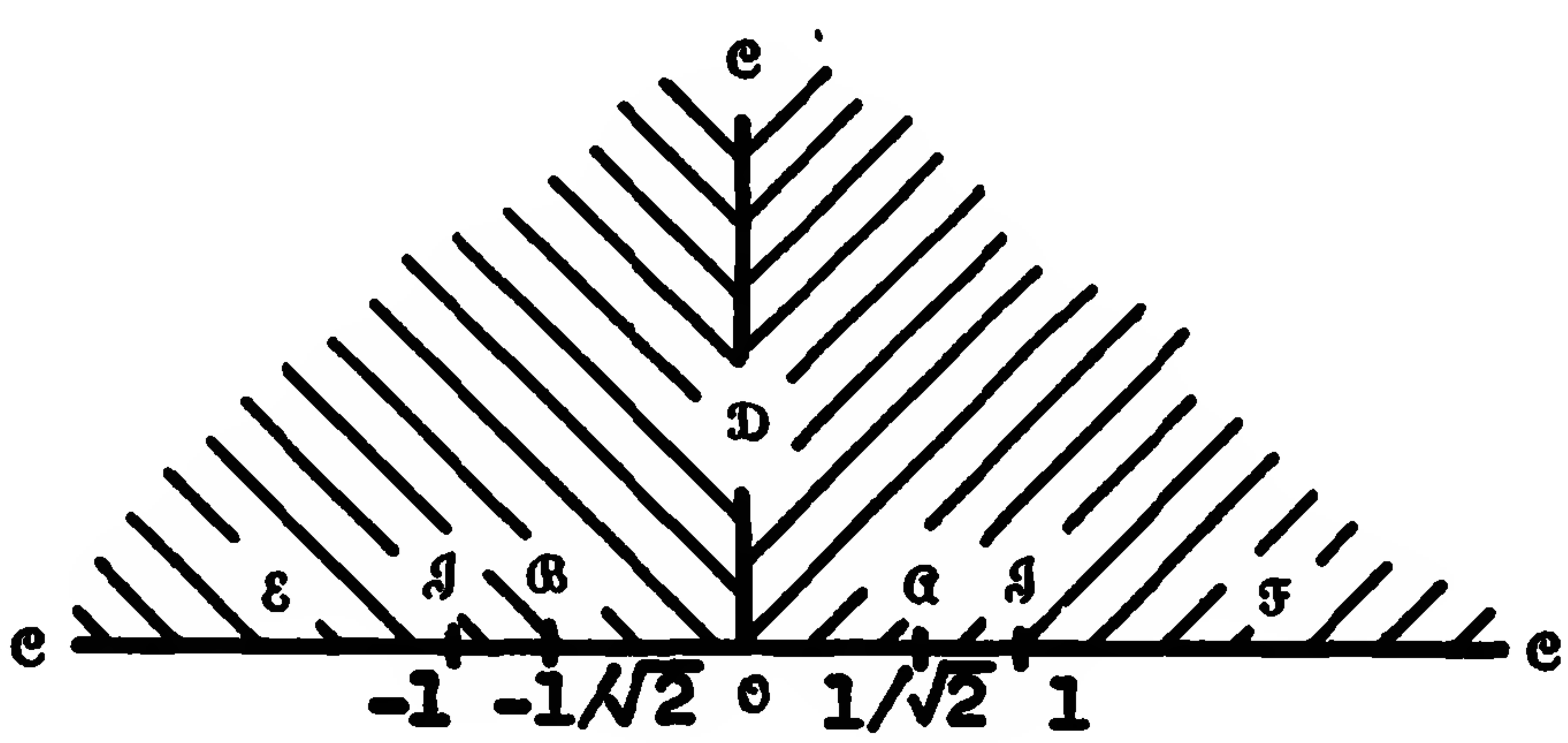
Upper half-plane, cut from $z = 0$ to $z = i$, on upper half-plane.

Critical points: $z = 0$, $z = \infty$, $z = i$.

z - plane	w - plane
point $z = 0$	points $w = 0$, $w = \infty$
points $z = \infty$; i	points $w = -1$; 1

z - plane	w - plane
	
half-line $y = 0, \ \infty > x > 0$	half-line $v = 0, \ -1 > u > -\infty$
line-segment $x = 0+, \ 0 < y \leq 1,$ i.e. $\mathfrak{A} K \mathfrak{F}$	half-line $v = 0, \ \infty > u \geq 1$
line-segment $x = 0-, \ 1 \geq y \geq 0,$ i.e. $\mathfrak{F} \mathfrak{G} \mathfrak{O}$	line-segment $v = 0, \ 1 \geq u \geq 0$
half-line $y = 0, \ 0 > x > -\infty$	line-segment $v = 0, \ 0 > u > -1$
half-line $x = 0, \ 1 \leq y \leq \infty$	semicircle $\lvert w \rvert = 1, \ v \geq 0$

Upper half-plane, cut from $z = i$ to $z = \infty$ along the imaginary axis, on upper half-plane.

z - plane	w = plane
	

Transformation: $\boxed{w = \frac{z}{\sqrt{z^2+1}}}$; $z = \frac{w}{\sqrt{1-w^2}}$. Critical points: $z = i; \infty$.

Interior of circle $|z| = c$, cut from 0 to c, on half-plane.

$$\boxed{w = \left(\frac{\sqrt{z/c}+1}{\sqrt{z/c}-1}\right)^2}\; ; \qquad \frac{z}{c} = \left(\frac{\sqrt{w}+1}{\sqrt{w}-1}\right)^2; \qquad \text{see §7.2, with } \alpha = 2\pi.$$

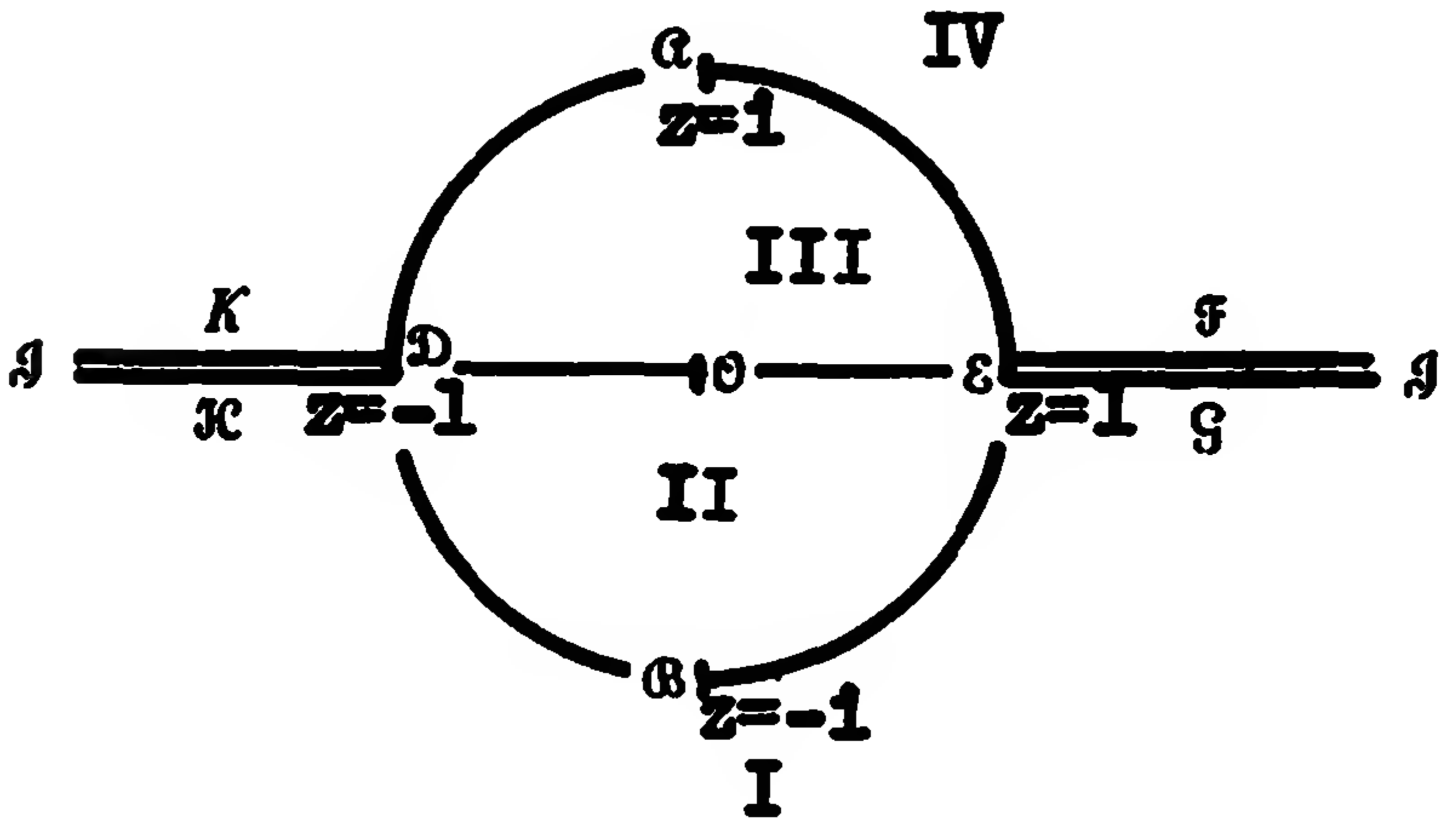
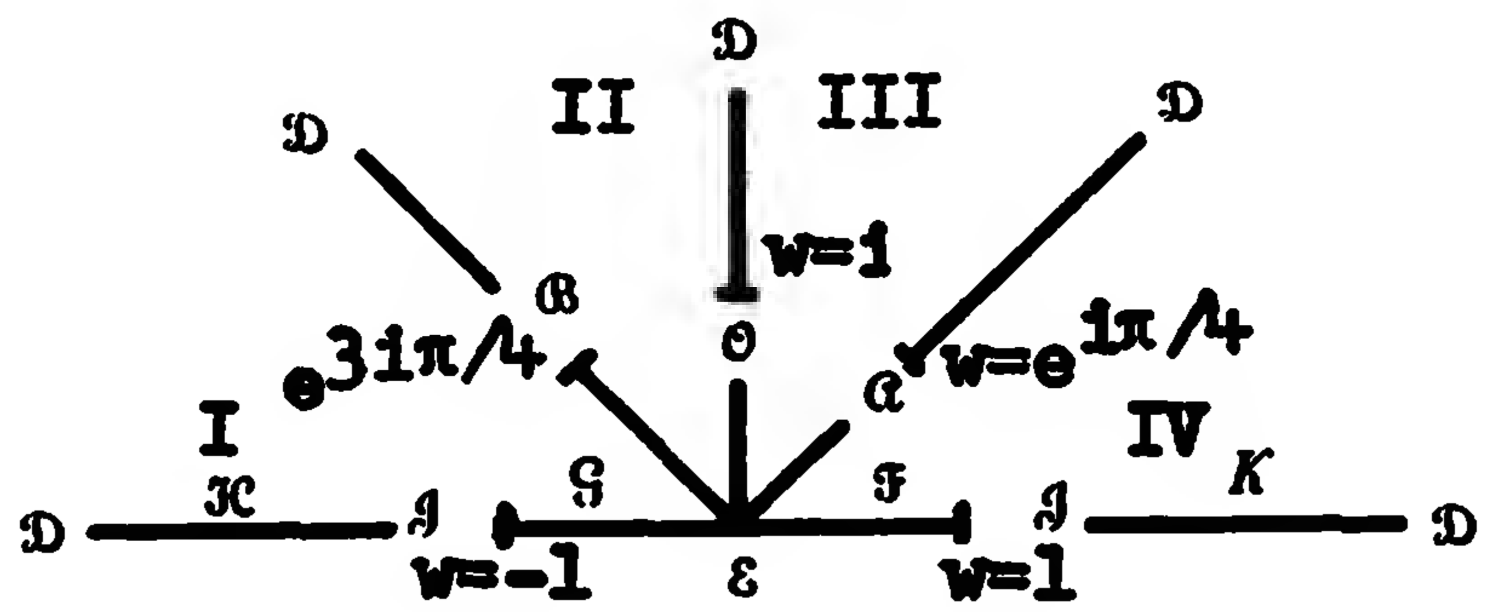
Cut plane (two cases) on half-plane. (cf. 8.1, first figure)

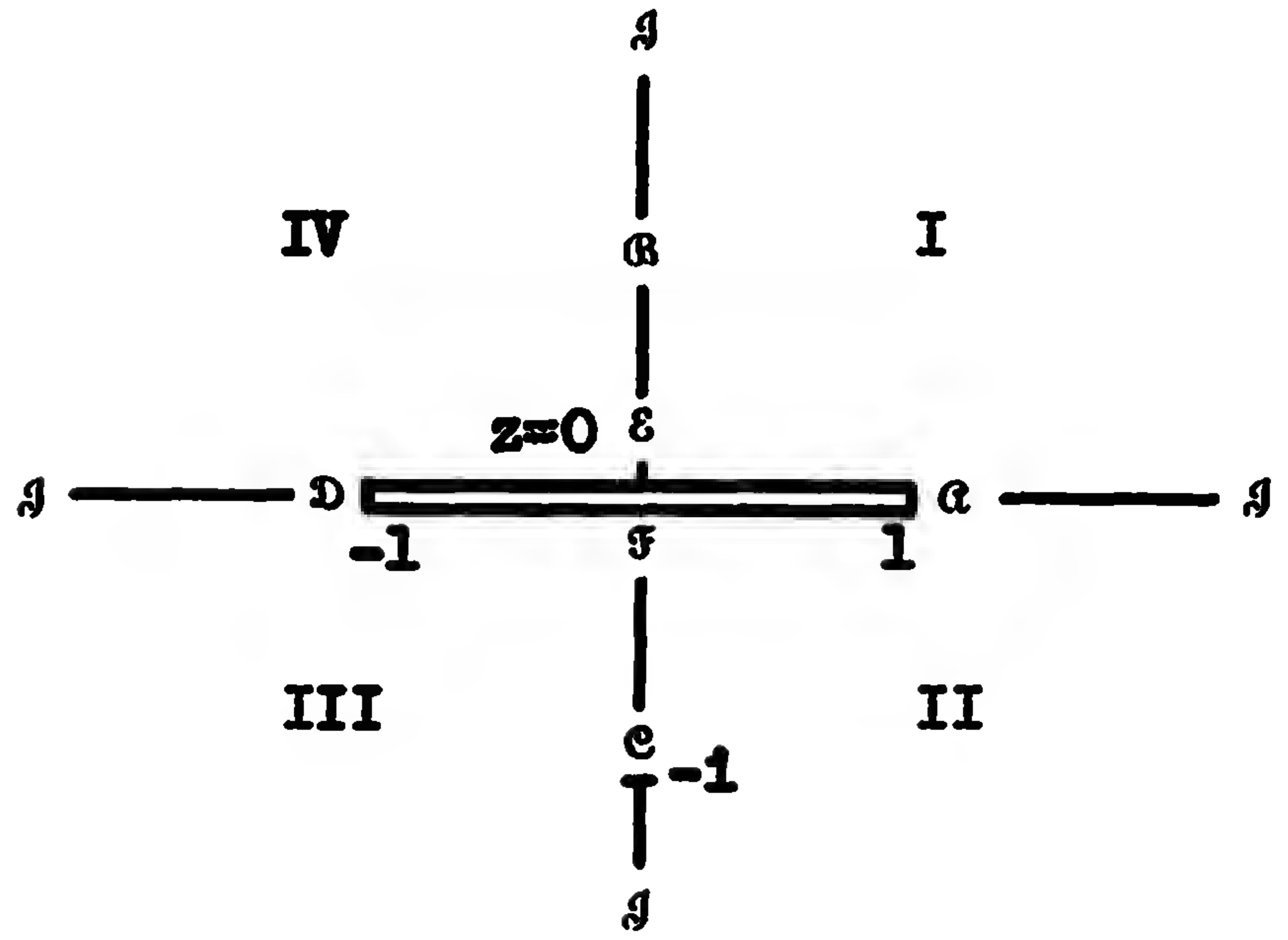
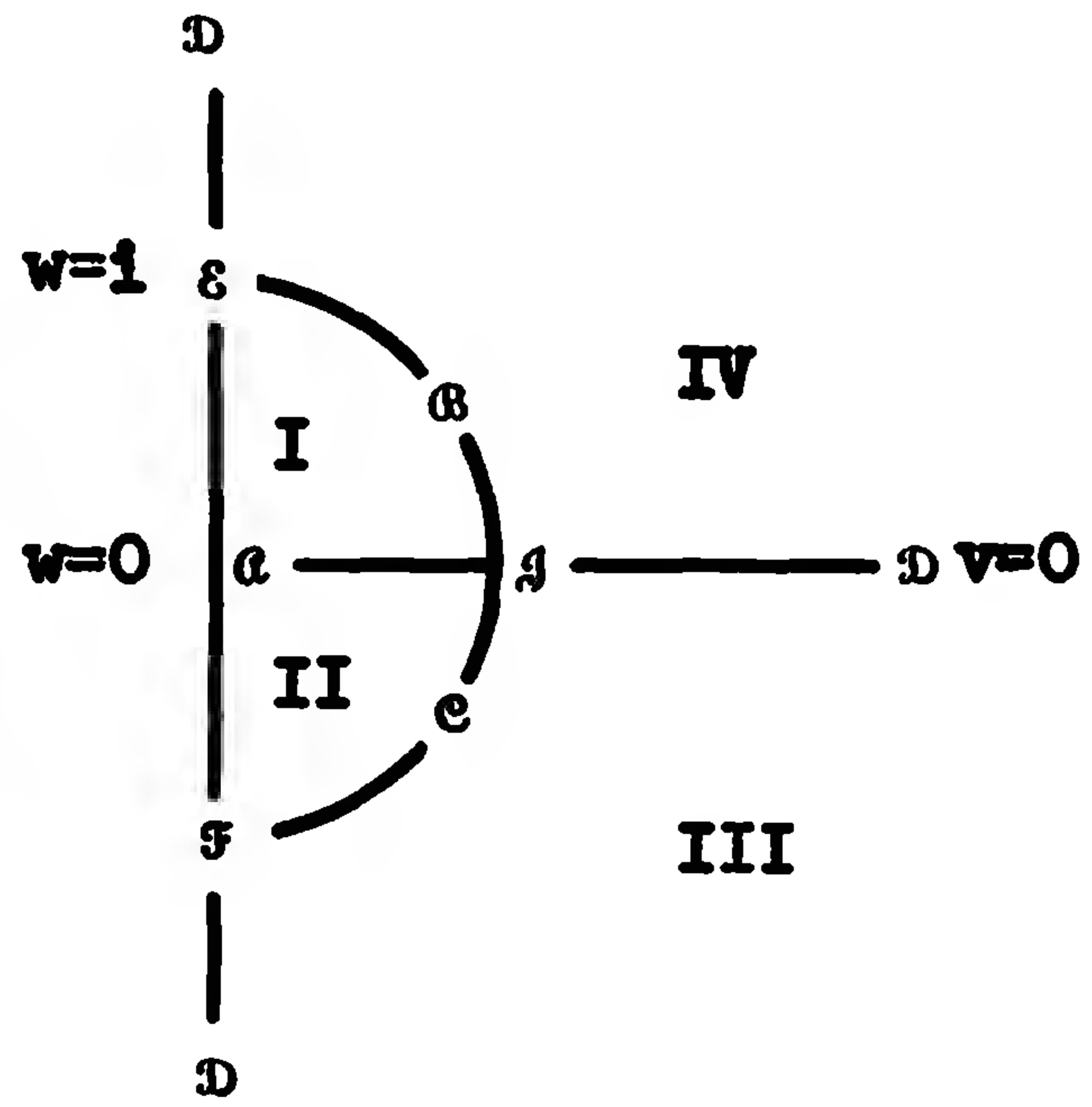
$$\boxed{w^2 = \frac{z-1}{z+1}}\; ; \qquad z = \frac{1+w^2}{1-w^2}; \quad \text{critical points:} \quad z = 1,\ -1,\ \infty.$$

The transformation is a combination of

$$w = \frac{\xi-1}{\xi+1} \quad \text{and} \quad z = \tfrac{1}{2}(\xi + 1/\xi); \quad \text{or of}$$

$$w = i\,\frac{\xi+1}{\xi-1} \quad \text{and} \quad -1/z = \tfrac{1}{2}(\xi + 1/\xi).$$

z - plane	w - plane
	
points 1; -1; 0; i; -i; ∞	points 0; ∞; i; $e^{i\pi/4}$; $e^{3i\pi/4}$; 1
segment $-1 < x < 1$, $y = 0$	half-line $\infty > v > 0$, $u = 0$
semi-circle $\lvert z\rvert = 1$, $y > 0$	half-line $\arg w = \pi/4$
semi-circle $\lvert z\rvert = 1$, $y < 0$	half-line $\arg w = 3\pi/4$
slit $1 < x < \infty$, $y = 0$	segment $0 < u < 1$, $v = 0$ segment $0 > u > -1$, $v = 0$

z - plane	w - plane
slit $-\infty < x < -1$, $y = 0$	segment $1 < u < \infty$, $v = 0$ segment $-1 > u > -\infty$, $v = 0$
	
line $x = 0$ slit $-1 < x < 1$, $y = 0$	semi-circle $\lvert w \rvert = 1$, $u \geq 0$ line $u = 0$

<u>Plane, cut along the negative part of the real axis, on half-plane:</u> $w = \sqrt{z}$, c.f. §6.1.

9.4 $$\boxed{w = \frac{(z-a_1)(z-a_2)\ldots \quad \ldots(z-a_n)}{(1-\bar{a}_1 z)(1-\bar{a}_2 z)\ldots \quad \ldots(1-\bar{a}_n z)}}$$

(i) $|a_j| < 1$ $(j = 1,2, \ldots,n)$

Interior of circle $|z| = 1$ on $|w| < 1$, counted n times.

(ii) $|a_j| > 1$ $(j = 1,2, \ldots,n)$.

Region $|z| > 1$ on $|w| < 1$, counted n times.

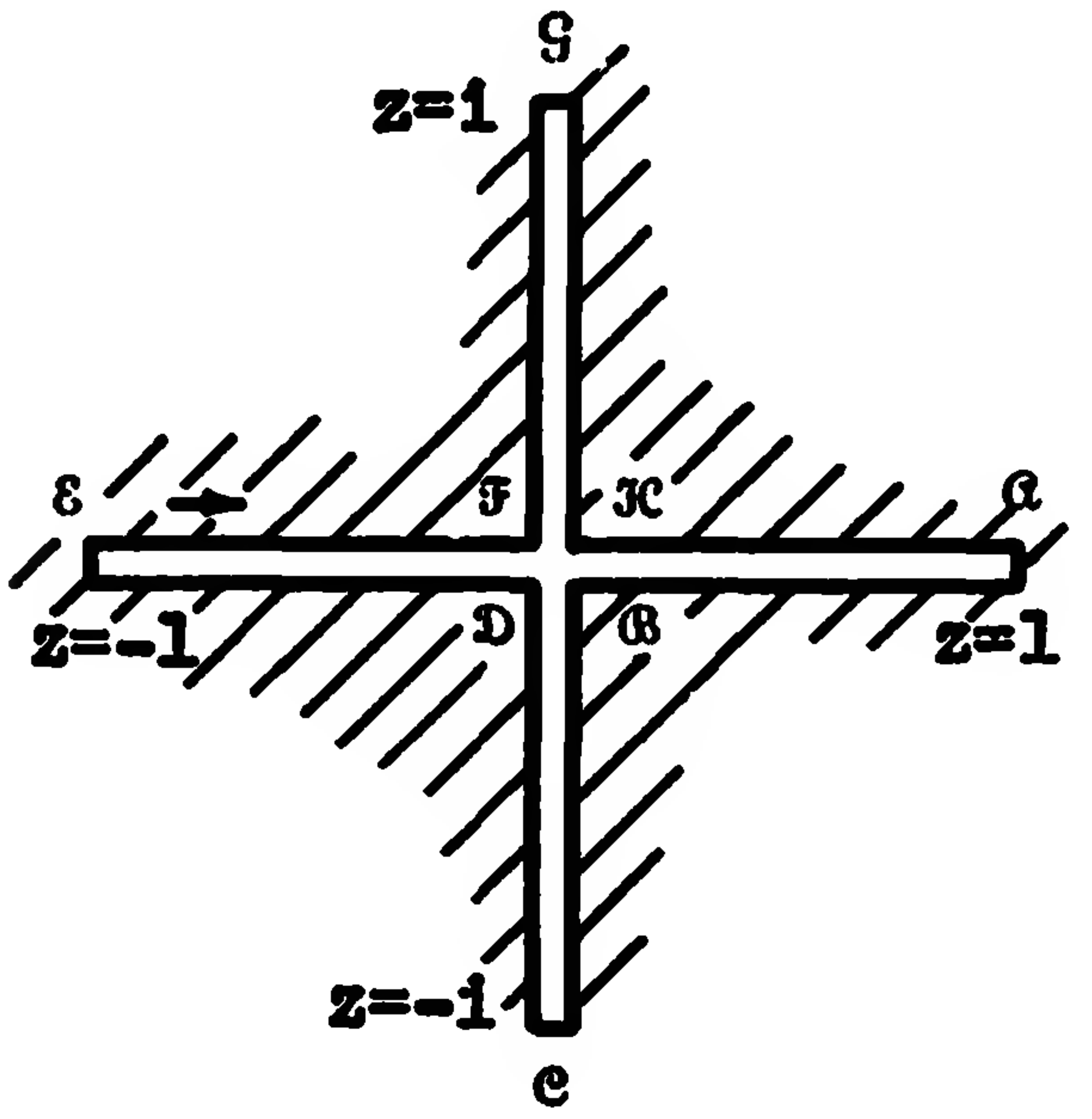
Plane with a number of equally spaced slits.

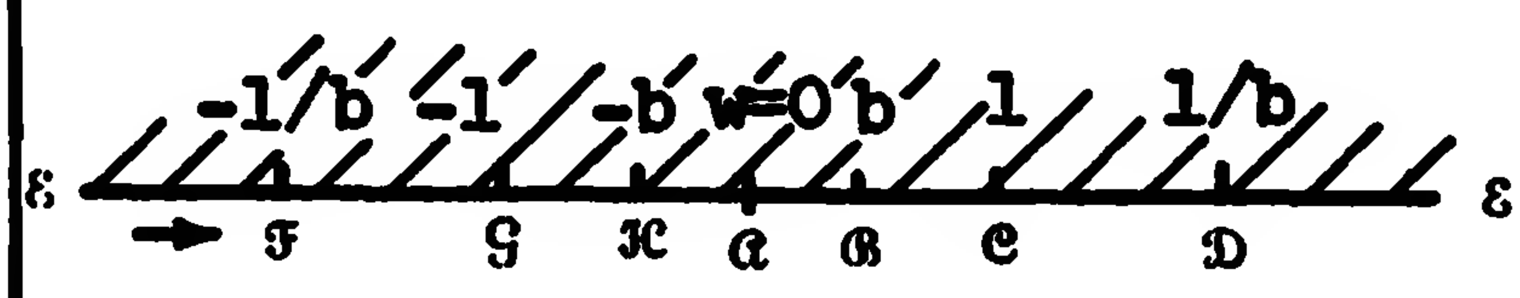
(i) Finite slits, starting from the origin.

$$\boxed{w = \tan\left(\frac{\cos^{-1} z^2}{4}\right)}, \qquad z = (w^4 - 6w^2 + 1)^{1/2}(1 + w^2)^{-1}.$$

Set $b = \tan\frac{\pi}{8} = \sqrt{2} - 1.$

z - plane	w - plane





region exterior to slits	half-plane $v > 0$

$$\boxed{w = \tan\left(\frac{1}{n}\cos^{-1} z^{n/2}\right)}, \qquad n \text{ a positive integer;}$$

$$z = \left\{\cos(n \tan^{-1} w)\right\}^{2/n}, \text{ algebraic function.}$$

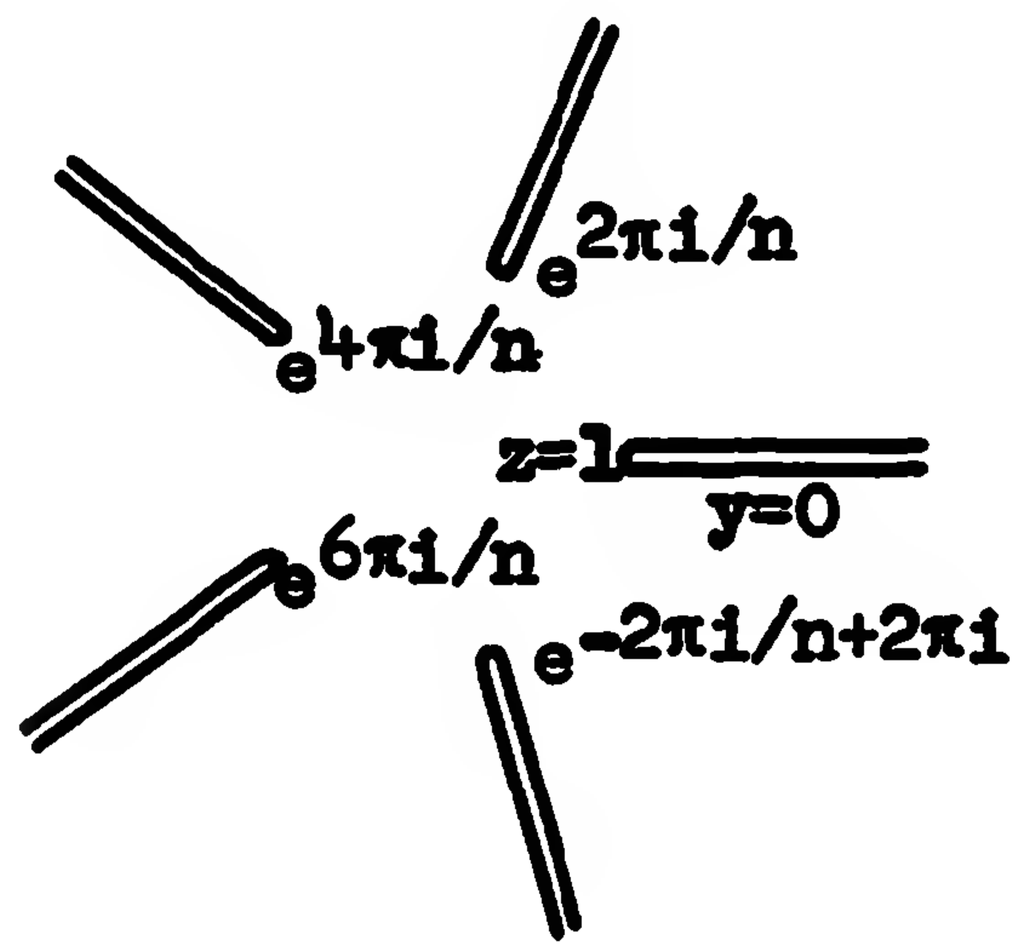
There are n equal and equally spaced slits.

Critical points: $z = 0;\ \infty;\ e^{2\pi ki/n}$ $(k = 1, 2, \ldots, n)$.

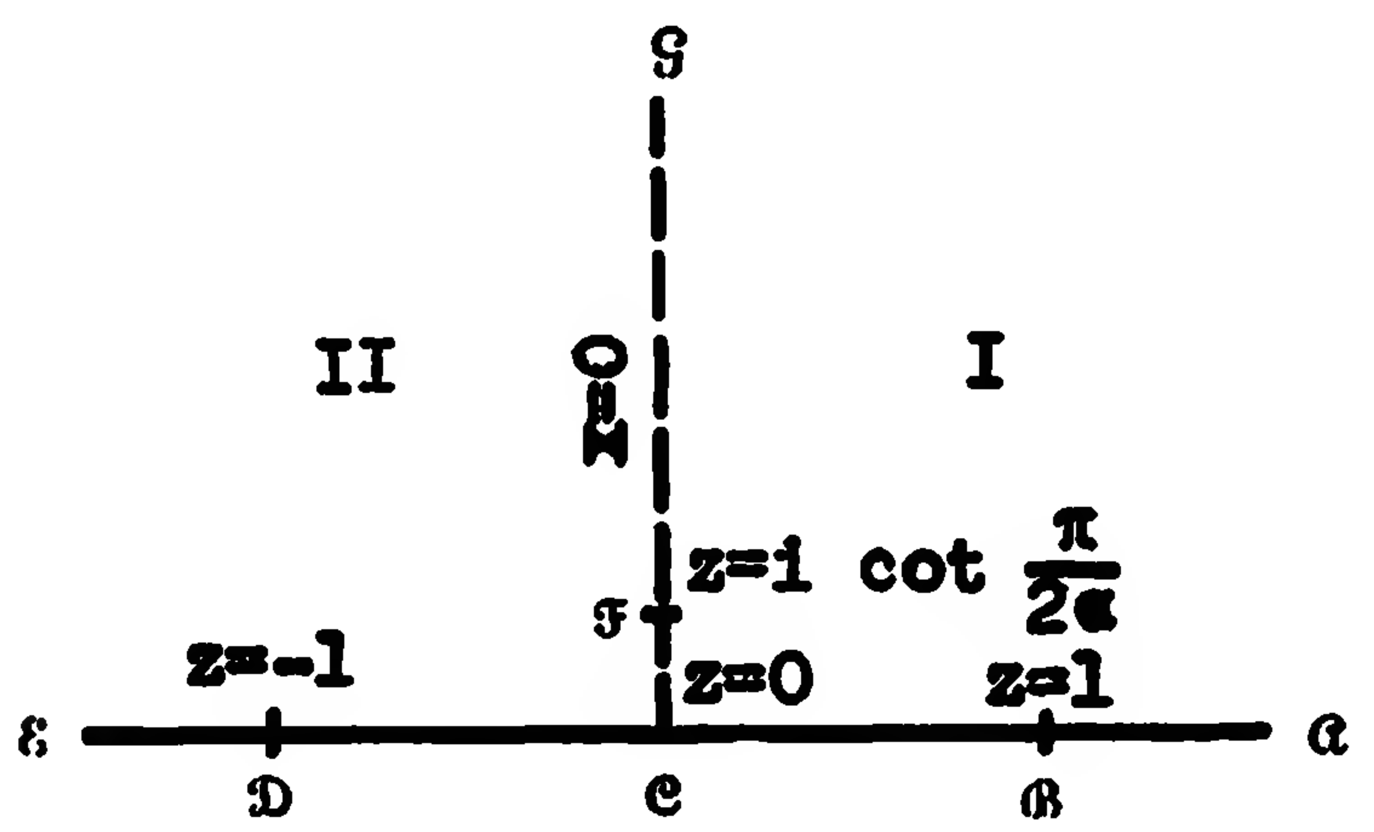
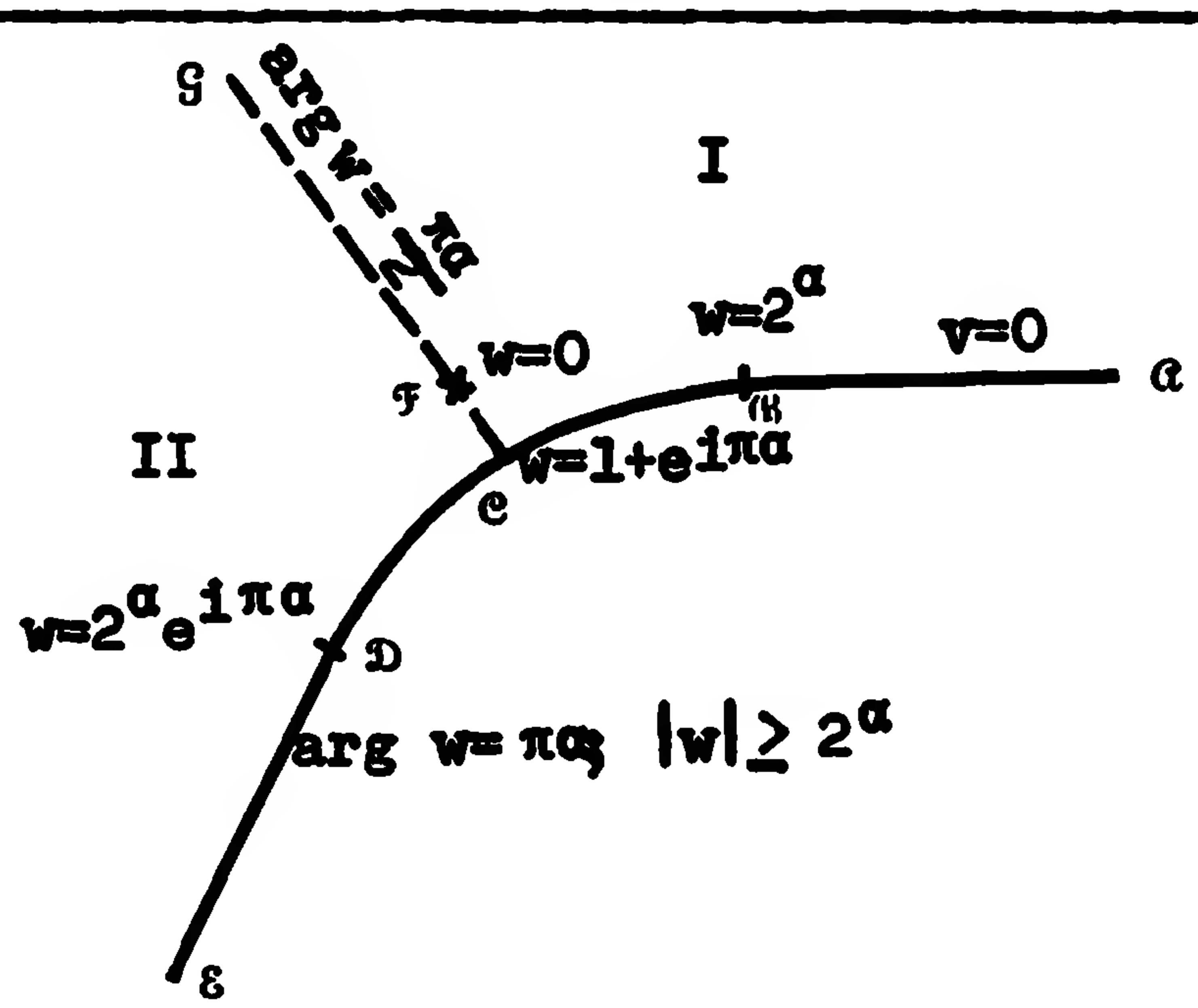
z - plane	w - plane
$z = 0$	$w = \tan\frac{(2k-1)\pi}{2n}$; $k = 1, 2, \ldots, n.$
$z = e^{-2\pi ki/n}$; $k = 1, 2, \ldots, n$	$w = \tan\frac{k\pi}{n}$
plane with n slits: $\arg z = \frac{2\pi ki}{n}$, $0 < \lvert z \rvert \le 1$	half-plane $v > 0$

(ii) Infinite slits, starting from points of the unit circle, along $\arg z = e^{2i\pi k/n}$, $|z| \geq 1$; $k = 0, 1, 2, \ldots, n-1$ (n positive integer).

$$\boxed{z = \left\{\cos(n \tan^{-1} w)\right\}^{-2/n}}$$

z - plane	w - plane
points $e^{2k\pi i/n}$; ∞	points $\tan \frac{k\pi}{n}$; $\tan \frac{2k+1}{2n}\pi$
	
cut plane	half-plane $v > 0$

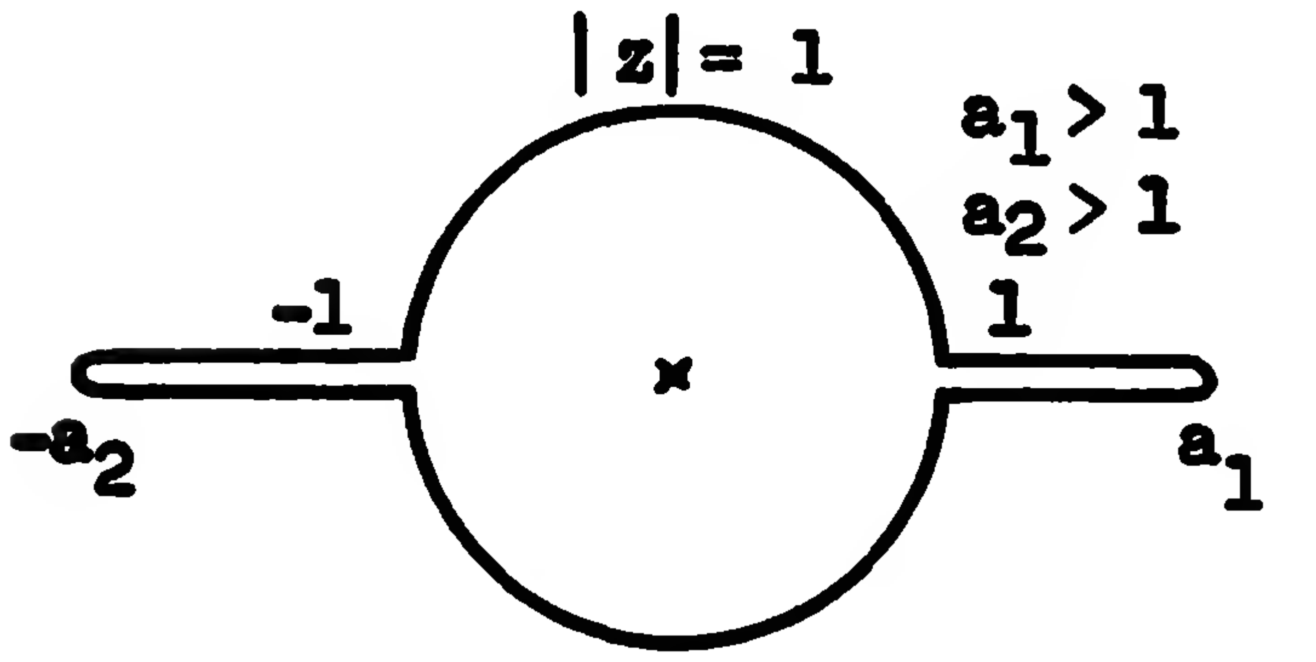
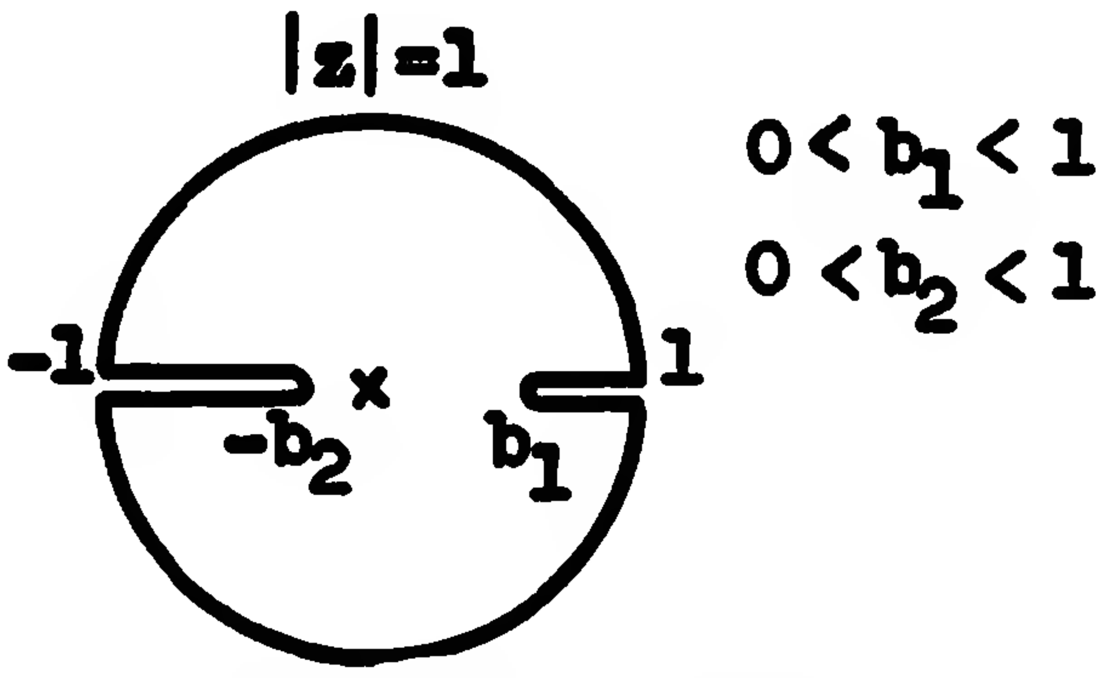
$\boxed{\text{Rounded corner; } w = (z+1)^\alpha + (z-1)^\alpha}$, $1 < \alpha < 2$.

z - plane	w - plane
	

z - plane	w - plane
line-segment $y = 0, -1 \leq x \leq 1$	part $-2^{\alpha} \cos \pi(\alpha-1) \leq u \leq 2^{\alpha}$, $-2^{\alpha} \sin \pi(\alpha-1) \leq v \leq 0$, i.e. part $\mathfrak{D}\ \mathfrak{C}\ \mathfrak{B}$, of $(v \cos \pi\alpha - u \sin \pi\alpha)^{1/\alpha} + (-v)^{1/\alpha} = 2(-\sin \pi\alpha)^{1/\alpha}$; for $\alpha = \frac{3\pi}{2}$, curve is asteroid $u^{2/3} + (-v)^{2/3} = 2$

APPENDIX

Interior mapping radius R_a of a region with respect to a finite point $z = a$ inside it. Exterior mapping radius R'_∞.*

	R_a	R'_∞
Interior of $\lvert z\rvert = r$; $\lvert a\rvert < r$	$r - \lvert a\rvert^2/r$	r
Exterior of $\lvert z\rvert = r$; $\lvert a\rvert > r$	$\lvert a\rvert^2/r - r$	
Angular region $0 < \arg z < \theta_o$ $(0 < \theta_o \leq 2\pi;\ 0 < \arg a < \theta_o)$	$\frac{2\lvert a\rvert\theta_o}{\pi} \sin \frac{\pi \arg a}{\theta_o}$	
Line-segment of length ℓ		$\frac{\ell}{4}$
Exterior of $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$		$\frac{\alpha+\beta}{2}$
Half-plane $y > 0$; $\Im(a) > 0$	$2\Im(a)$	
Infinite strip of width D; d = smallest distance of point a from boundary.	$\frac{2D}{\pi} \sin \frac{\pi d}{D}$	
		$\frac{(a_1+a_2)(1+a_1a_2)}{4a_1a_2}$
 Interior of circle, except slits; $a = 0$	$\frac{4b_1b_2}{(b_1+b_2)(1+b_1b_2)}$	

* Details and examples: see Pólya-Szegő, Part IV, Chapter 2.

Plane, cut along $y = 0$ from $x = \frac{1}{4}$ to $x = \infty$; $-\infty < a < \frac{1}{4}$	$1 - 4a$	
The two line-segments $y = 0$, $-\alpha \le x \le \alpha$; and $x = 0$, $-\beta \le y \le \beta$ $[\alpha > 0, \quad \beta > 0]$		$\frac{1}{2}(\alpha^2+\beta^2)^{1/2}$

PART THREE

THE EXPONENTIAL FUNCTION AND RELATED FUNCTIONS AND COMBINATIONS

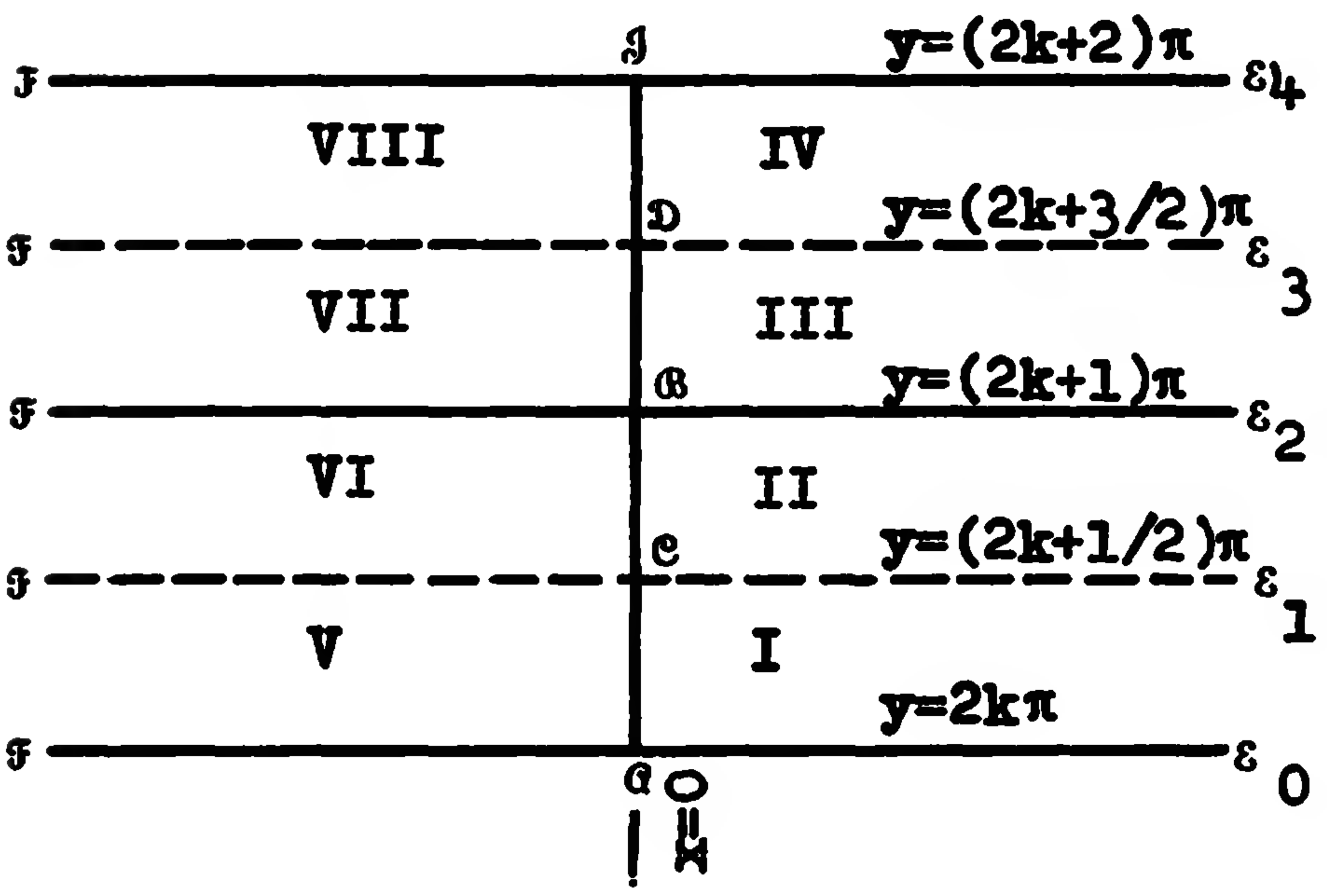
10. ELEMENTARY FUNCTIONS.

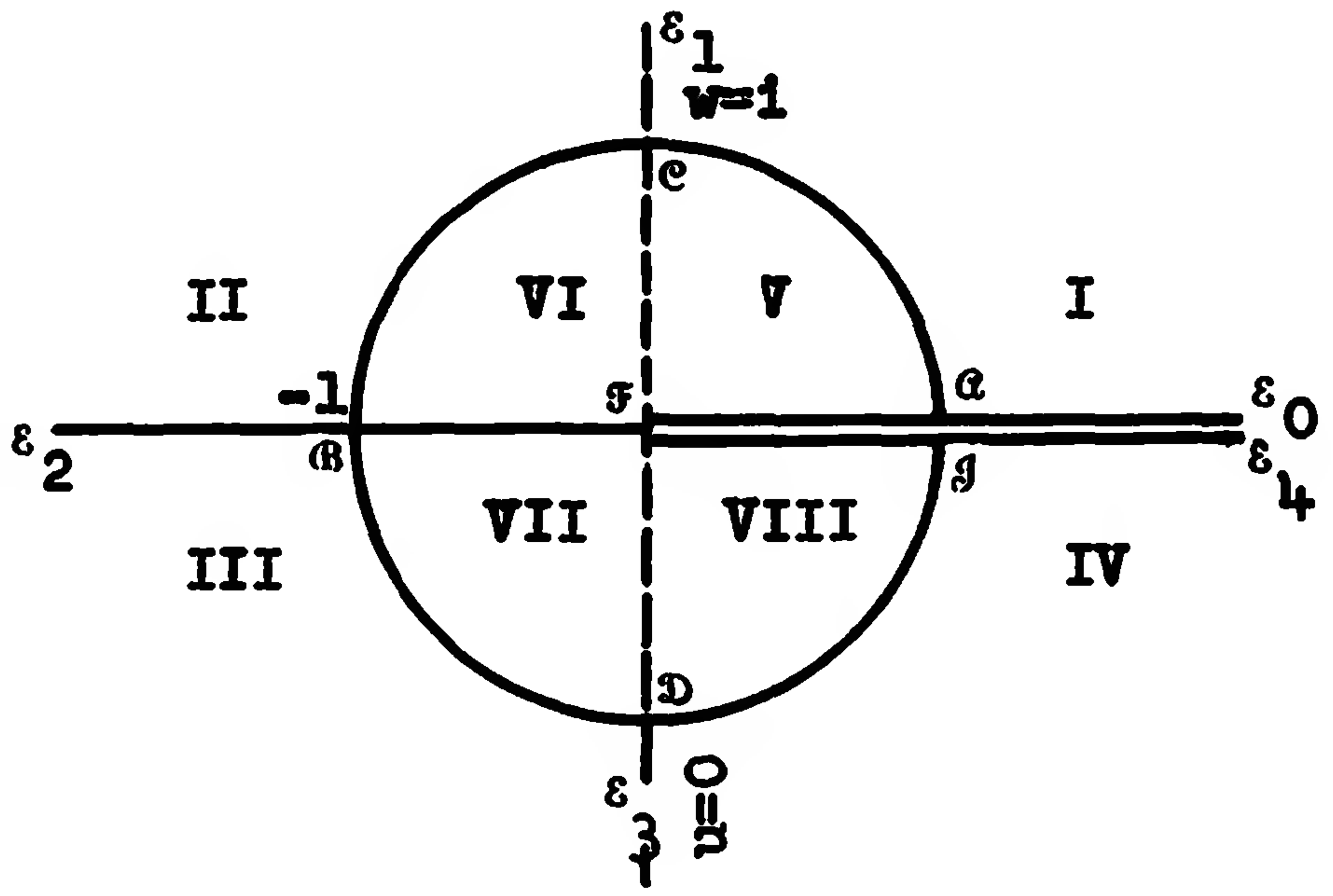
10.1 $\boxed{w = e^z}$ $\boxed{z = \log w}$

Critical points: for $w = e^z$: only $z = \infty$; for $z = \log w$: $w = 0, \infty$.

p, q, c, R, θ real, $R > 0$; $k = 0, \pm 1, \pm 2, \ldots$

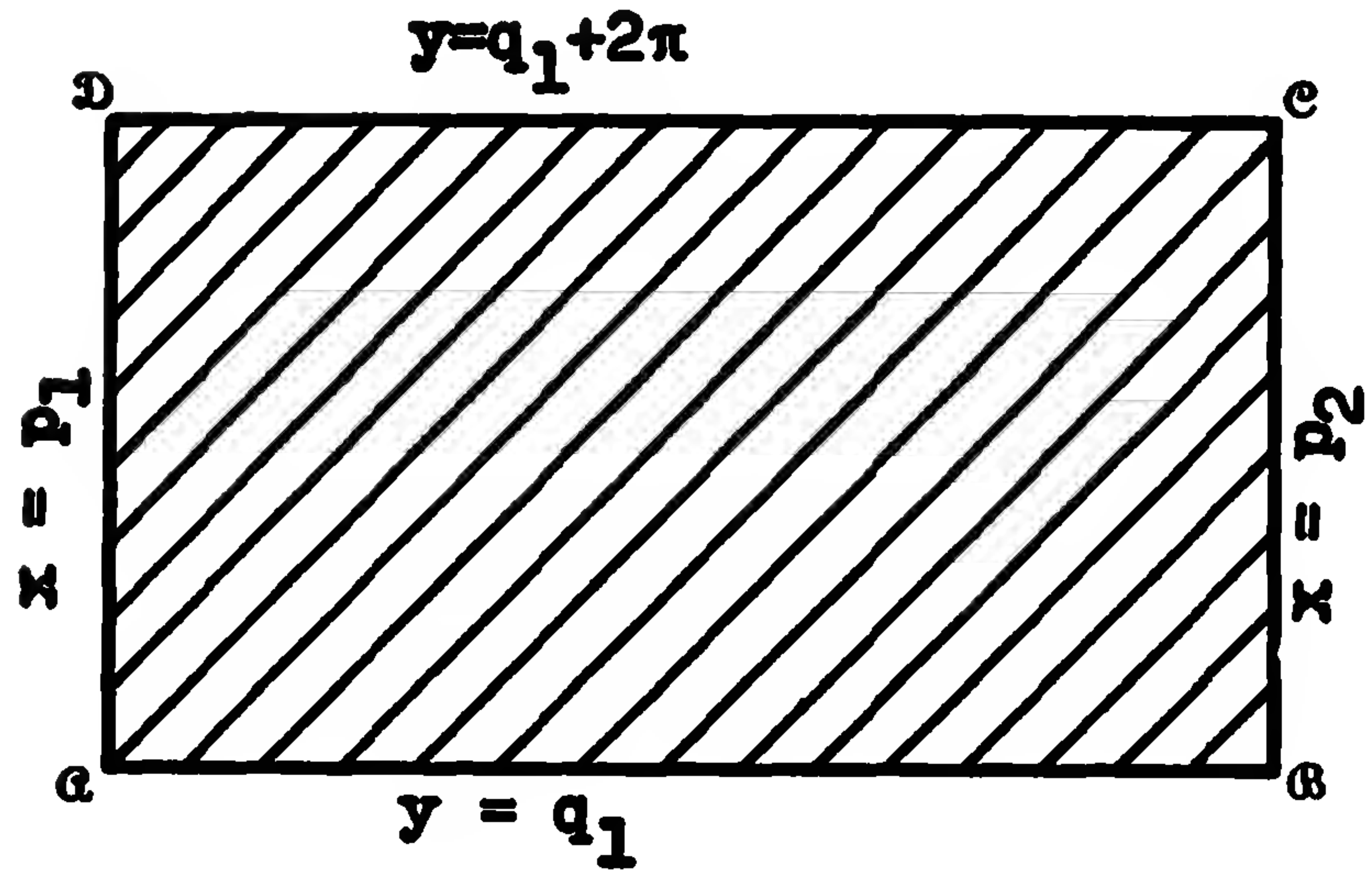
z - plane	w - plane
line-segment $x = p$, $c \leq y < c + 2\pi$	circle $\lvert w \rvert = e^p$
lines $y = 2k\pi$; $y = (2k+1)\pi$ $(-\infty < x < \infty)$	half-lines $v = 0$, $0 < u < \infty$; $v = 0$, $0 > u > -\infty$
line $y = q + 2k\pi$, $-\infty < x < \infty$	half-line $\arg w = q$, $0 < \lvert w \rvert < \infty$

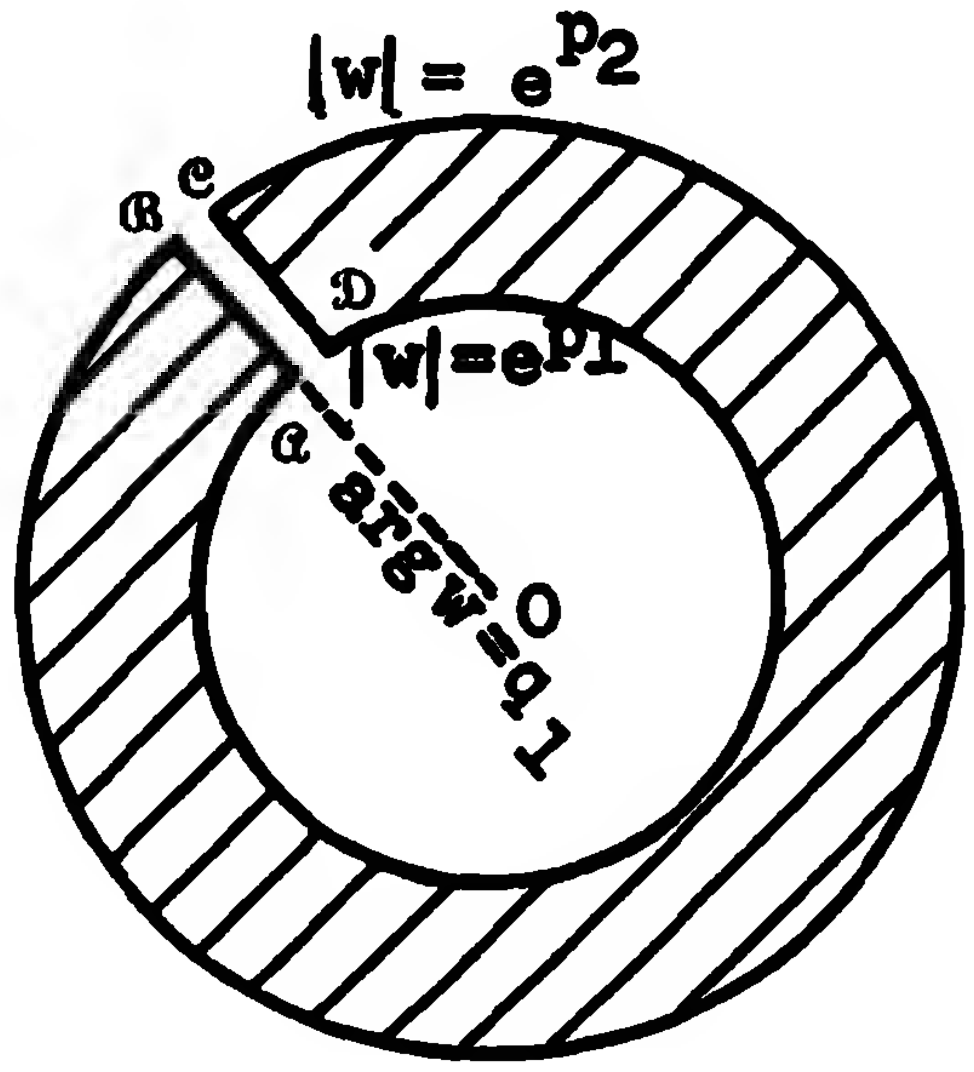




z - plane	w - plane
infinite strip $2k\pi < y < 2(k+1)\pi$	plane, cut along positive real axis
infinite strip $(2k-1)\pi < y < (2k+1)\pi$	plane, cut along negative real axis
infinite strip $c < y < c + 2\pi$	plane, cut along half-line $\arg w = c$

z - plane	w - plane
curve‡ $\cos(y-c) = pe^{-x}$; $\log\lvert p\rvert \gtrless x < \infty$, $(2k-\frac{1}{2})\pi < y-c < (2k+\frac{1}{2})\pi$ for $p > 0$ $(2k+\frac{1}{2})\pi < y-c < (2k+\frac{3}{2})\pi$ for $p < 0$ (asymptotes $y = c+(2k\pm\frac{1}{2})\pi$ [$p > 0$], or $y = c+(2k+1\pm\frac{1}{2})\pi$ [$p < 0$]).	line $\Re(e^{-ic}w) = p$; $p \geq 0$
line $x + my = p$ ($m \gtrless 0$)	spiral $R = e^{p}e^{-m\theta}$ ($w = Re^{i\theta}$)
rectangular region bounded by $y = q_1, q_2$; $x = p_1, p_2$ $(0 < q_2 - q_1 < 2\pi)$	part, bounded by $\arg w = q_1$, $\arg w = q_2$, of annular region bounded by $\lvert w\rvert = e^{p_1}$ and $\lvert w\rvert = e^{p_2}$.
the same rectangle, but with $q_2 - q_1 = 2\pi$	annular region bounded by $\lvert w\rvert = e^{p_1}$ and $\lvert w\rvert = e^{p_2}$, cut along $\arg w = q_1$.





‡For the diagram of $\cos y = pe^{-x}$ see p. 105. Take there a=2, b=1, c=2p > 0.

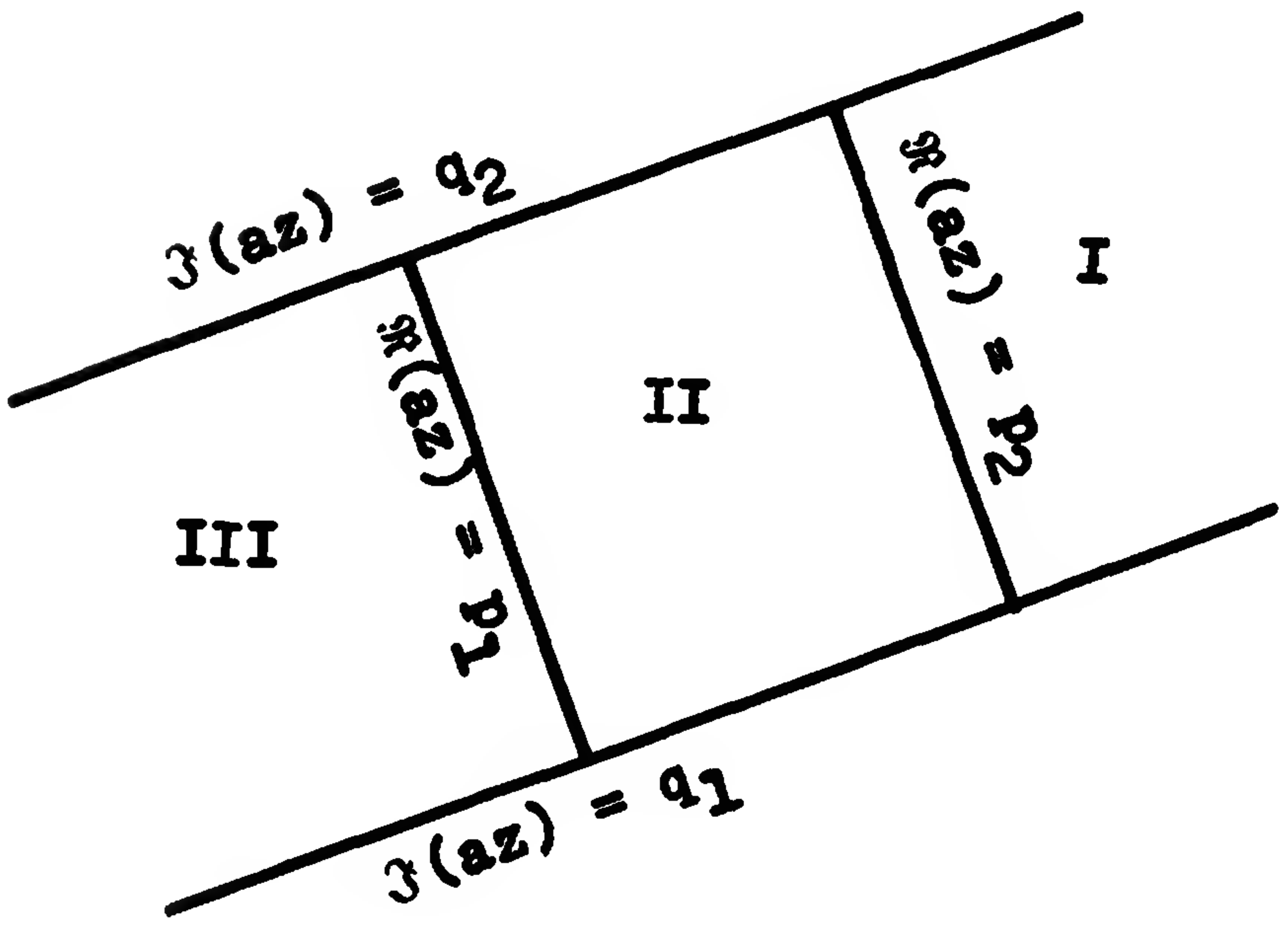
$\boxed{w = A + e^{az}}$, $a \neq 0$; $\boxed{z = b \log(w-A)}$, $b = 1/a$.

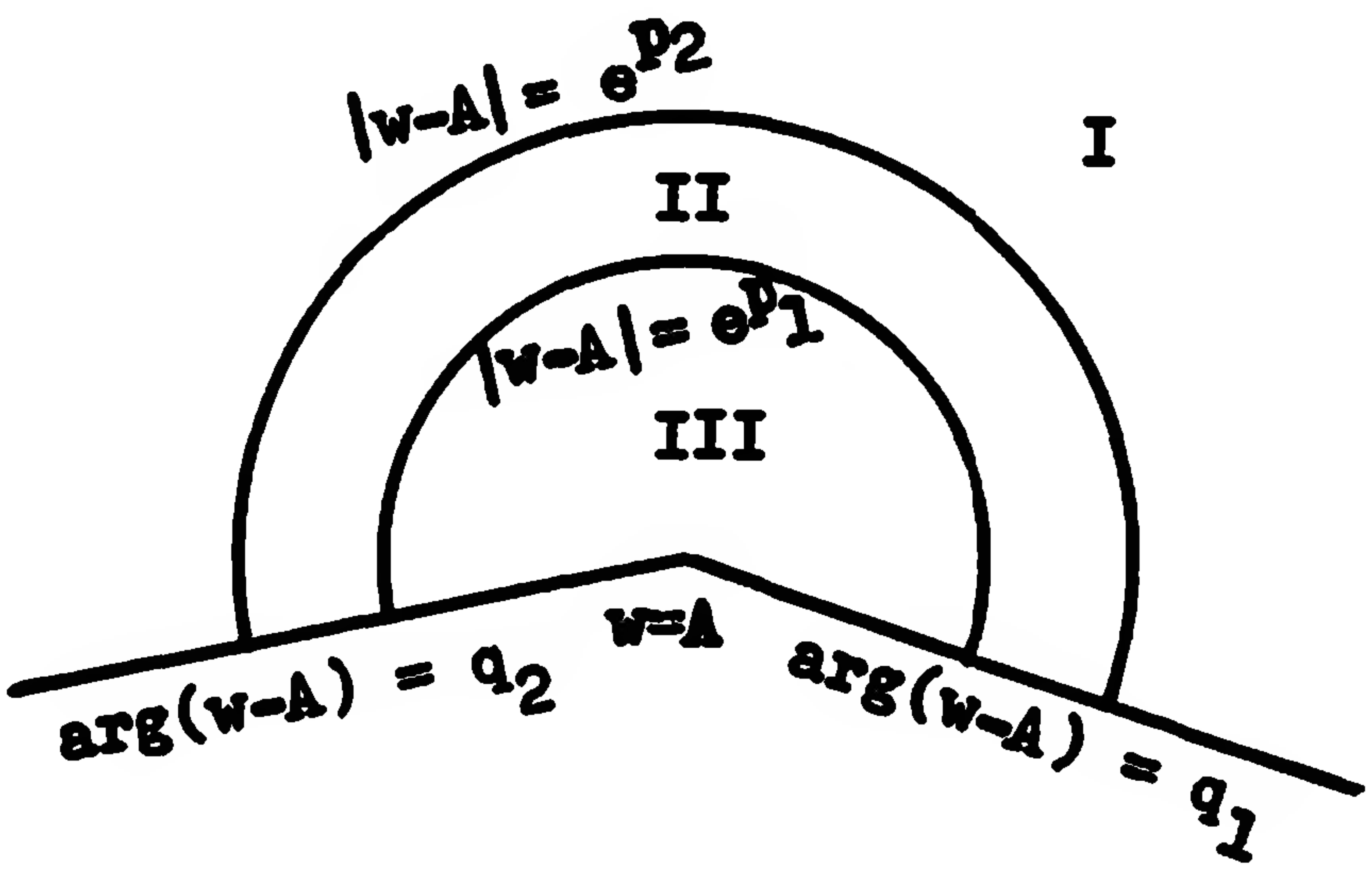
Critical point for $w = A + e^{az}$: $z = \infty$.

Critical points for $z = b \log(w-A)$: $w = A, \infty$.

p, q real; $k = 0, \pm 1, \pm 2, \ldots$

z - plane	w - plane
line $\mathfrak{I}(az) = q + 2k\pi$	half-line $\arg(w-A) = q$
line-segment $\mathfrak{R}(az) = p$, $q \leq \mathfrak{I}(az) < q + 2\pi$	circle $\lvert w-A \rvert = e^p$
infinite strip $q_1 < \mathfrak{I}(az) < q_2$, where $0 < q_2 - q_1 < 2\pi$	angular region $q_1 < \arg(w-A) < q_2$
infinite strip $q < \mathfrak{I}(az) < q+2\pi$	plane cut along $\arg(w-A) = q$
rectangular region bounded by $\mathfrak{I}(az) = q_1$, $\mathfrak{I}(az) = q_2$, $\mathfrak{R}(az) = p_1$, $\mathfrak{R}(az) = p_2$, where $0 < q_2 - q_1 < 2\pi$, $p_2 > p_1$	part, bounded by $\arg(w-A) = q_1, q_2$, of annular region bounded by $\lvert w-A \rvert = e^{p_1}, e^{p_2}$



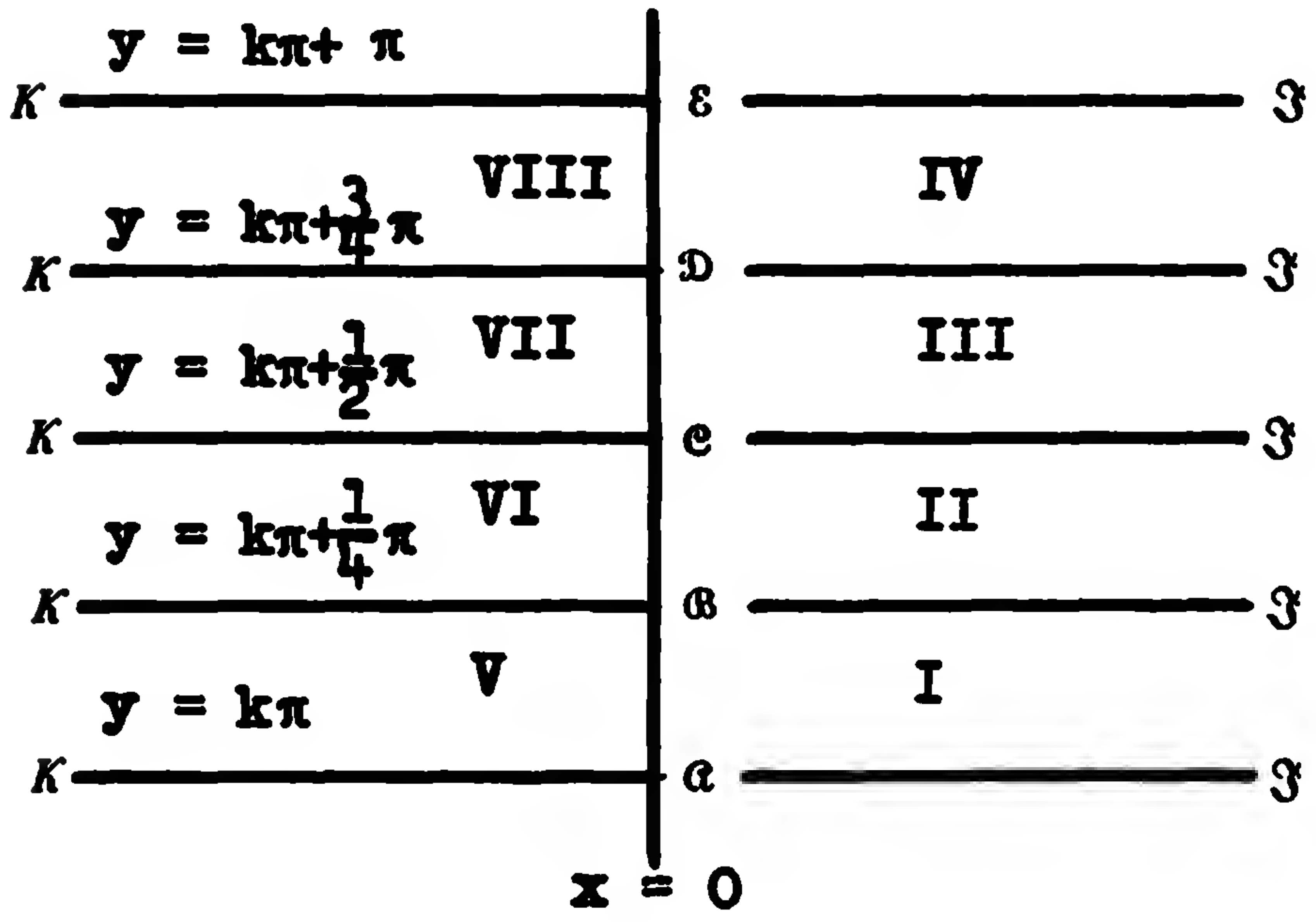
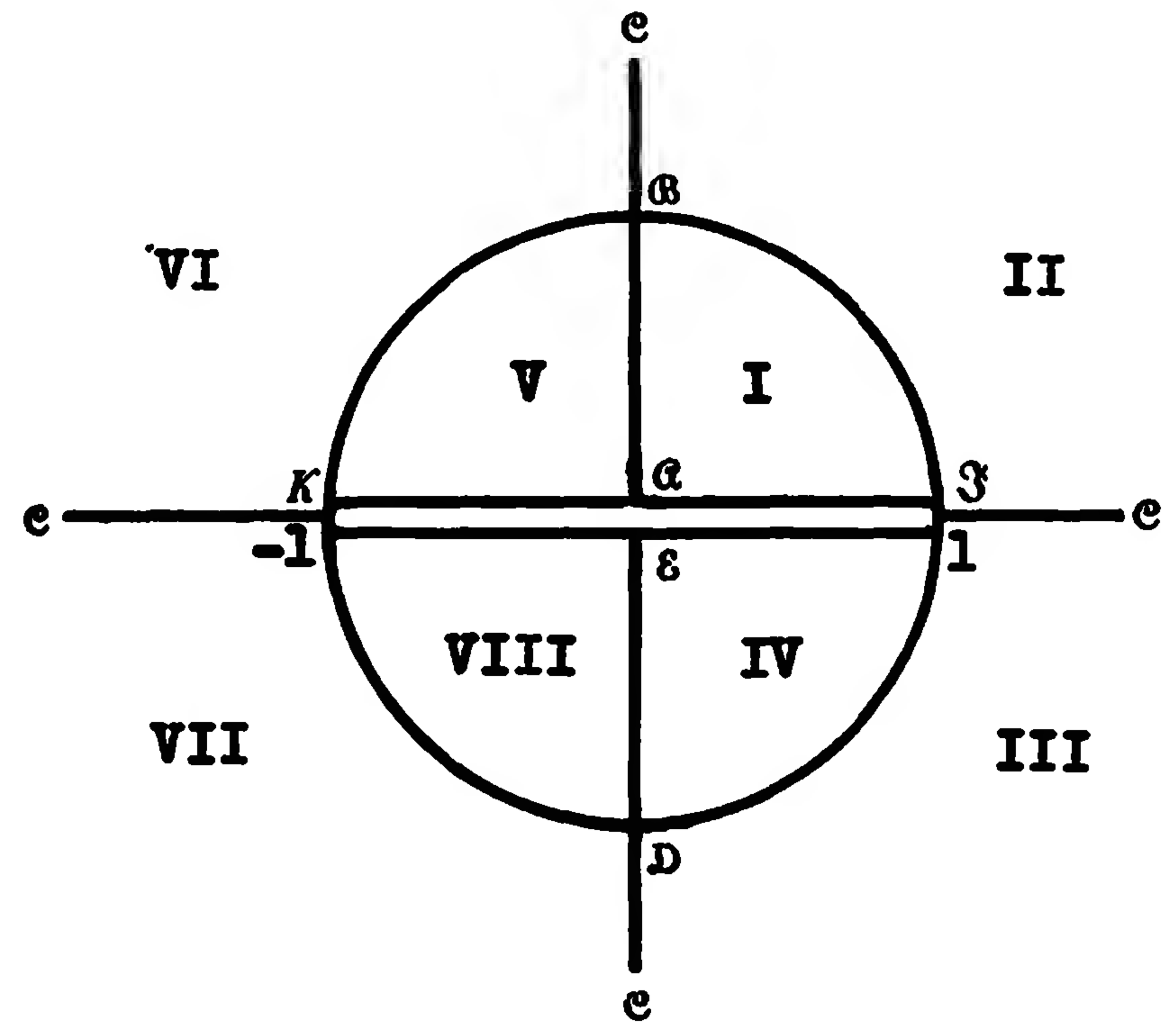
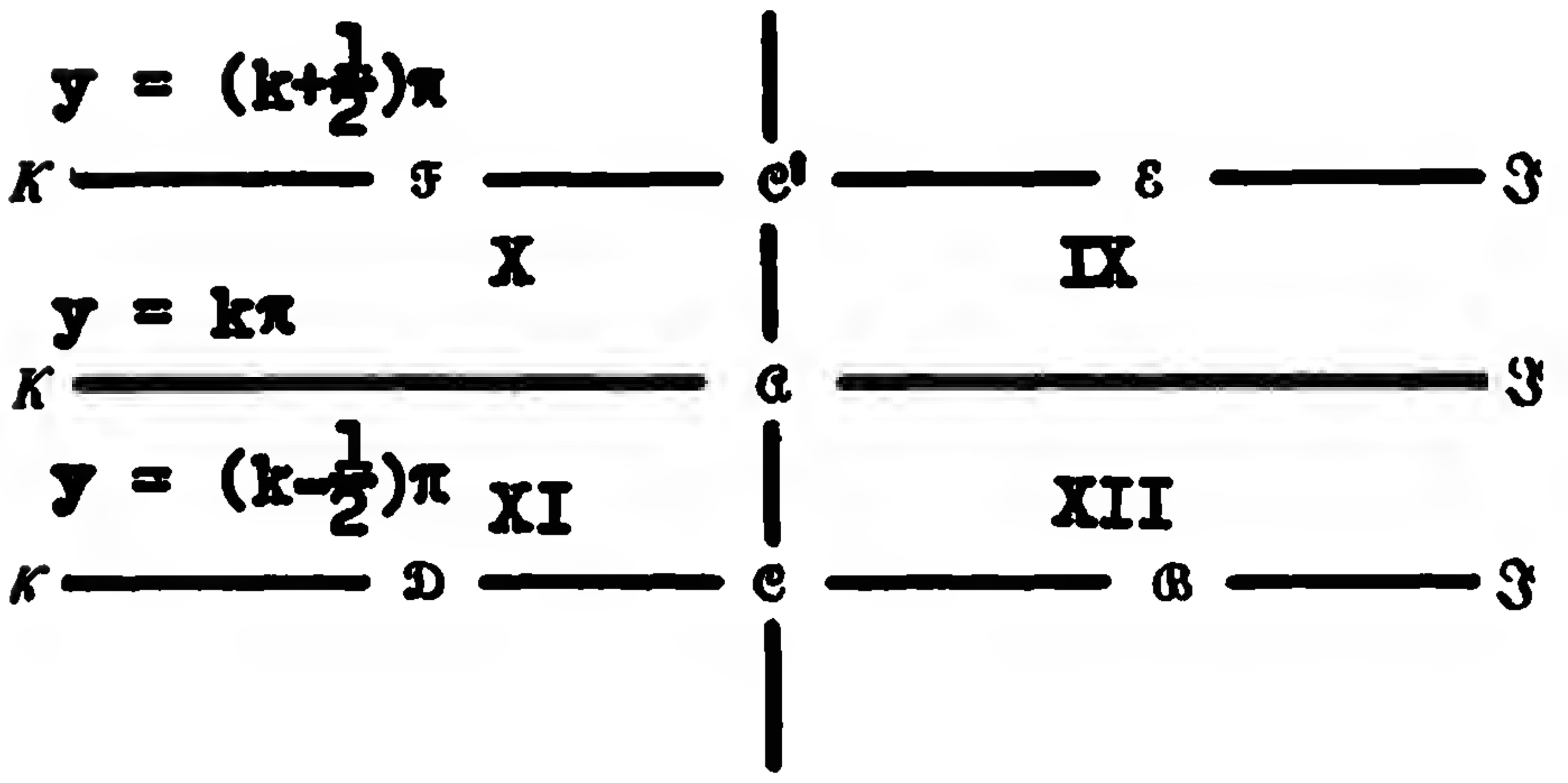
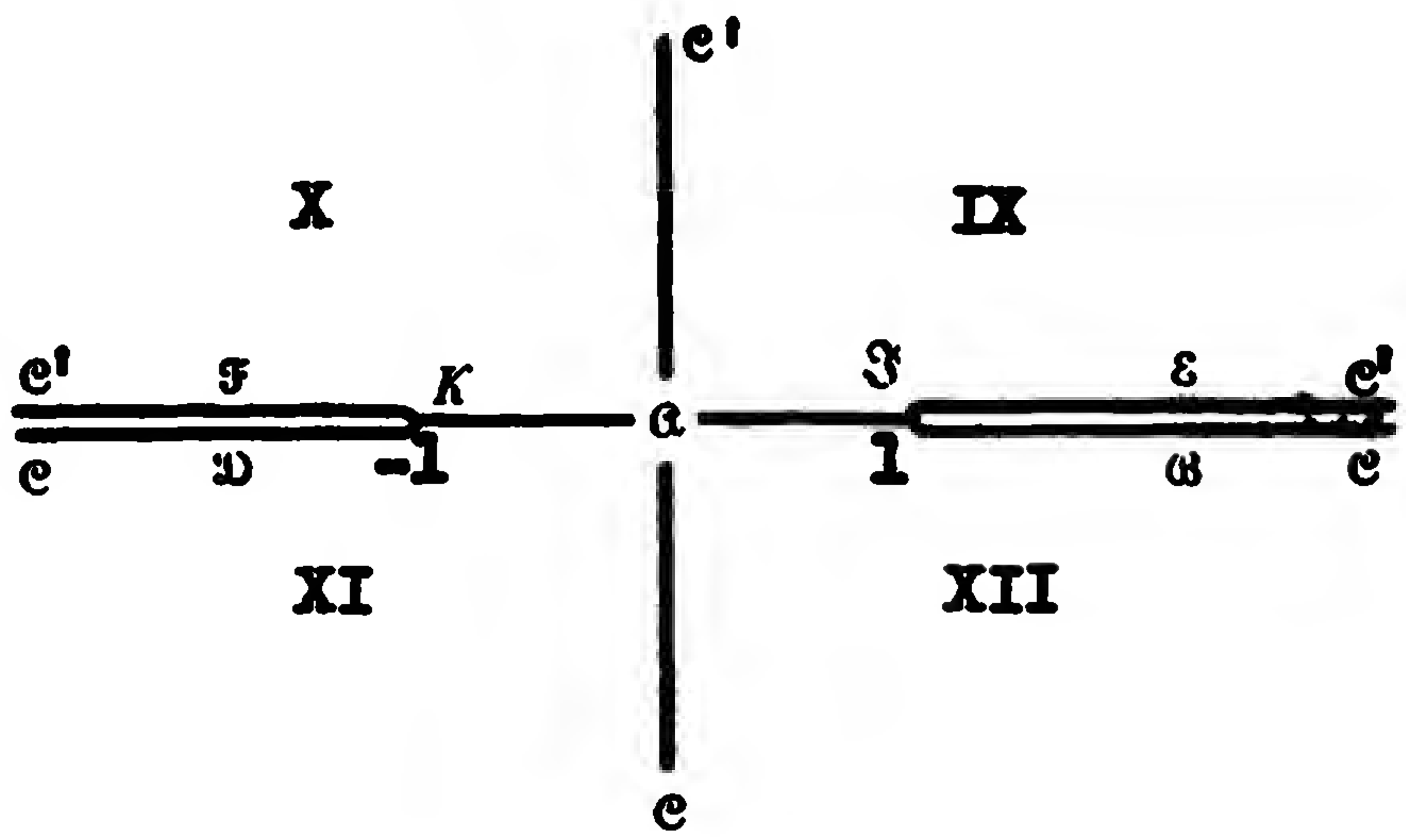


10.2 $\boxed{w = \tanh z = \frac{e^{2z}-1}{e^{2z}+1}}$; $\boxed{z = \frac{1}{2}\log\frac{1+w}{1-w} = \tanh^{-1} w}$

Critical points: $z = \pm \frac{i\pi}{2}, \pm \frac{3i\pi}{2}, \ldots;\quad z = \infty.$

<u>Infinite strip and semi-infinite strip.</u>

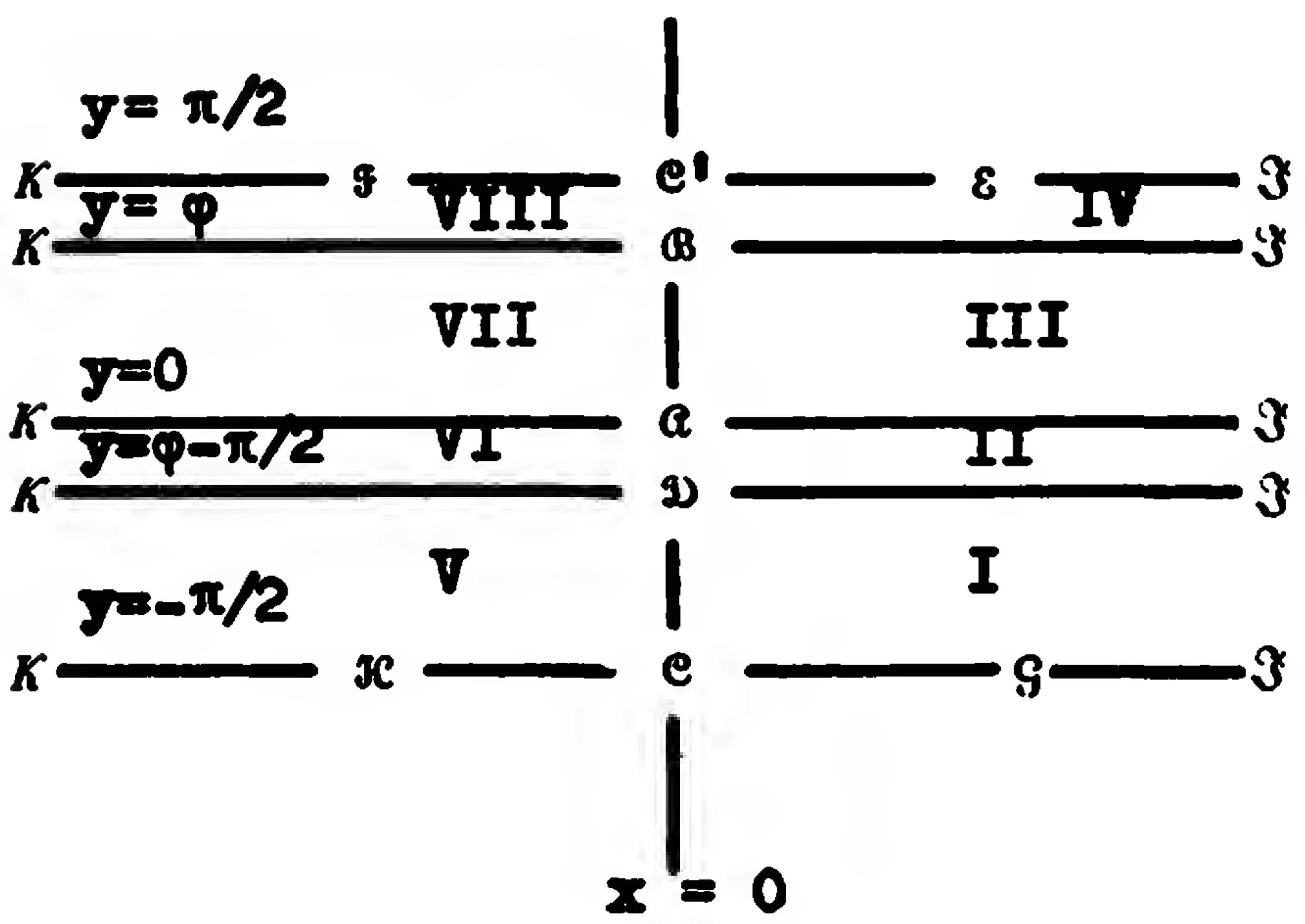
$k = 0, \pm 1, \pm 2, \ldots$; φ real and arbitrarily chosen

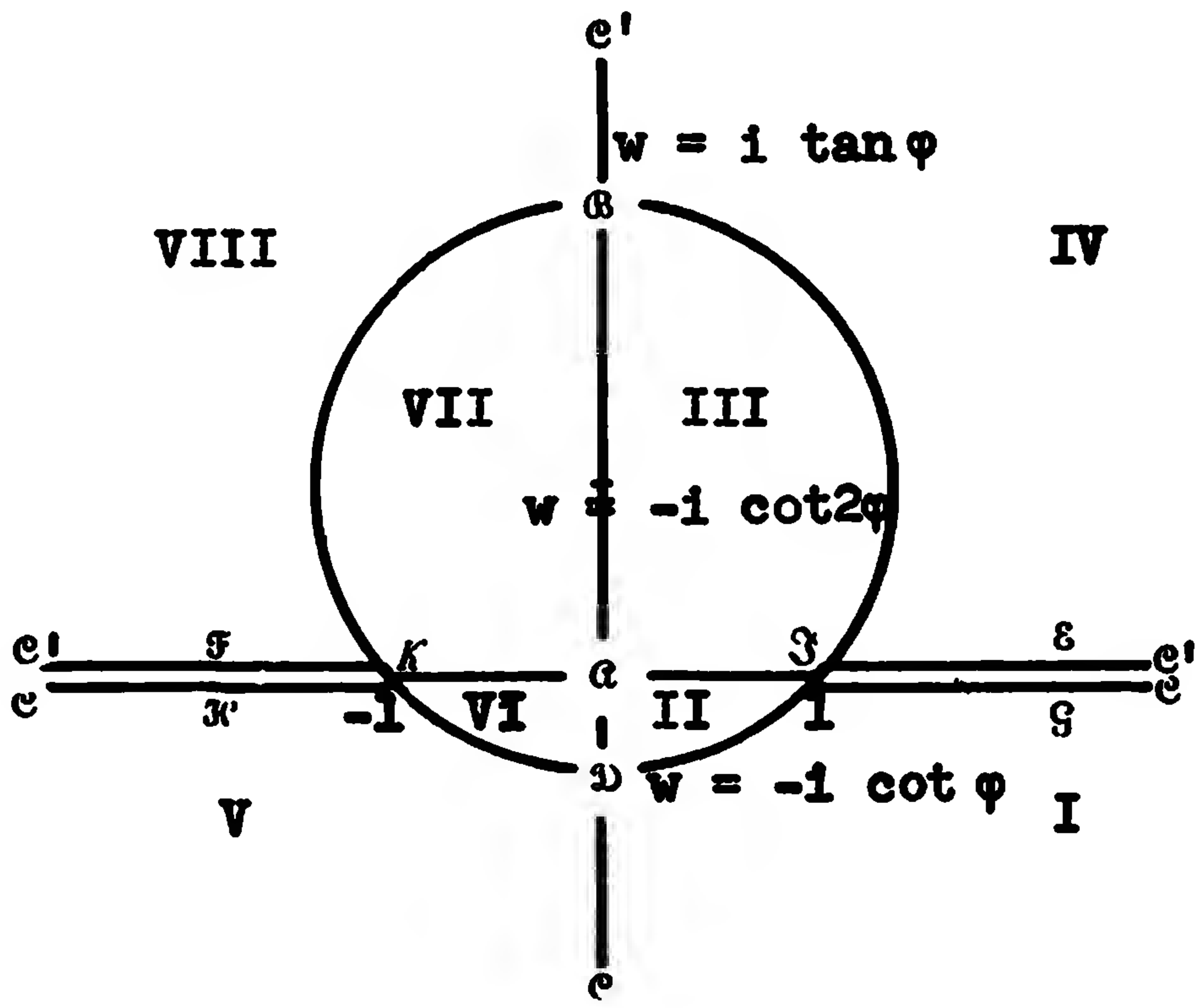
z - plane	w - plane
	
infinite strip $k\pi < y < (k+\frac{1}{2})\pi$	half-plane $v > 0$
infinite strip $k\pi < y < (k+1)\pi$	cut plane
infinite half-strip $k\pi < y < (k+\frac{1}{2})\pi$, $x > 0$	quadrant $u > 0,\ v > 0$
infinite half-strip $k\pi < y < (k+1)\pi$, $x > 0$	half-plane $u > 0$, cut from $w = 1$ to $w = 0$
	

z - plane	w - plane
points $z = i\varphi + k\pi i$; $k\pi i$; $(k+\frac{1}{2})\pi i$	$w = i \tan\varphi$; 0; ∞
line segment $x = 0$, $\varphi \leq y < \varphi + \pi$	line $u = 0$, $-\infty < v < \infty$; point $v = \pm\infty$ corresponding to $y_1 = (k_1+\frac{1}{2})\pi$ (k_1 integer)
line $y = k\pi$, $-\infty < x < \infty$	line segment $v = 0$, $-1 < u < 1$
line $y = (k+\frac{1}{2})\pi$, $-\infty < x < \infty$	line $v = 0$, excluding the segment $-1 < u < 1$; $u = \infty$ corresponding to $x = 0$

Set of coaxal circles, passing through $w = -1$ and $w = 1$.

z - plane	w - plane
lines $y = (k+\frac{1}{4})\pi$; $y = (k+\frac{3}{4})\pi$	semi-circle $\|w\| = 1$, $v > 0$, or $\|w\| = 1$, $v < 0$, respectively



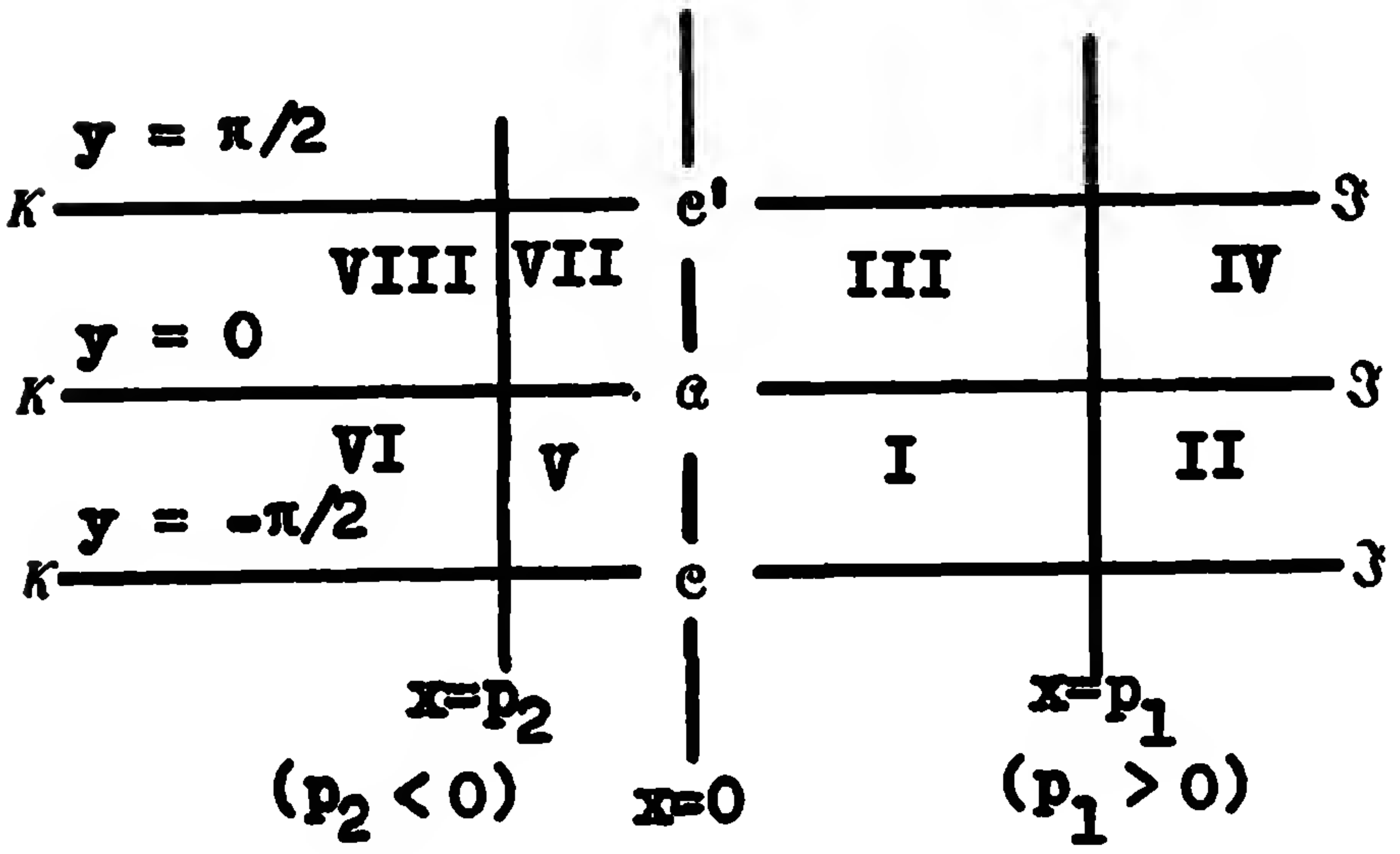


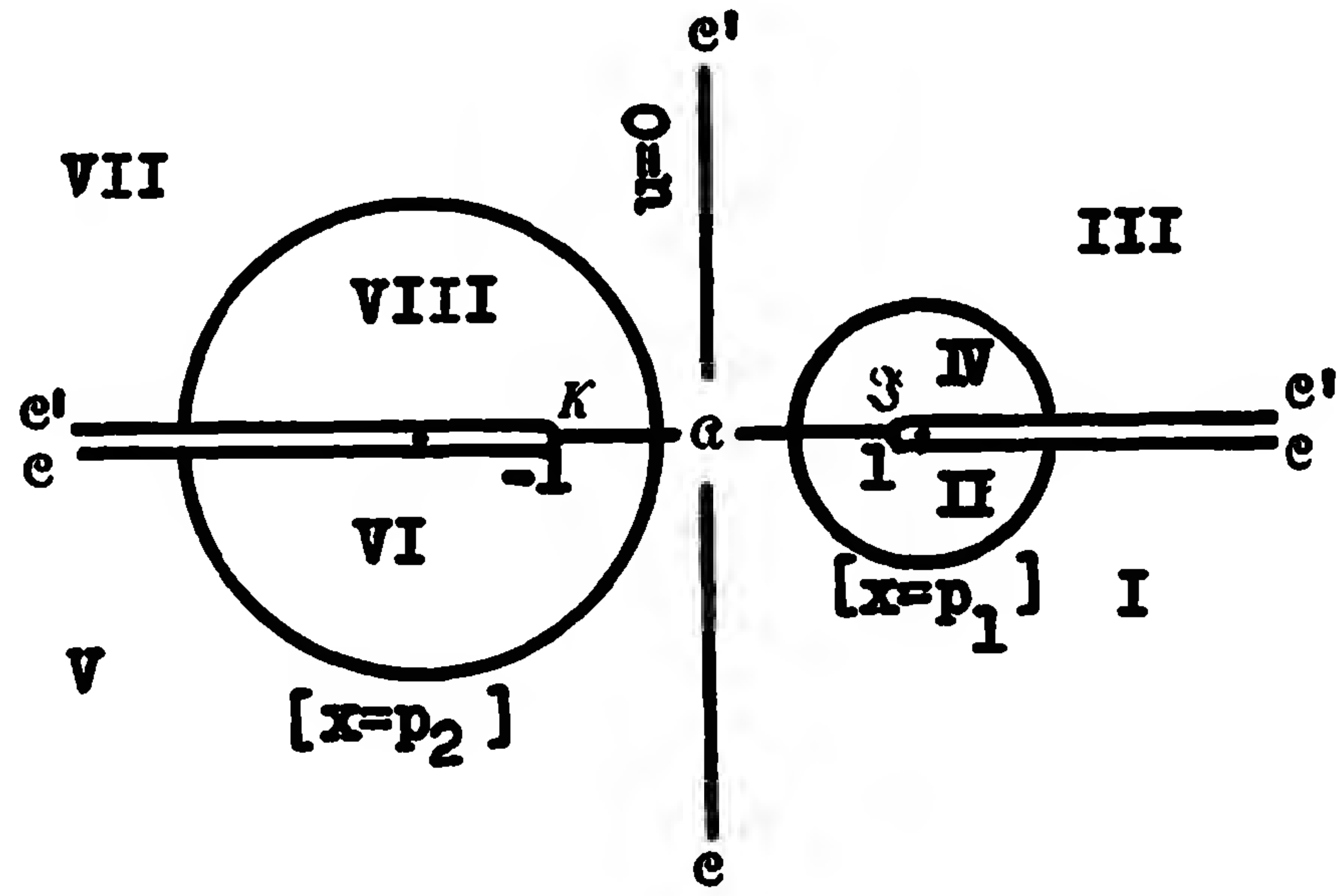
z - plane	w - plane
line $y = \varphi + k\pi$; $2\varphi/\pi$ not an integer, φ constant	arc $(-1, i\tan\varphi, 1)$ of circle $\|w + i\cot 2\varphi\| = \|\operatorname{cosec} 2\varphi\|$

z - plane	w - plane
line $y = \varphi + (k \pm \frac{1}{2})\pi$ [in the figure, $0 < \varphi < \frac{1}{2}\pi$, $k = 0$]	arc $(-1, -i \cot \varphi, 1)$ of the same circle

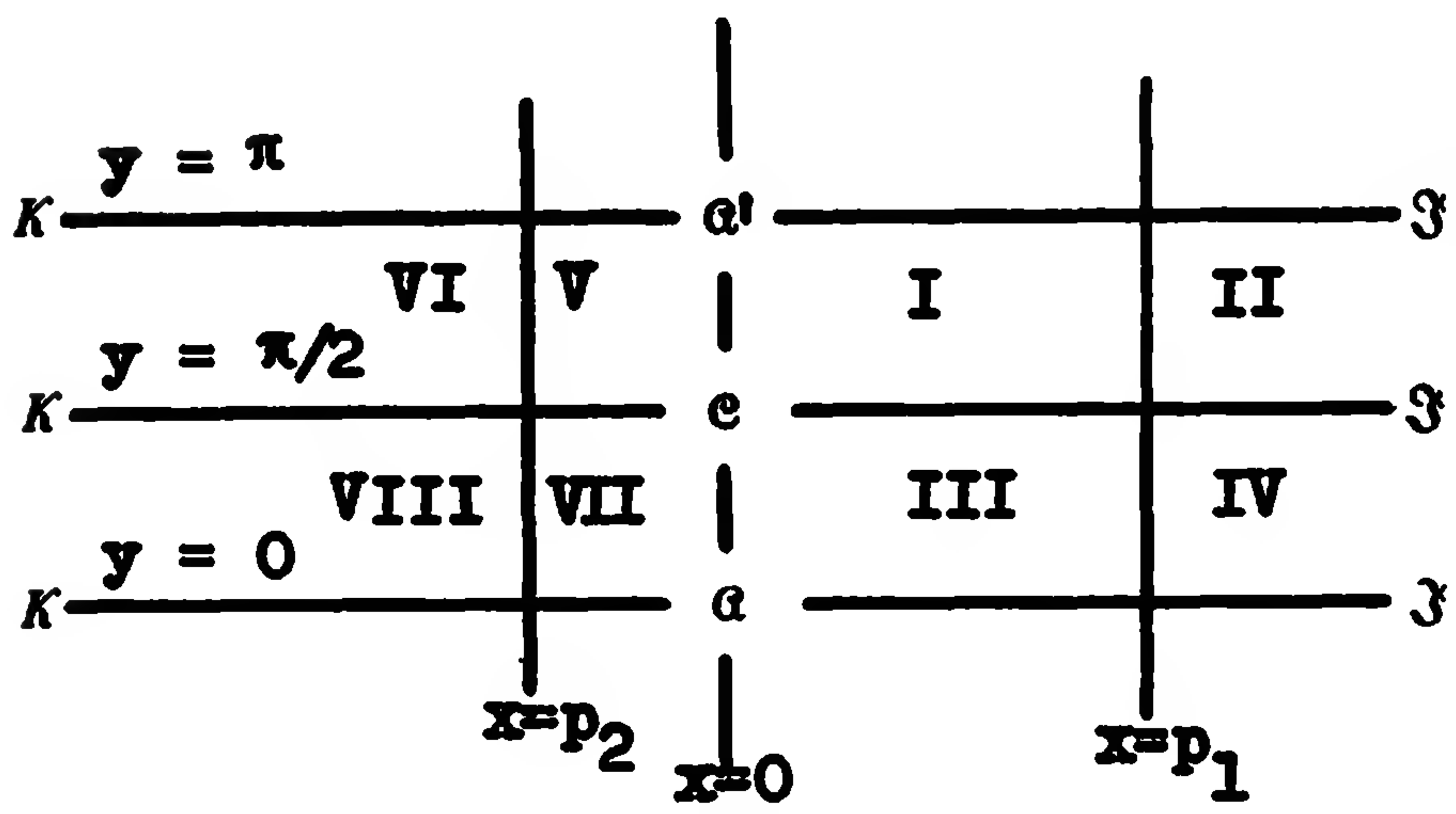
Set of coaxal circles $|w - \coth 2p| = |\sinh 2p|^{-1}$, with limiting points $-1, 1$.

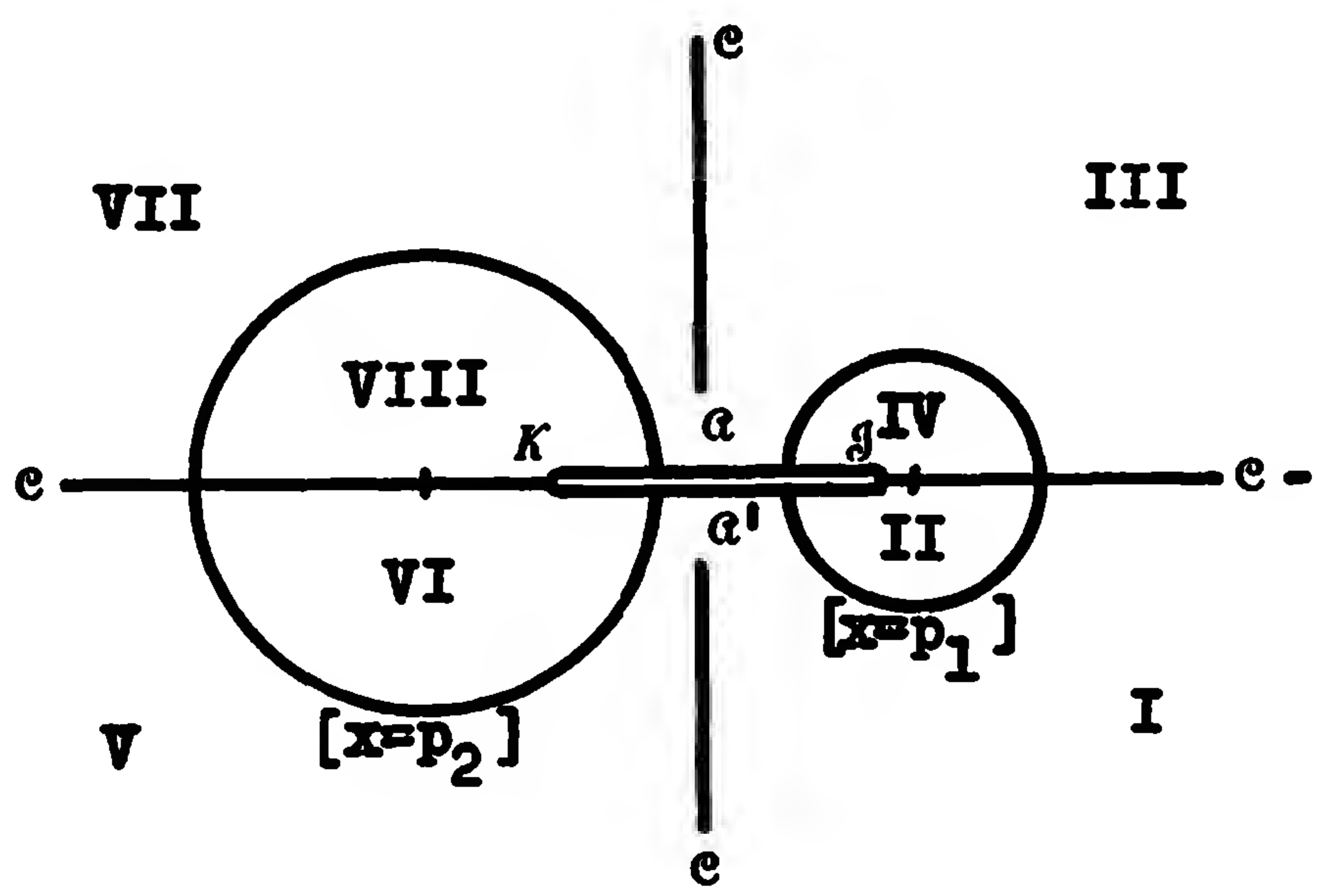
z - plane	w - plane
line segment $x = p$ $(p \gtrless 0)$, $\varphi \leq y < \varphi + \pi$	circle $\lvert w - \coth 2p \rvert = \lvert \sinh 2p \rvert^{-1}$



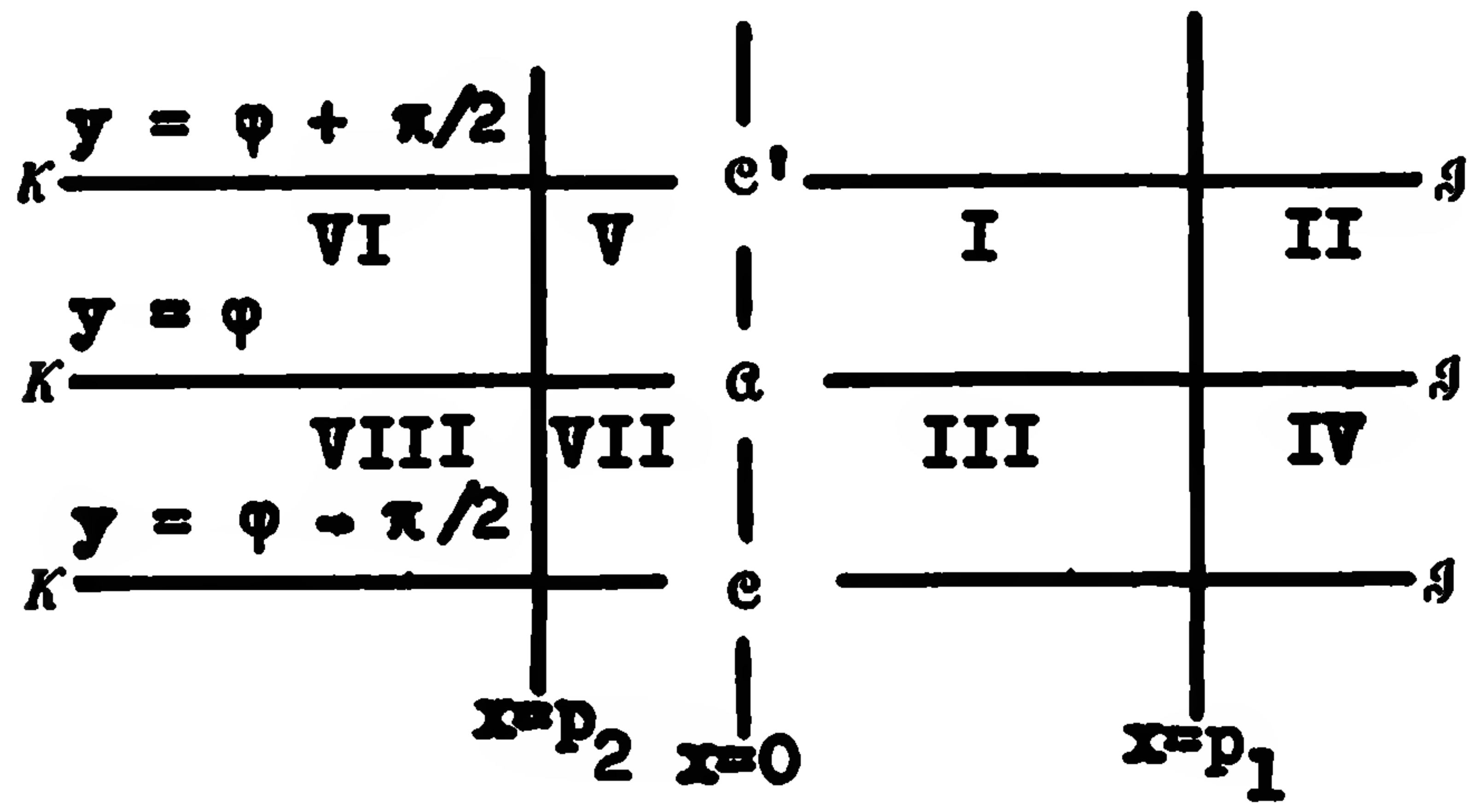
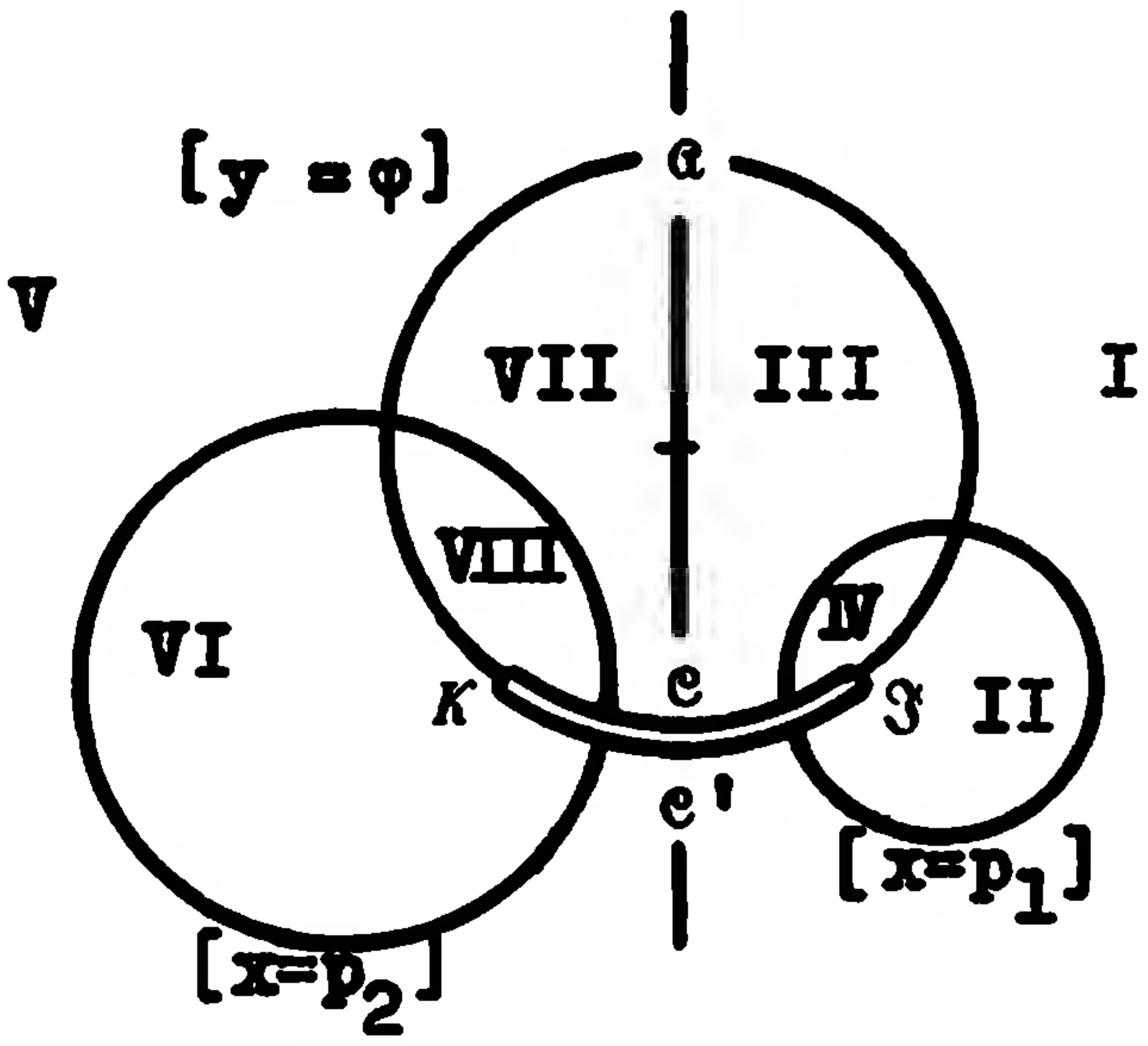


rectangle (I + III + V + VII) case $\varphi = 0$	exterior of the two circles, with two slits





rectangle (I + III + V + VII) case $\varphi = \pi/2$	exterior of the two circles, with one slit

z - plane	w - plane
	
here $\varphi \neq k\pi/2$	
rectangle (I + III + V + VII)	exterior of the two smaller circles, with slit along arc of $\lvert w + i \cot 2\varphi \rvert = \lvert \operatorname{cosec} 2\varphi \rvert$.

10.3 Functions related to $w = \tanh z$.

(1) $\boxed{w = \coth z} = \tanh (z + i\pi/2)$; $\boxed{z = \frac{1}{2} \log \frac{w+1}{w-1}}$

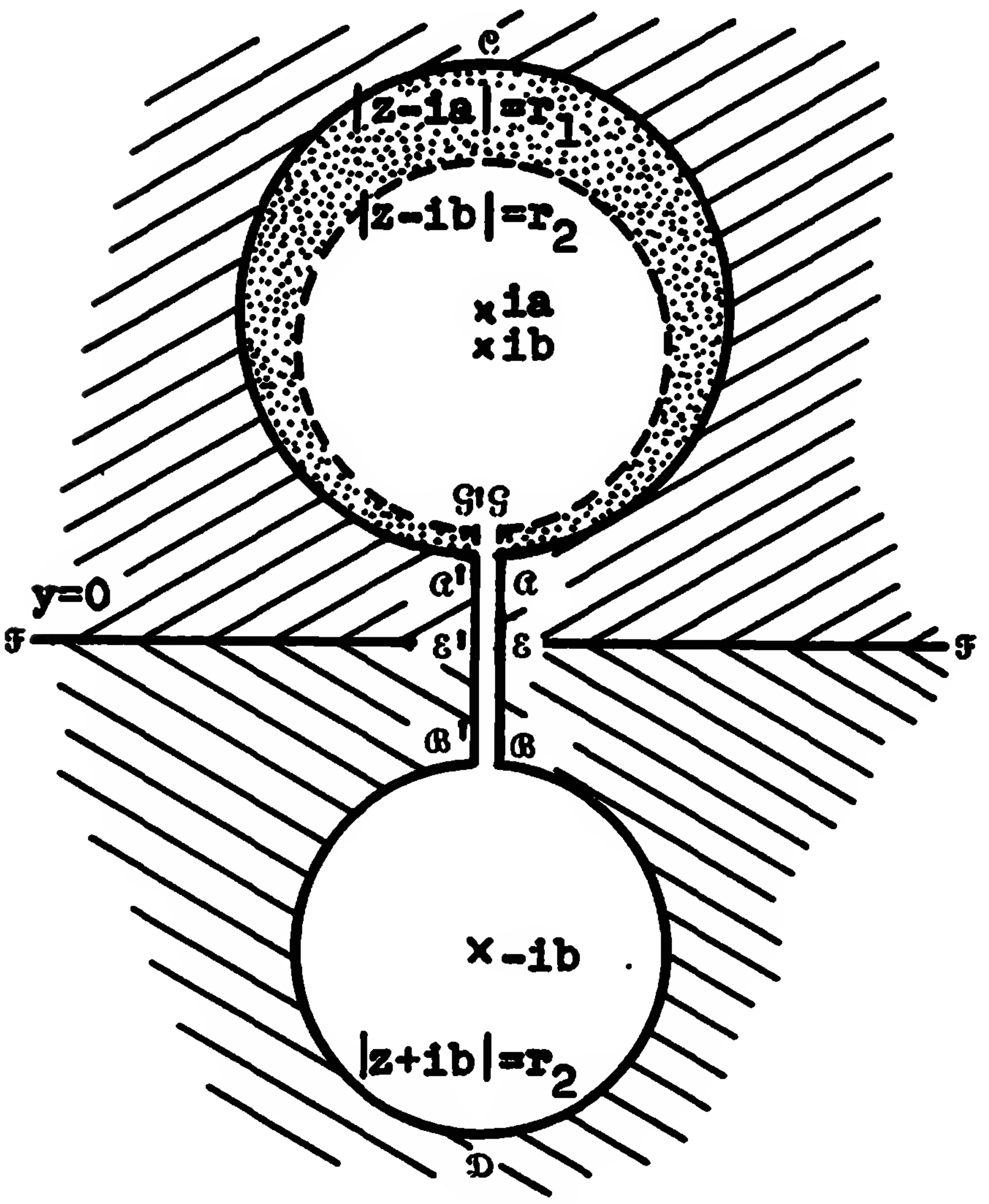
(2) $\boxed{w = \tan z} = -i \tanh iz$; $z = \tan^{-1} w = \frac{1}{2i} \log \frac{1+iw}{1-iw}$.

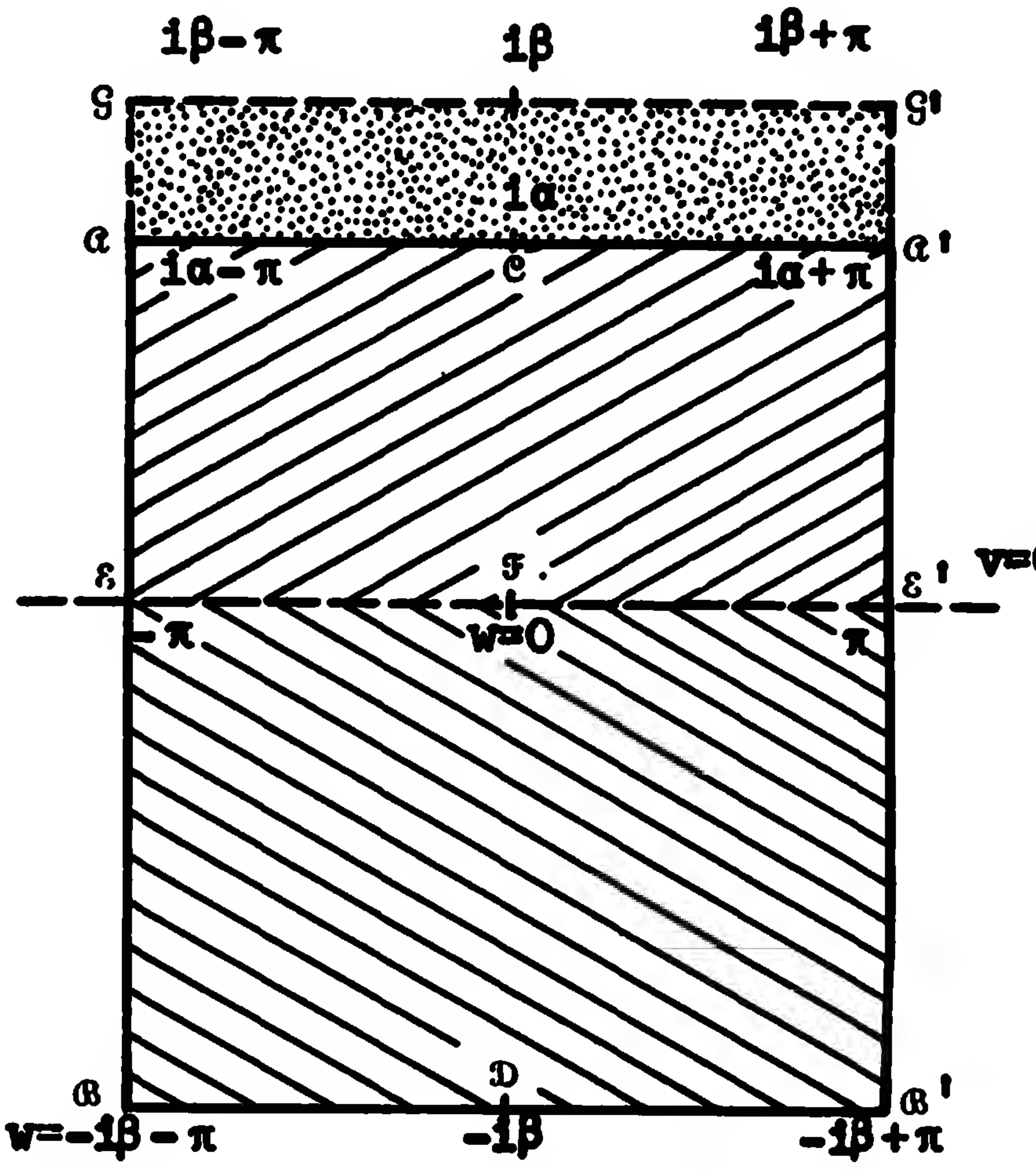
In all the figures of §10.2, both the z-plane and the w-plane are turned through $\pi/2$ about the origin. For the lines $u = p$ and $v = q$, see E. Jahnke and F. Emde, page 71 of appendix, figure 30. For the lines R = constant and θ = constant, where $\tan z = Re^{i\theta}$ (i.e., $\log \tan z = \log R + i\theta$), see §11.7.

(3) $\boxed{w = \cot z} = i \tanh (iz + i\pi/2)$; $z = \cot^{-1} w = \frac{i}{2} \log \frac{iw+1}{iw-1}$.

Example: Region exterior to two non-intersecting circles; cut annular region. $w = -2 \cot^{-1} (z/c)$.

z - plane	w - plane
$0 < c < a$, $c < b$; $r_1^2 = a^2 - c^2$, $r_2^2 = b^2 - c^2$; the circles have $y = 0$ as radical axis	$\alpha = \log \frac{r_1}{a-c} > 0$; $\beta = \log \frac{r_2}{b-c} > 0$.





z - plane	w - plane
cut region exterior to $\lvert z-ia \rvert = r_1$ and $\lvert z+ib \rvert = r_2$.	interior of rectangle with vertices $i\alpha \pm \pi$, $-i\beta \pm \pi$.
annular region between $\lvert z-ia \rvert = r_1$ and $\lvert z-ib \rvert = r_2$, cut along $x = 0$, $a-r_1 \leq y \leq b-r_2$, i.e., $\alpha\, \mathcal{G}$ $(a \neq b)$.	interior of rectangle with vertices $i\alpha \pm \pi$, $i\beta \pm \pi$.

$\boxed{w = A \log \frac{az+b}{cz+d}}$; $Aac \neq 0$, $ad-bc \neq 0$; $\boxed{z = \frac{a'e^{\alpha w} + b}{ce^{\alpha w} + d'}}$,

where $\alpha = A^{-1}$, $a' = -d$, $d' = -a$. Combination of $w = 2A\xi + A(\log \frac{a}{c} + \pi i)$,

$z = -\frac{ad+bc}{2ac} + \frac{bc-ad}{2ac}\zeta$, and $\xi = \tanh \zeta$.

$\boxed{w = \frac{\cos(z+a)}{\cos(z+b)}}$, $\frac{a-b}{\pi}$ not an integer; $z = -b + \tan^{-1} \frac{w-\beta}{\alpha}$,

where $\alpha = \sin(b-a)$, $\beta = \cos(b-a)$.

Critical points: $z = (k + \frac{1}{2})\pi i - b$; ∞ $[k = 0, \pm 1, \ldots]$.

This is a combination of: $w = \alpha\xi + \beta$, $\zeta = z + b$, and $\xi = \tan \zeta$.

z - plane	w - plane
points $z = -\frac{a+b}{2} + k\pi$;	points $w = 1$;
$z = -a + (k + \frac{1}{2})\pi$; $-b + (k + \frac{1}{2})\pi$	0; ∞
strip $k\pi - \Re(b) < x < (k + \frac{1}{2})\pi - \Re(b)$	half-plane $\Re(\frac{w-\beta}{\alpha}) > 0$
strip $(k - \frac{1}{2})\pi - \Re(b) < x < k\pi - \Re(b)$	half-plane $\Re(\frac{w-\beta}{\alpha}) < 0$

$\boxed{w = \frac{\sin(z + a)}{\sin(z + b)}} = \frac{\cos(z + a')}{\cos(z + b')}$, where $a' = a - \frac{\pi}{2}$, $b' = b - \frac{\pi}{2}$.

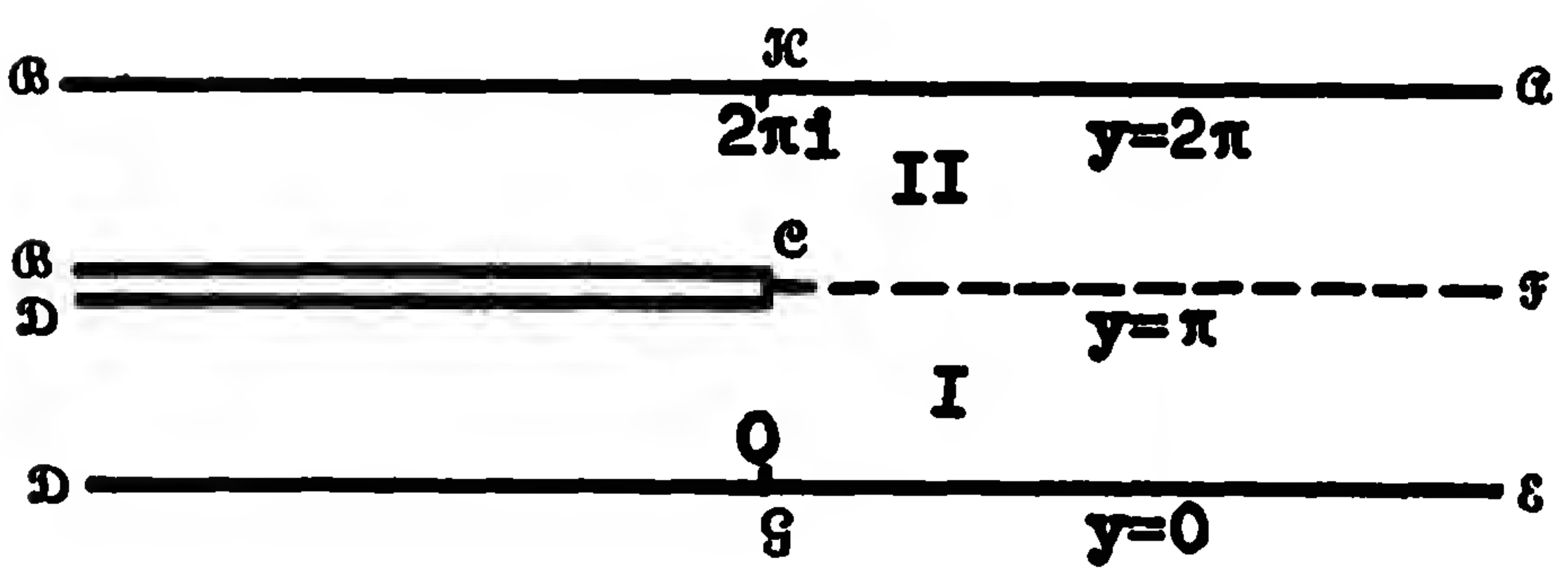
Or $w = \beta + a \tan(z + b - \frac{\pi}{2})$.

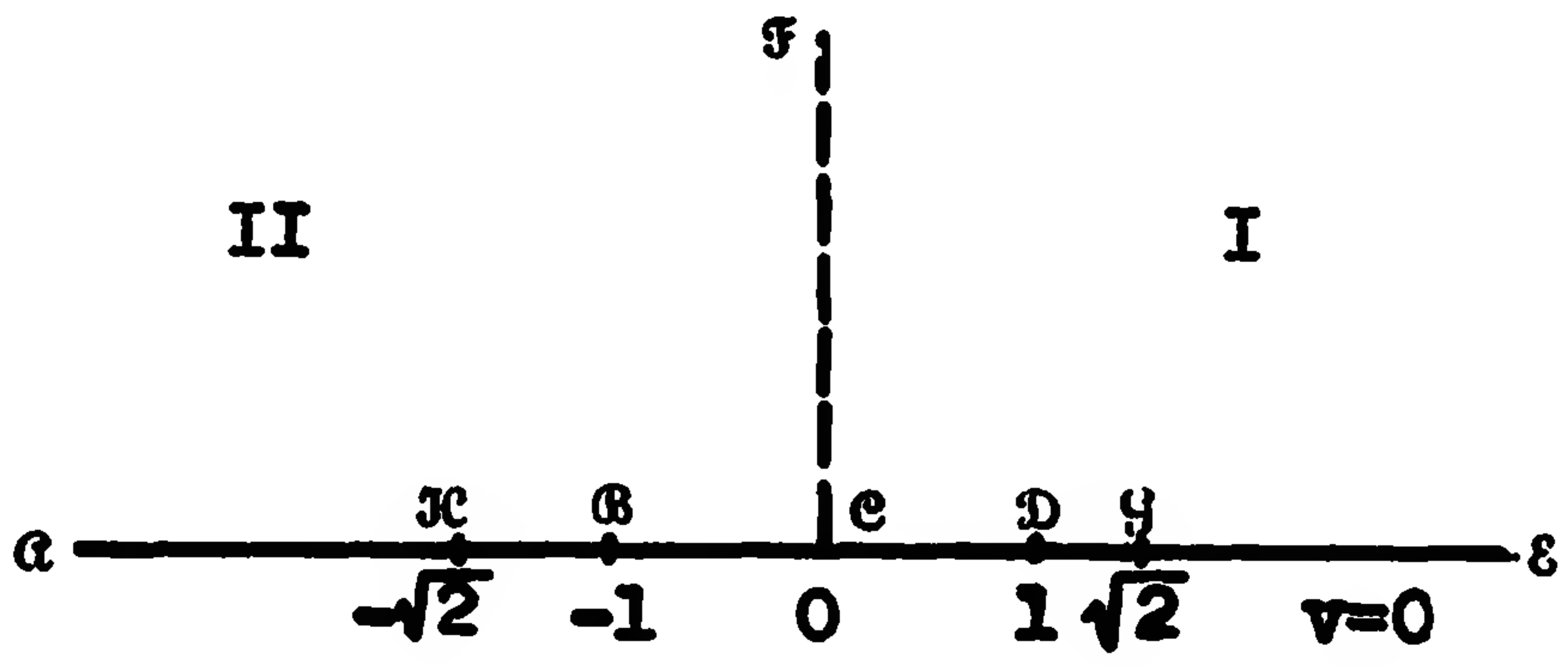
10.4 $\boxed{w = \sqrt{(1 + e^z)}}$, $\boxed{z = \log(w + 1)(w - 1)}$, cf. §11.2.

Critical points: $z = (2k + 1)\pi i$; ∞.

$k = 0, \pm 1, \pm 2, \ldots$

z - plane	w - plane
point $z_o + 2k\pi i$	points $\pm\sqrt{(1 + e^{z_o})}$
$0 \leq y \leq 2\pi$	$v \geq 0$
strip $0 < y < 2\pi$, cut along $y = \pi$, $x \leq 0$	half-plane $v > 0$





z - plane	w - plane
lines $y = 0$ or $y = 2\pi$, respectively, $(-\infty < x < \infty)$	half-line $v = 0$, $1 < u < \infty$ or $-1 > u > -\infty$
half-line $y = \pi$, $0 \geq x > -\infty$	segment $v = 0$, $0 \leq u < 1$ segment $v = 0$, $0 \geq u > -1$
half-line $y = \pi$, $0 \leq x < \infty$	half-line $u = 0$, $0 \leq v < \infty$
line-segment $x = p$, $0 < y < 2\pi$; $p > 0$	part $v > 0$ of Cassinian $\lvert w+1\rvert\lvert w-1\rvert = e^p$
line-segment $x = p$, $0 < y < \pi$; $p \leq 0$	part $u > 0$, $v > 0$ of $\lvert w+1\rvert\lvert w-1\rvert = e^p$

z - plane	w - plane
line-segment $x = p$, $\pi < y < 2\pi$; $p \leq 0$	part $u < 0$, $v > 0$ of the same Cassinian
line $y = q$, $-\infty < x < \infty$; $q \neq \pi$, $0 < q < 2\pi$	part of rectangular hyperbola $\Re\left\{w^2/(1-e^{2iq})\right\} = 1/2$
line $y = q$, $0 < q < \pi$	part $u > 0$, $v > 0$ of this hyperbola
line $y = q$, $\pi < q < 2\pi$	part $u < 0$, $v > 0$ of this hyperbola

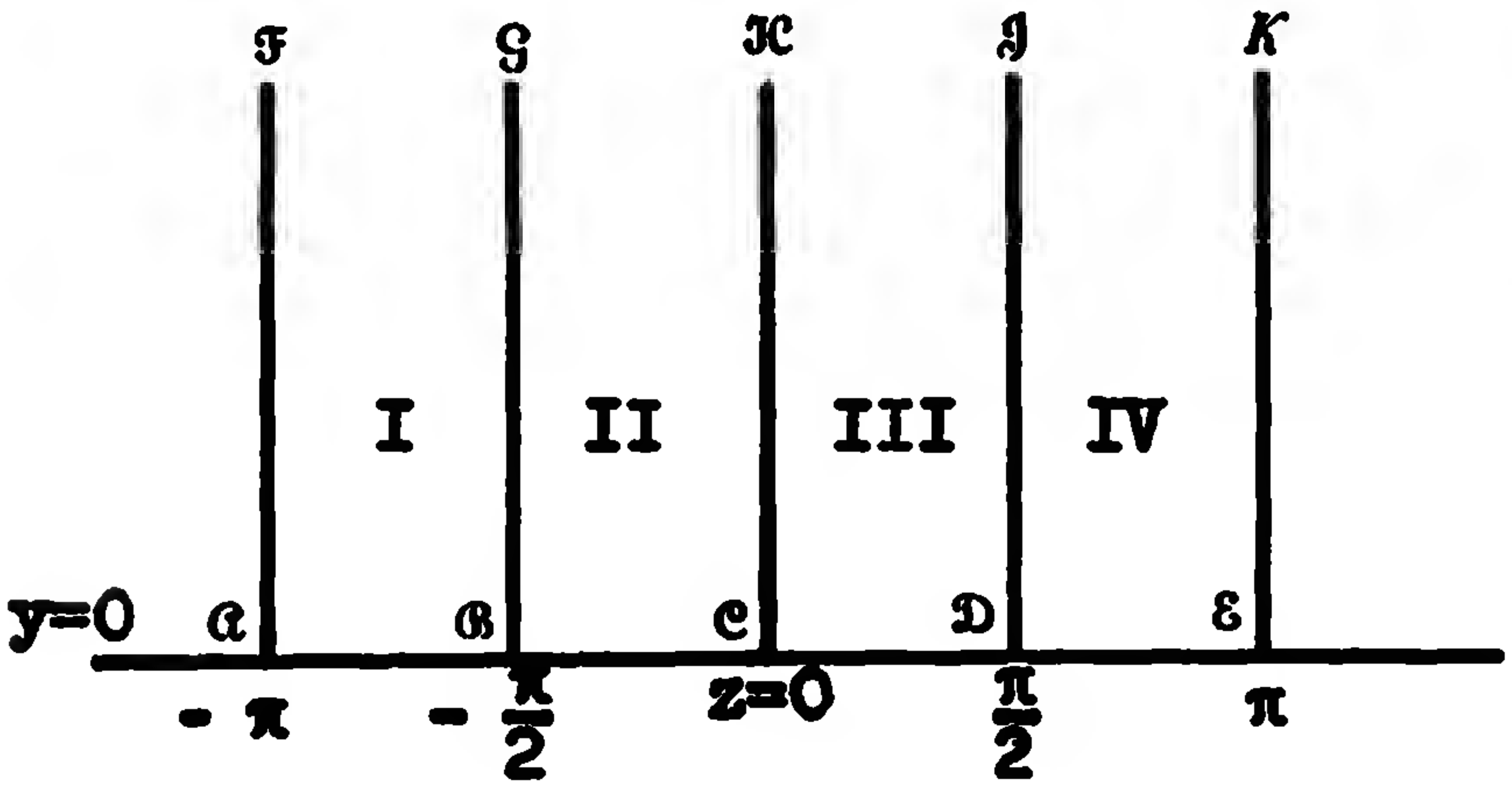
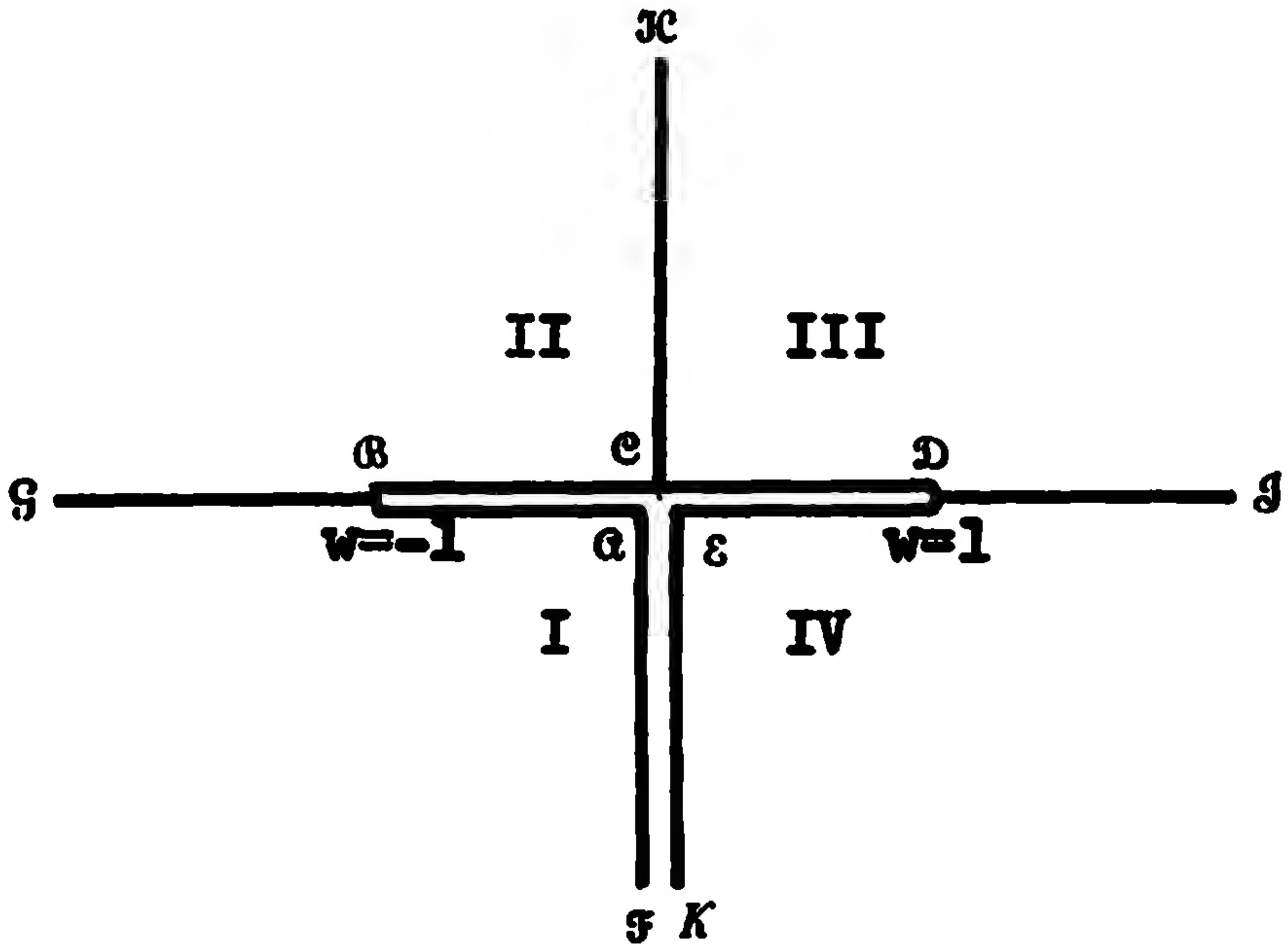
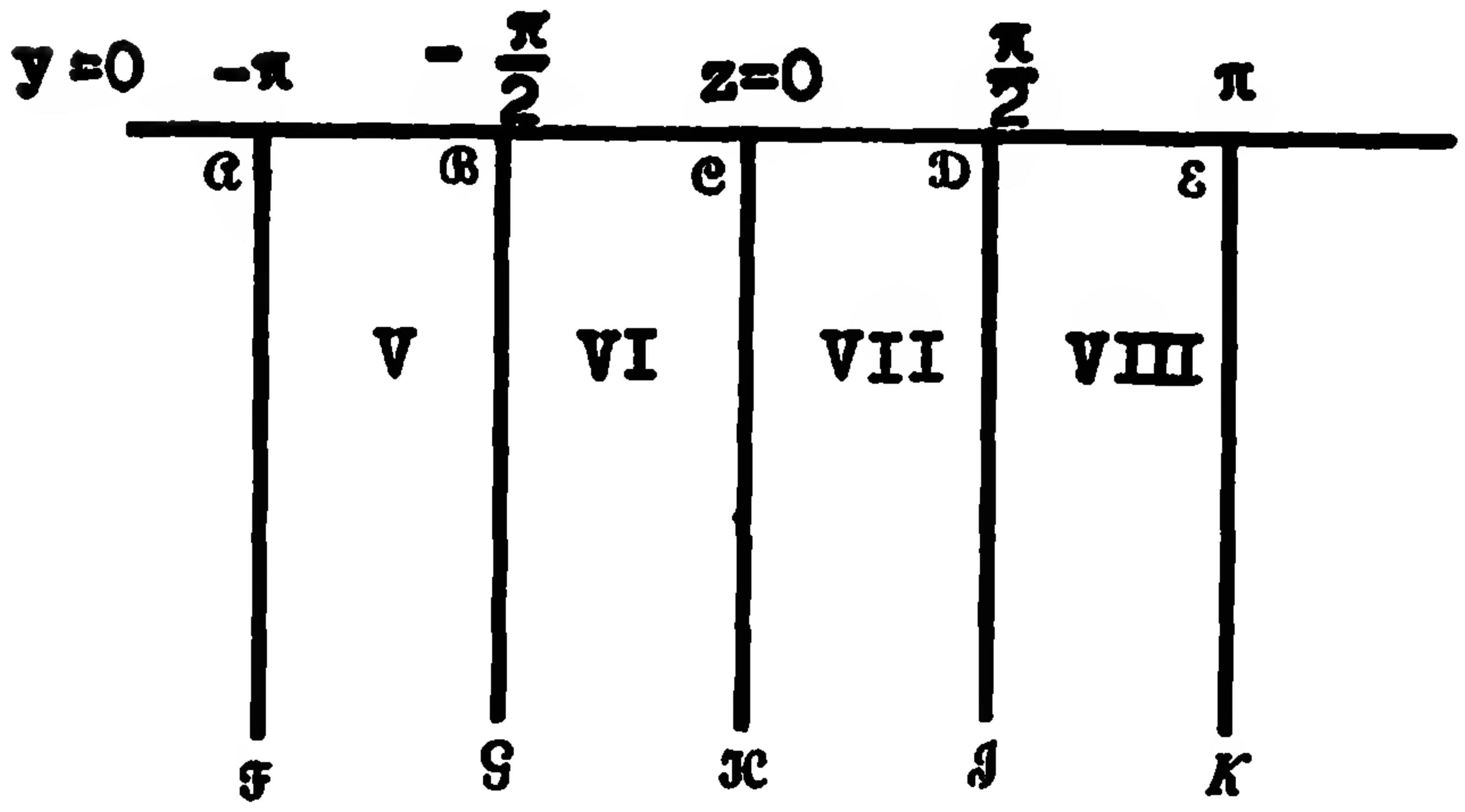
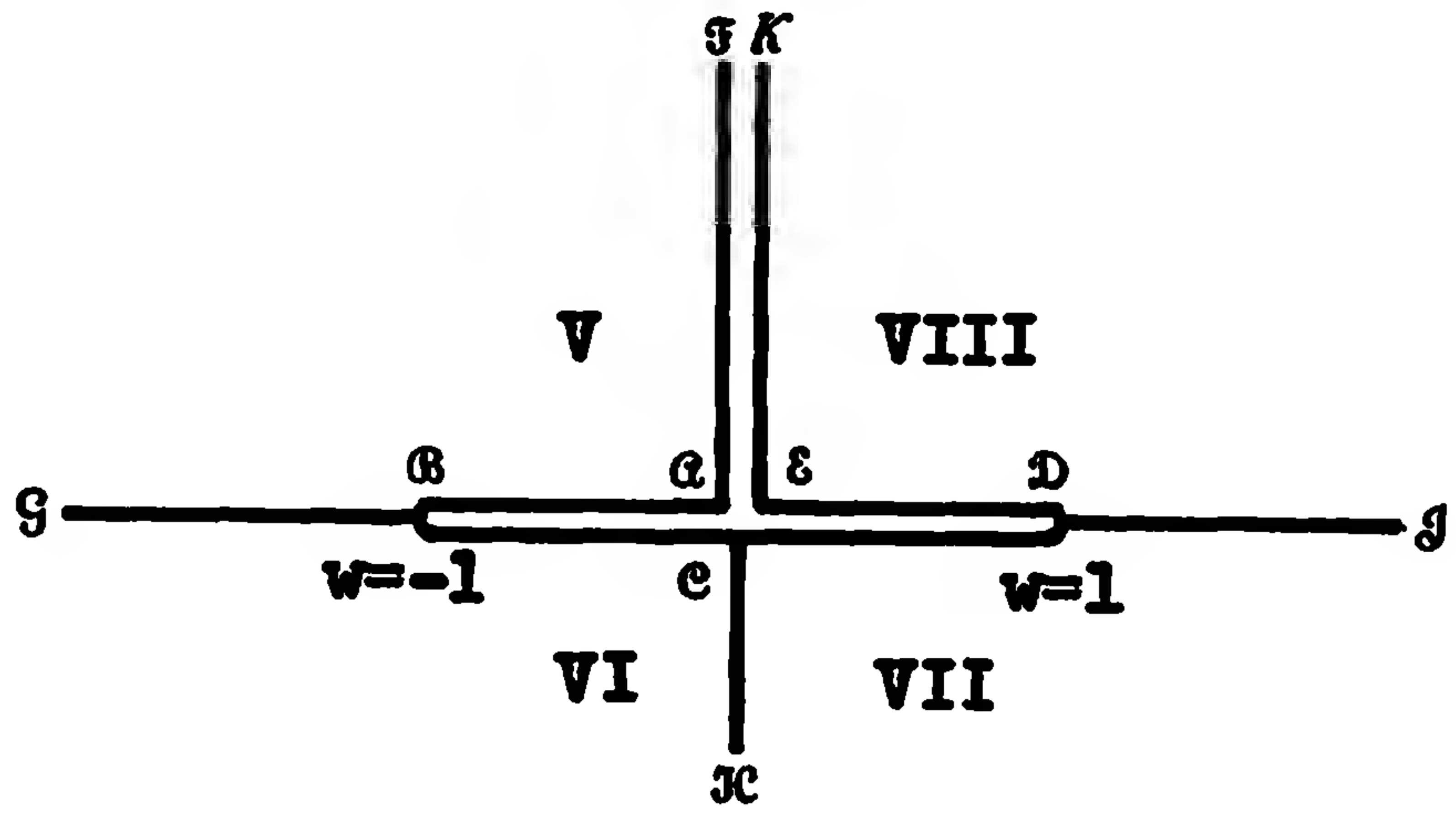
$w = c + \sqrt{(d+e^{\alpha z})}$ $(\alpha \neq 0$; c, d real, $d > 0)$; $\boxed{z = A \log(w-a)(w-b)}$ $(a, b$ real$)$, where $a = c - \sqrt{d}$, $b = c + \sqrt{d}$, $A = 1/\alpha$. Combination of $w = \frac{b+a}{2} + \frac{b-a}{2}\xi$ and $z = A\zeta + 2A \log \frac{b-a}{2}$ and $\zeta = \log(\xi+1)(\xi-1)$.

10.5 $\boxed{w = \sin z}$; $z = \sin^{-1} w = -i \log\left\{iw + \sqrt{(1-w^2)}\right\}$.

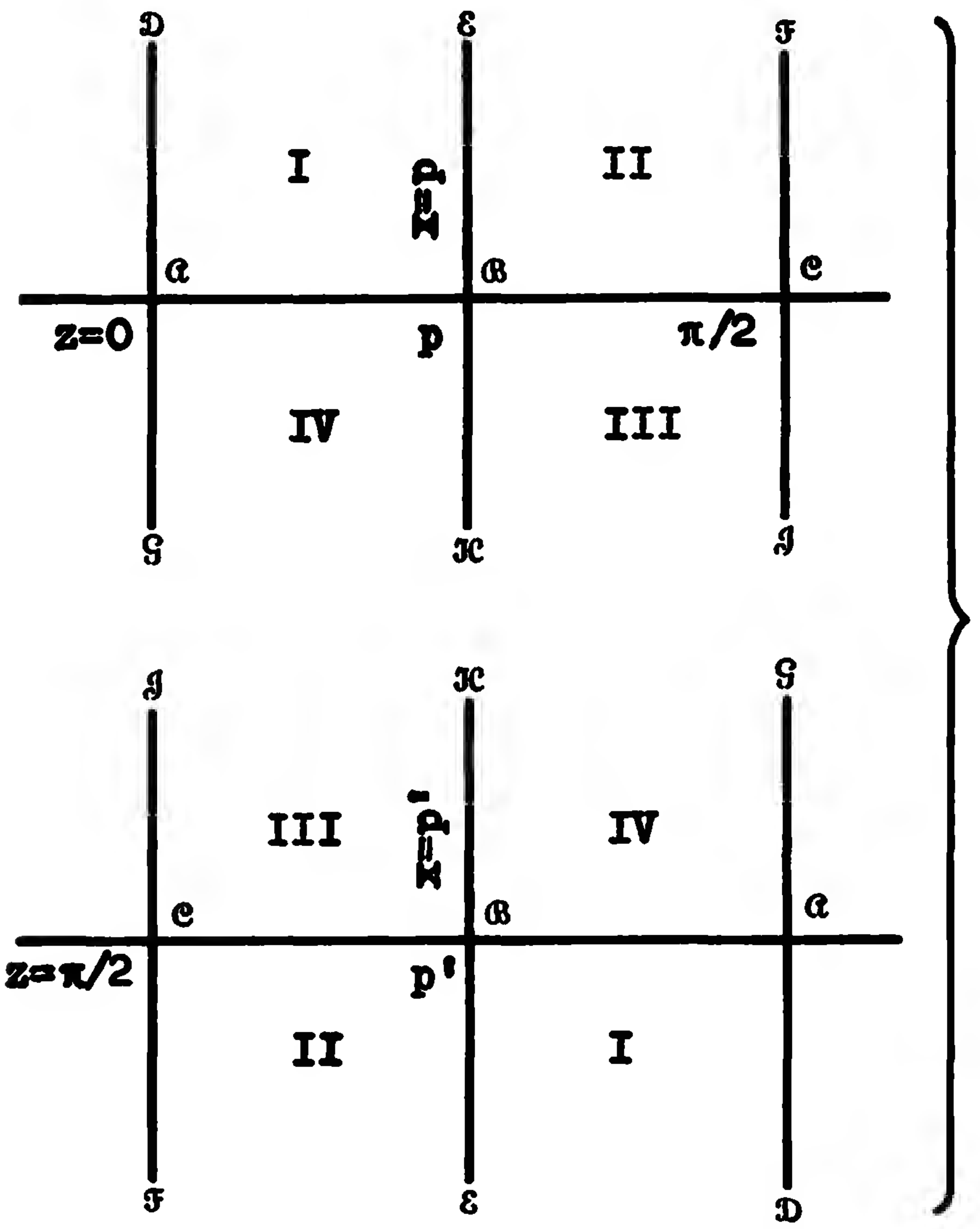
Critical points: $z = \infty$, $z = (2k + 1/2)\pi$.

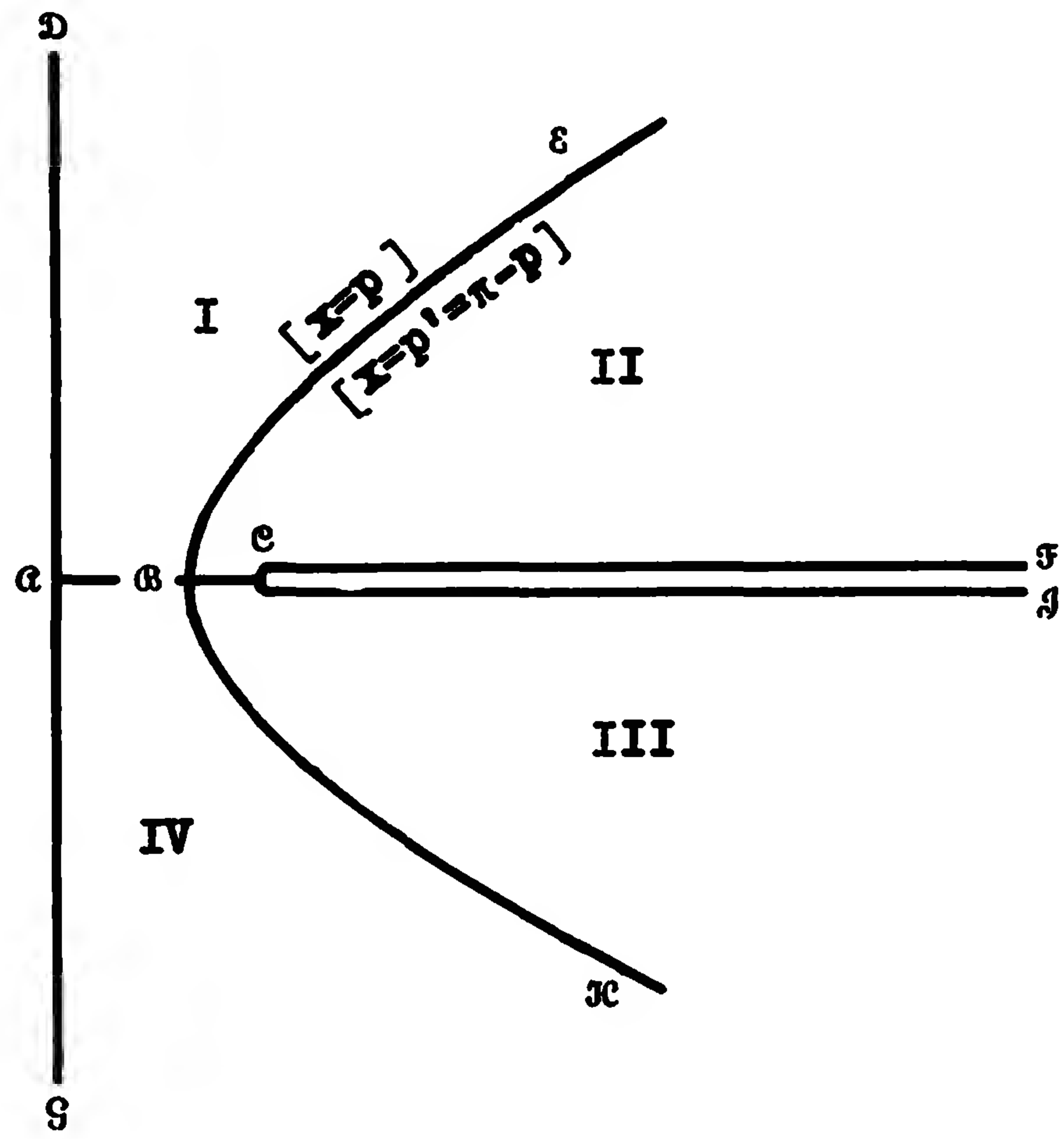
$k = 0, \pm 1, \pm 2, \ldots$; p, q real.

z - plane	w - plane
points $z = k\pi$; $(k + \frac{1}{2})\pi$; $(k + \frac{1}{2} \pm \frac{1}{3})\pi$; $(k + \frac{1}{2} \pm \frac{1}{6})\pi$	points $w = 0$; $(-1)^k$; $(-1)^k/2$; $(-1)^k \sqrt{3}/2$
line $x = k\pi$, $-\infty < y < \infty$	line $u = 0$, $\begin{matrix} -\infty < v < \infty \\ \infty > v > -\infty \end{matrix}$ for $\begin{matrix}\text{even}\\ \text{odd}\end{matrix}$ k.
$\left.\begin{matrix}\text{half-line } x = (k + \frac{1}{2})\pi,\ 0 \leq y < \infty \\ \text{half-line } x = (k + \frac{1}{2})\pi,\ 0 \geq y > \infty\end{matrix}\right\}$	half-line $v = 0$, with $\begin{matrix} 1 \leq u < \infty \\ -1 \geq u > -\infty\end{matrix}$ if k is $\begin{matrix}\text{even}\\ \text{odd}\end{matrix}$
$\left.\begin{matrix}\text{strip } 2k\pi < x < (2k + \frac{1}{2})\pi \\ \text{strip } (2k + \frac{1}{2})\pi < x < (2k + 1)\pi\end{matrix}\right\}$	half-plane $u > 0$, cut along D $\mathfrak{J}$

z - plane	w - plane
	
	
strip $(2k + 1)\pi < x < (2k + \frac{3}{2})\pi$ strip $(2k + \frac{3}{2})\pi < x < (2k + 2)\pi$	half-plane $u < 0$, cut along $\mathcal{B}\,\mathcal{G}$
line-segment $y = q$ $(q \neq 0)$, $2k\pi \leq x < (2k + 2)\pi$	ellipse $\|w+1\| + \|w-1\| = 2 \cosh q$
interior of rectangle $z = \pm\pi$, $\pm\pi + iq$ $(q > 0)$	interior of this ellipse, except for slit from -1 to 1 and for slit along negative part of $u = 0$
line $x = p + 2k\pi$ $(2p/\pi$ not an integer)	branch $\|w+1\| - \|w-1\| = 2 \sin p$ of hyperbola

z - plane	w - plane

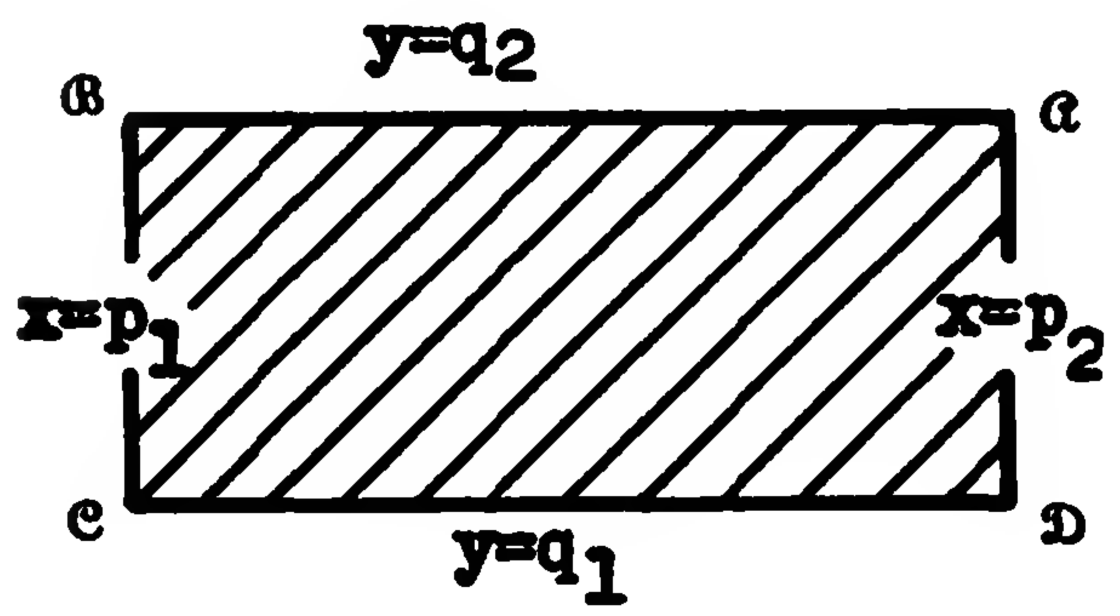




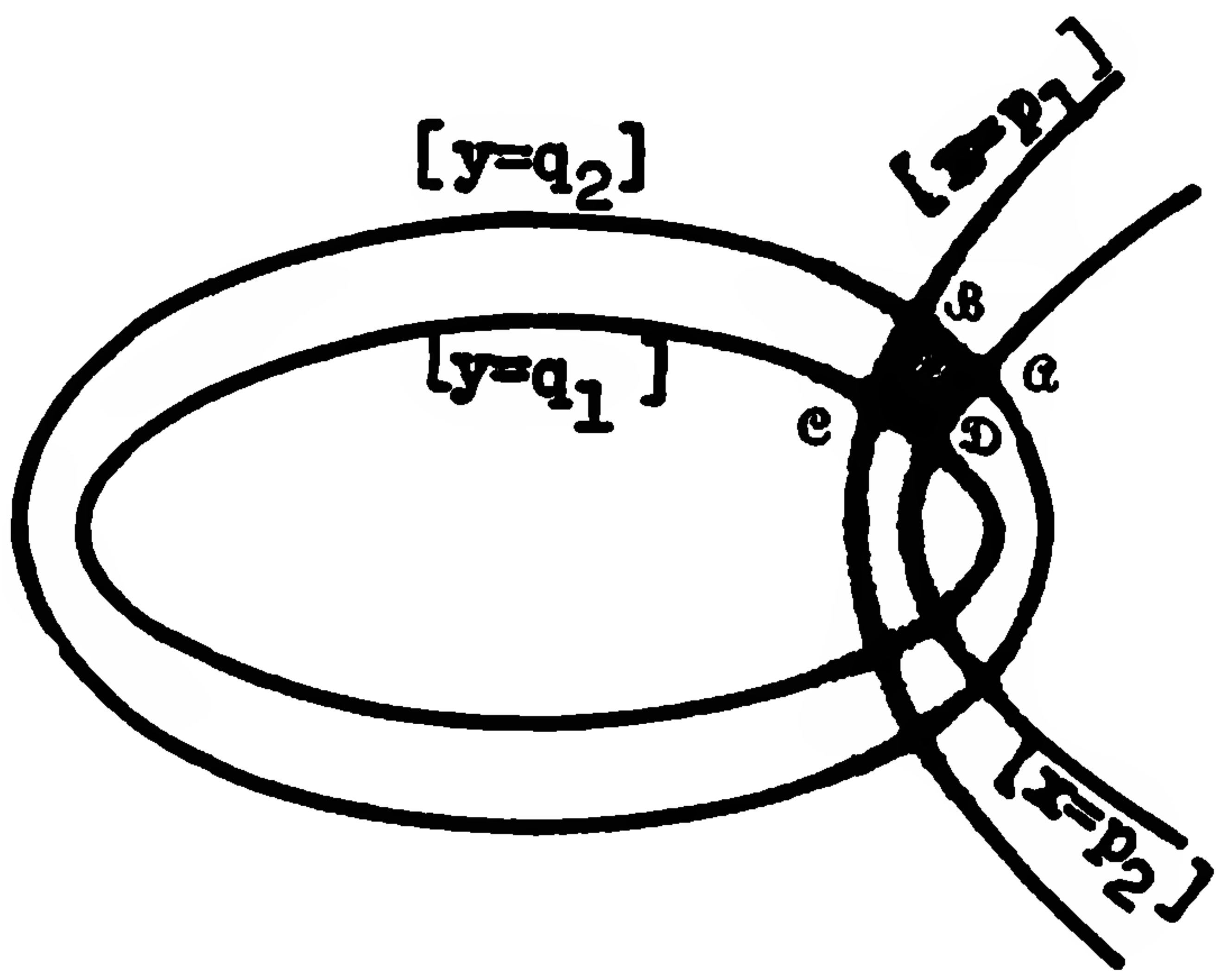
($p' = \pi - p$; $0 < p < \pi/2$; in the figure, $p = \pi/4$)

Interior of rectangle bounded by $x = p_1, p_2$, $y = q_1, q_2$, where $q_1 q_2 > 0$ and, for some k, $k\pi < p_1 < p_2 < (k + \frac{1}{2})\pi$ or $(k - \frac{1}{2})\pi < p_1 < p_2 < k\pi$

curvilinear quadrilateral 𝔄𝔅ℭ𝔇, with four right angles

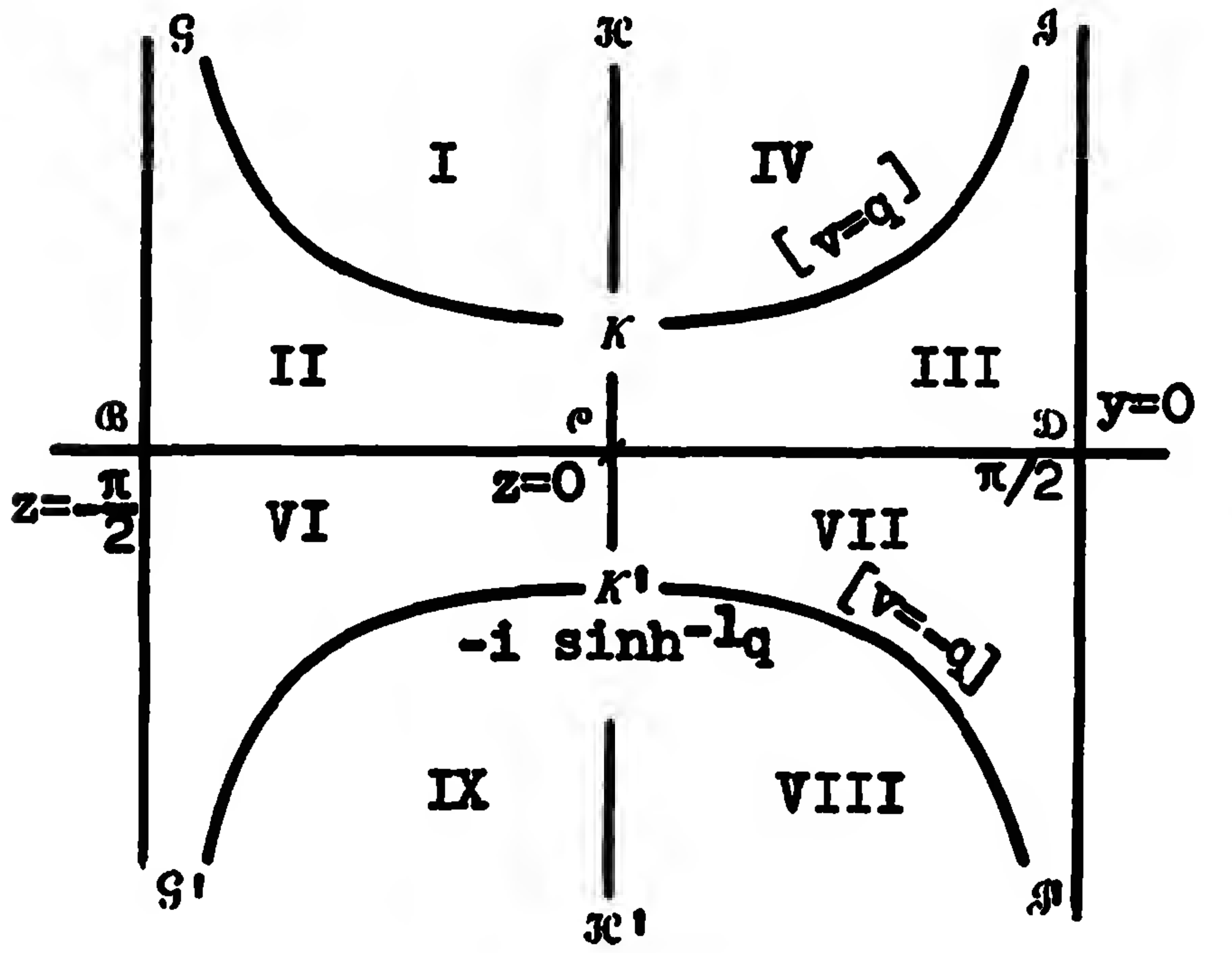
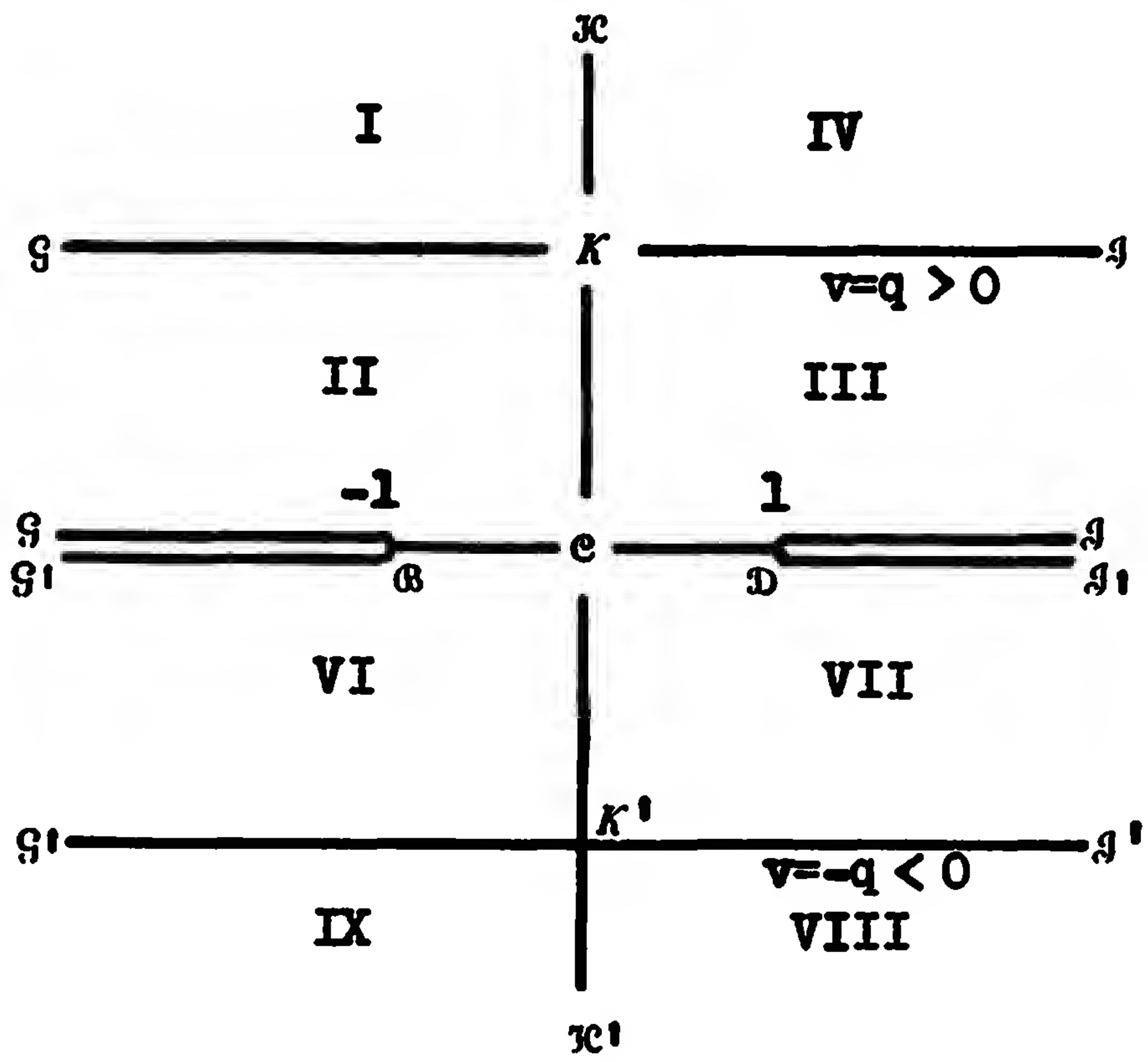
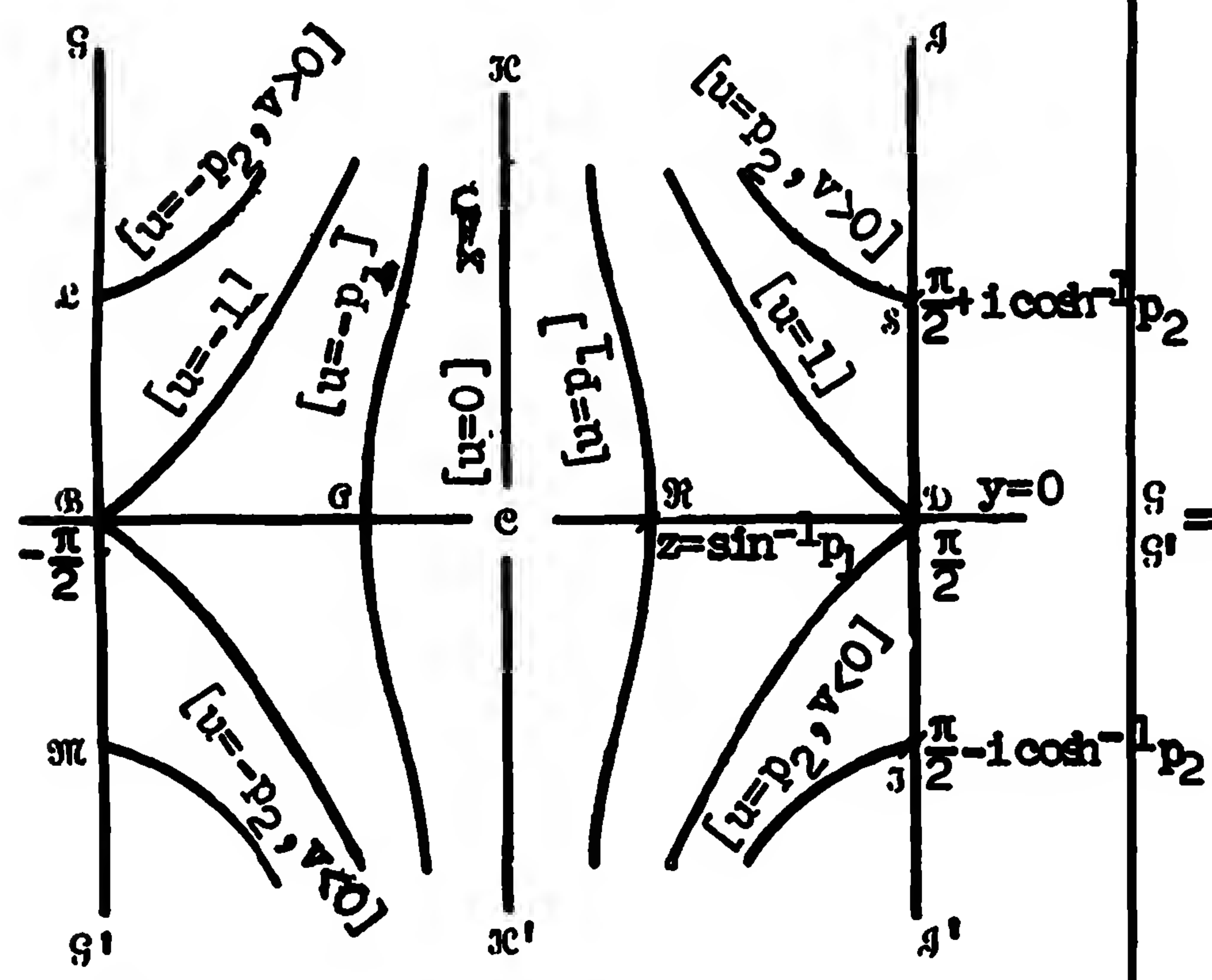
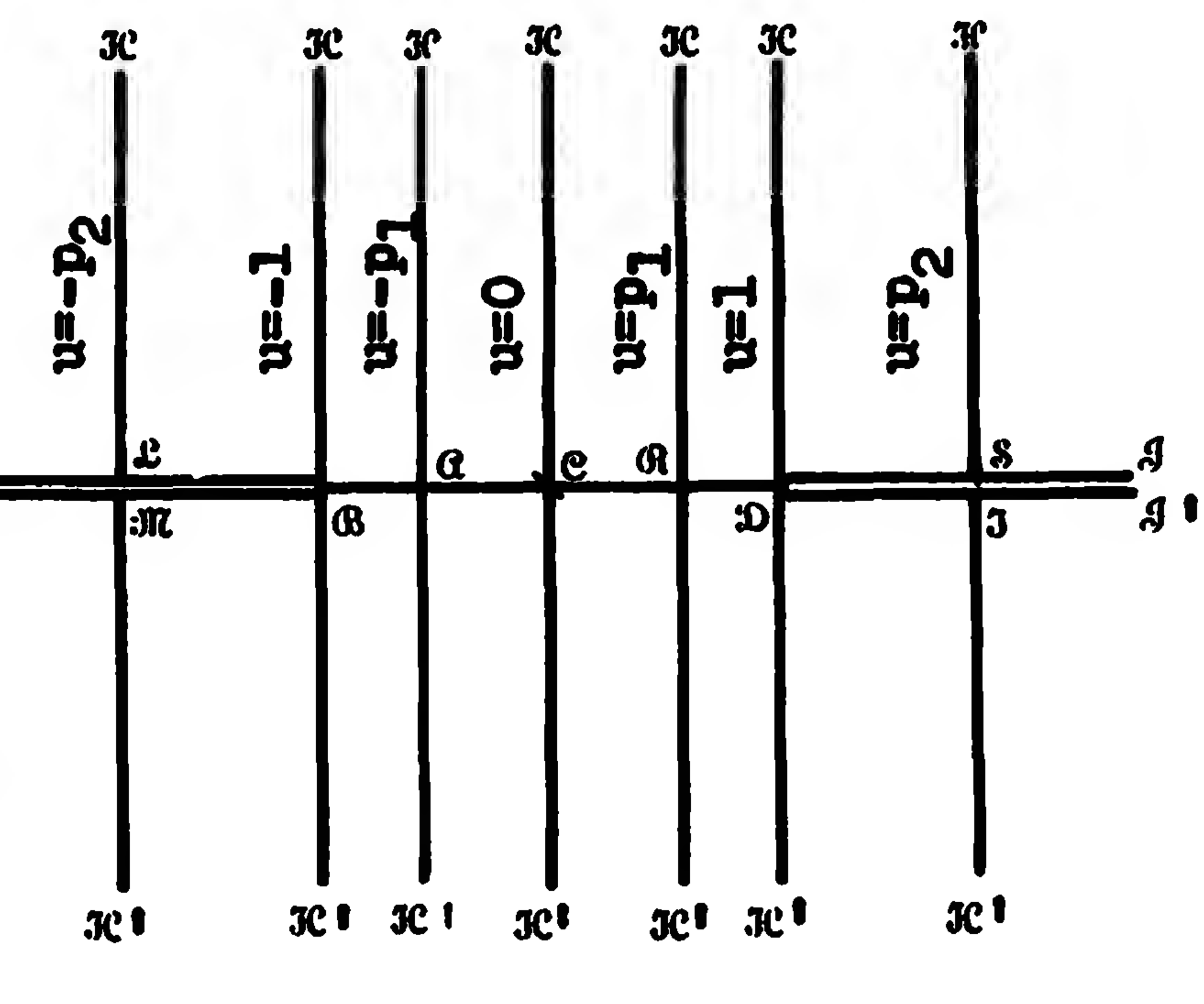


(In the figure, $0 < p_1 < p_2 < \pi/2$, $q_2 > q_1 > 0$)



For the curves in the z-plane which are mapped on R = constant or Θ = constant ($w = Re^{i\Theta}$, i.e., $\log \sin z = \log R + i\Theta$), see §11.6.

<u>Lines $u = p$; $v = q$.</u> (p, q real).

z - plane	w - plane
	
curve $\cos x = q/\sinh y$ ($\lvert y\rvert \geq \sinh^{-1} \lvert q\rvert$, $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$	line $v = q$, $-\infty < \mu < \infty$ [$q \gtrless 0$; in the diagram, $q > 0$].
	
The curves are reprinted from Jahnke-Emde, fig. 27.	

z - plane	w - plane
curve $\sin x = p_1/\cosh y$ $(\sin^{-1}p_1 \geq x > 0, -\infty < y < \infty)$	line $u = p_1$; $0 < p_1 < 1$
$\sin x = 1/\cosh y$ $(\frac{1}{2}\pi \geq x > 0, -\infty < y < \infty)$	line $u = 1$
$\sin x = p_2/\cosh y$ $(\cosh^{-1} p_2 \leq \lvert y \rvert < \infty, \frac{1}{2}\pi \geq x > 0)$	line $u = p_2$; $p_2 > 1$.

10.6 <u>Functions related to sin z.</u>

$\boxed{w = \sinh z}$, $\quad z = \sinh^{-1} w = \log\left\{w + \sqrt{(w^2 + 1)}\right\}$.

Combination of $\zeta = iz$, $\xi = iw$ and $\xi = \sin\zeta$; in all the figures of §10.5, both planes are turned through an angle of $\pi/2$.

$\boxed{w = \cos z} = \sin(z + \frac{\pi}{2})$; $\quad z = \cos^{-1} w = \frac{1}{i}\log\left\{w + i\sqrt{(1-w^2)}\right\}$.

$\boxed{w = \cosh z} = \sin(iz + \frac{\pi}{2})$; $z = \cosh^{-1} w = \log\left\{w + \sqrt{(w^2-1)}\right\}$

$$= 2\log\frac{\sqrt{(w+1)} + \sqrt{(w-1)}}{\sqrt{2}}.$$

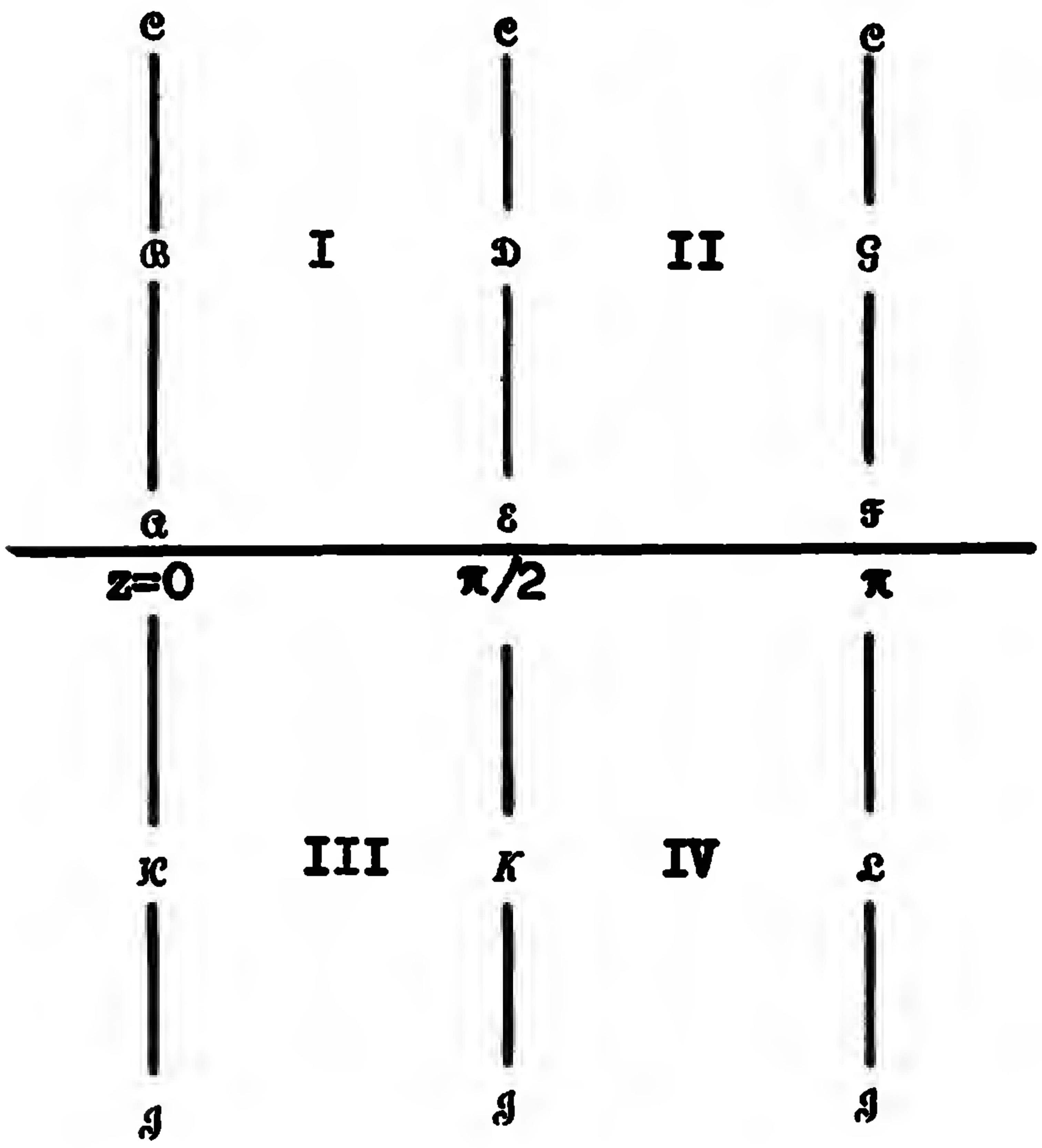
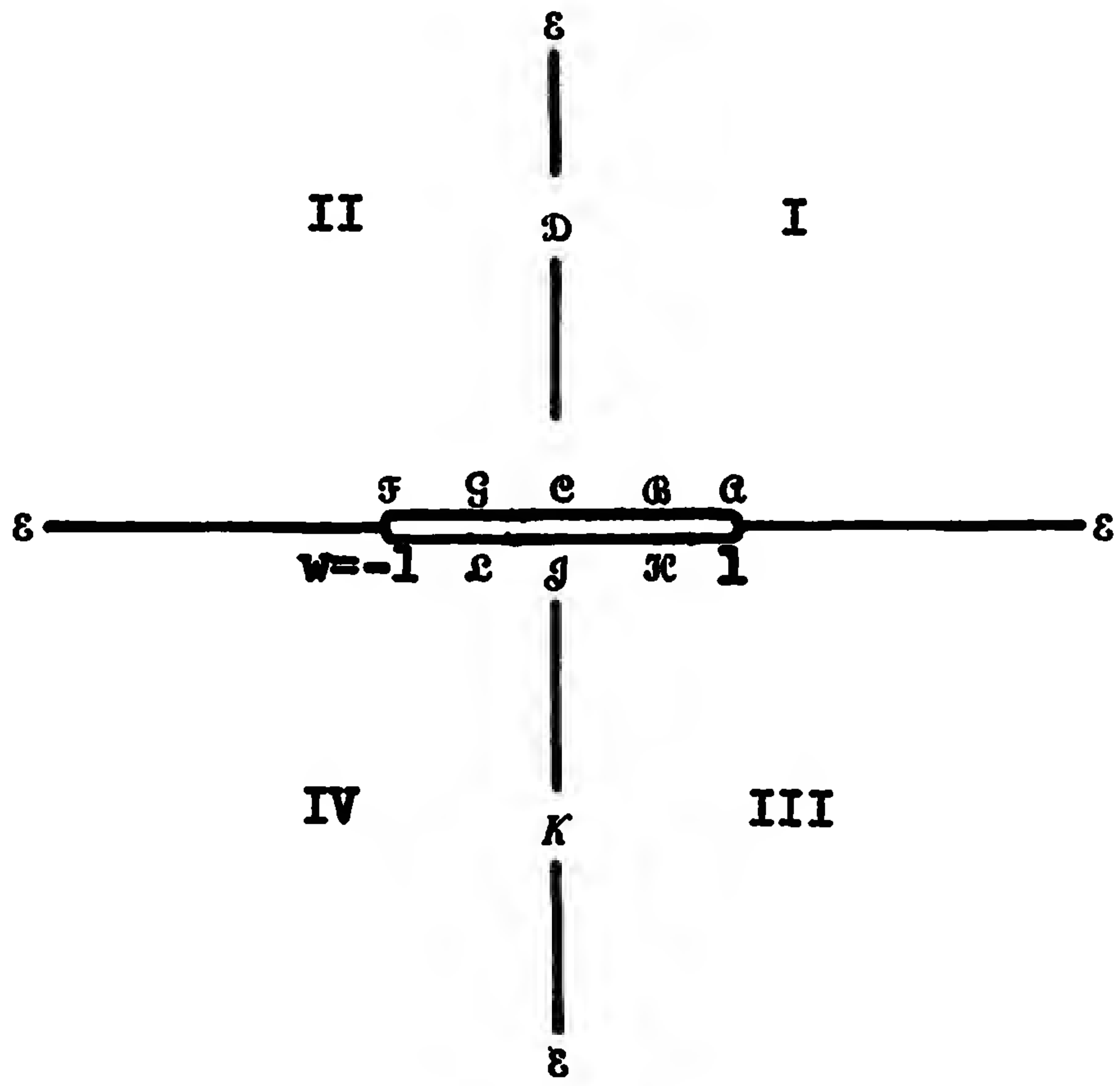
$\boxed{z = \log\frac{\sqrt{(w-a)} + \sqrt{(w-b)}}{\sqrt{(a-b)}}}$; $\quad a \neq b$, $\quad w = \frac{a+b}{2} + \frac{a-b}{2}\cosh 2z$.

z - plane $(-\pi/2 \leq y \leq 0)$	w - plane
points $z = 0$; $-i\pi/2$; ∞	points $w = a$; b; ∞
semi-strip $0 < x < \infty$, $-\pi/2 < y < 0$	half-plane, bounded by the line through a and b
semi-strip $0 > x > -\infty$, $-\pi/2 < y < 0$	the other half-plane, bounded by this line.

$$\boxed{w = \sec z}\ ,\qquad \boxed{z = \cos^{-1}(1/w)} = \frac{1}{i}\log\frac{1+\sqrt{(1-w^2)}}{w}\ .$$

Critical points: $z = k\pi;\quad (k+\frac{1}{2})\pi;\quad \infty$.

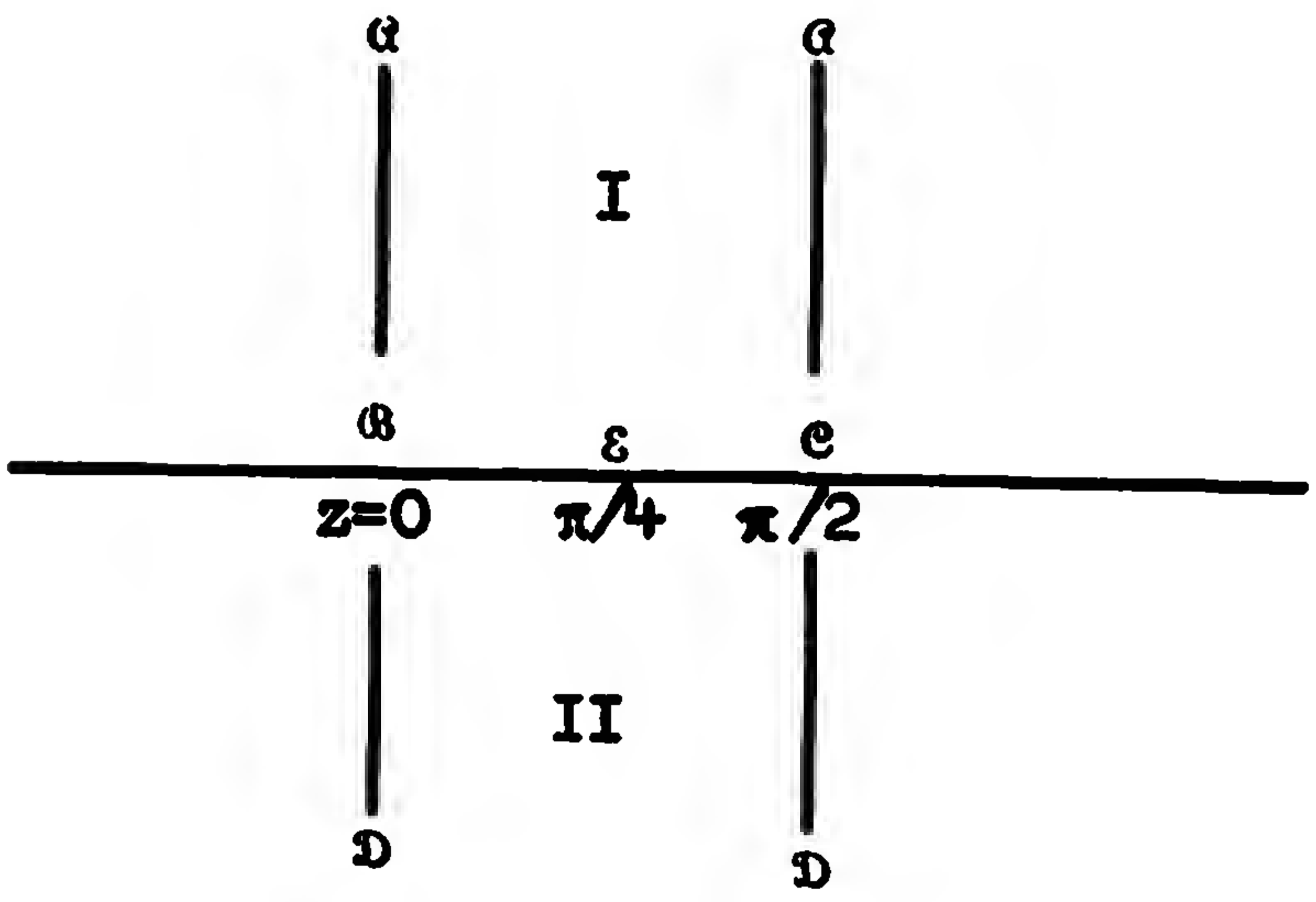
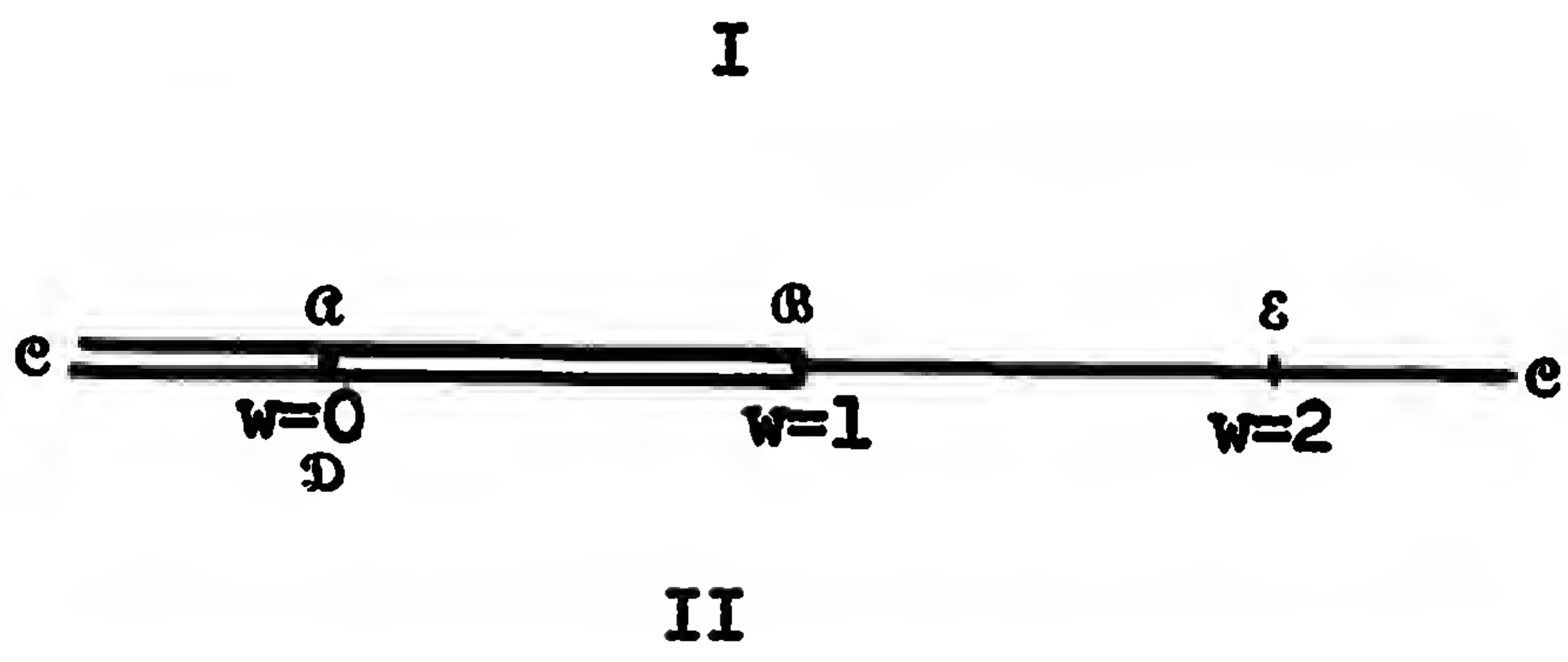
$k = 0, \pm 1, \pm 2, \ldots;\ w = Re^{i\theta};\quad q > 0.$

z - plane	w - plane
points $z = 2k\pi;\ (2k+1)\pi;\ (k+\frac{1}{2})\pi$.	points $w = 1;\ -1;\ \infty$.
	
line $x = p\quad (0 < p < \pi/2)$	part $-p < \theta < p$ of curve $$2R^2 = \frac{\cos 2\theta - \cos 2p}{\sin^2 2p}$$
line $x = p\quad (\pi/2 < p < \pi)$	part $p < \theta < 2\pi - p$ of $$2R^2 = \frac{\cos 2\theta - \cos 2p}{\sin^2 2p}$$
line $x = \pi/4$	part $u > 0$ of lemniscate $$\lvert w-1\rvert\lvert w+1\rvert = 1$$
line $x = 3\pi/4$	part $u < 0$ of this lemniscate

z - plane	w - plane
line segment $y = q,\ 0 < x < \pi$	part $0 < \theta < \pi$ of $2R^2 = \dfrac{\cosh 2q - \cos 2\theta}{\sinh^2 2q}$
line segment $y = -q,\ \pi > x > 0$	part $\pi < \theta < 2\pi$ of this curve

10.7 $\boxed{w = \sec^2 z}$, $\boxed{z = \cos^{-1}(1/\sqrt{w})}$ Or $w = \dfrac{2}{1 + \cos 2z}$, cf. p. 102

Critical points: as at $w = \sec z$.

z - plane	w - plane
	

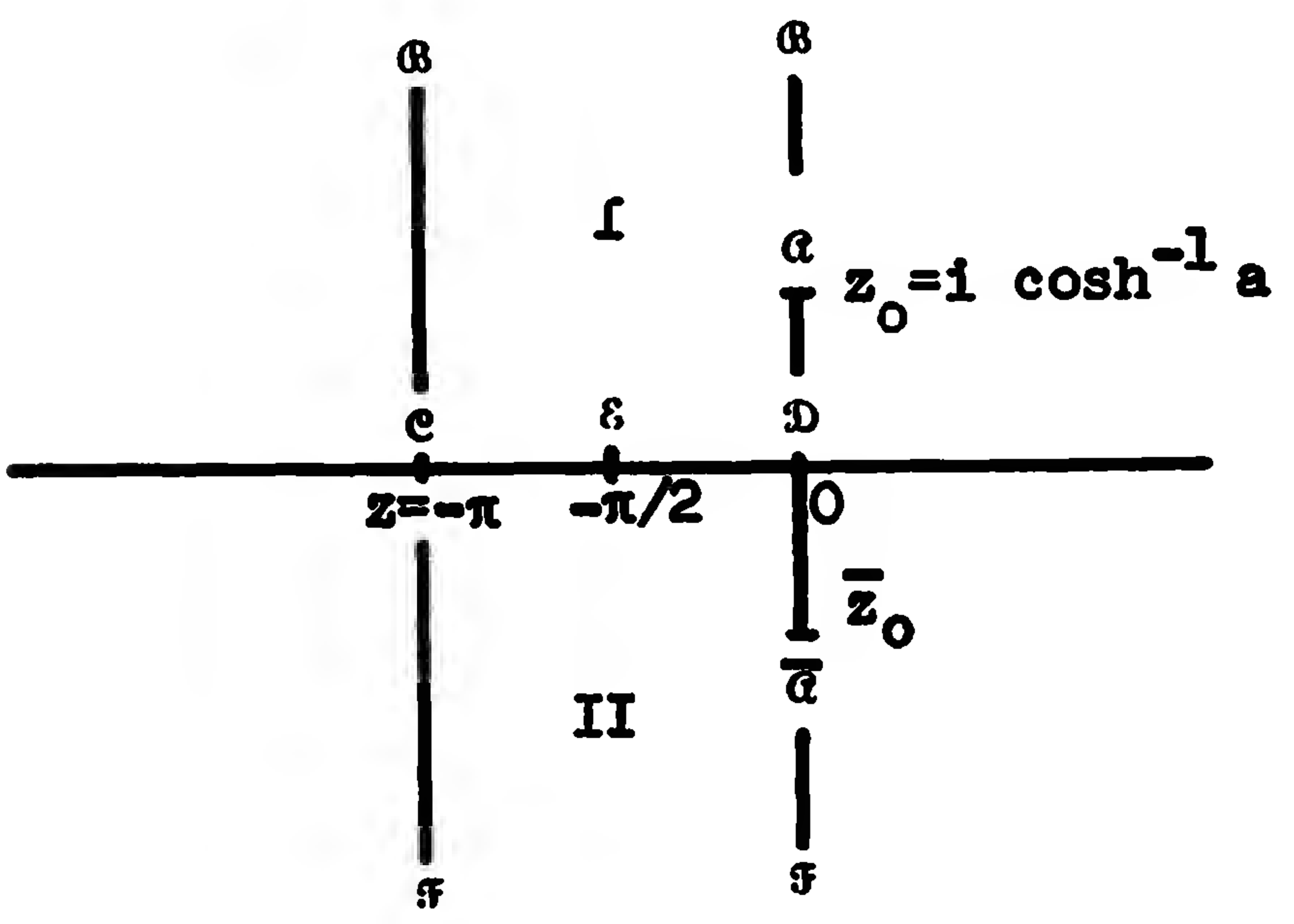
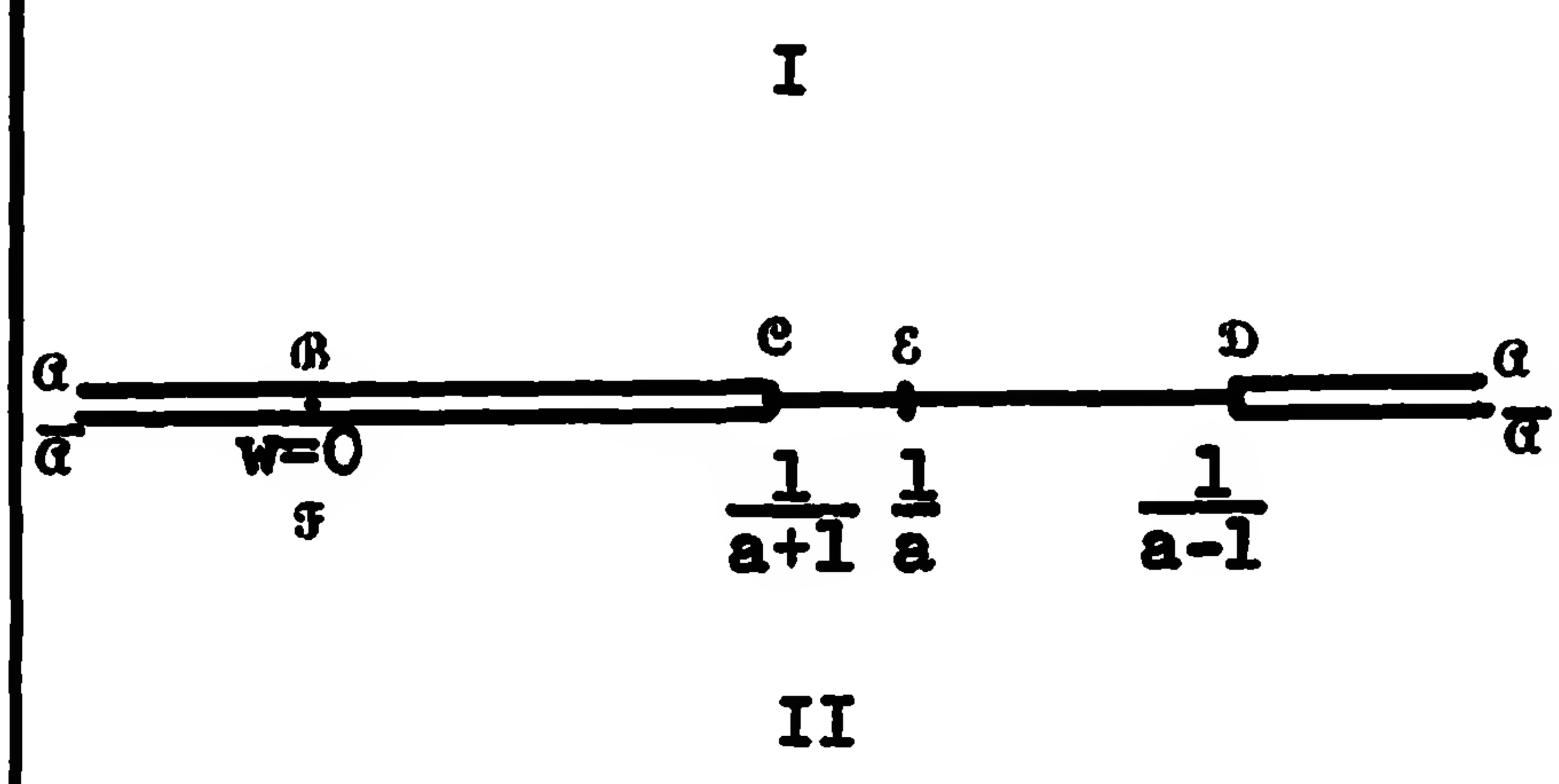
$\boxed{w = \tan^2 z}$ $= \sec^2 z - 1;\quad z = \tan^{-1}\sqrt{w} = \cos^{-1}\dfrac{1}{\sqrt{(w+1)}}$.

$\boxed{w = \tanh^2 z}$ $= 1 - \sec^2(iz);\quad z = \log\dfrac{1+\sqrt{w}}{\sqrt{(1-w)}}$.

$\boxed{w = \dfrac{1}{a - \cos z}}$; a real, $a \neq \pm 1$; $z = \cos^{-1}(a - 1/w)$.

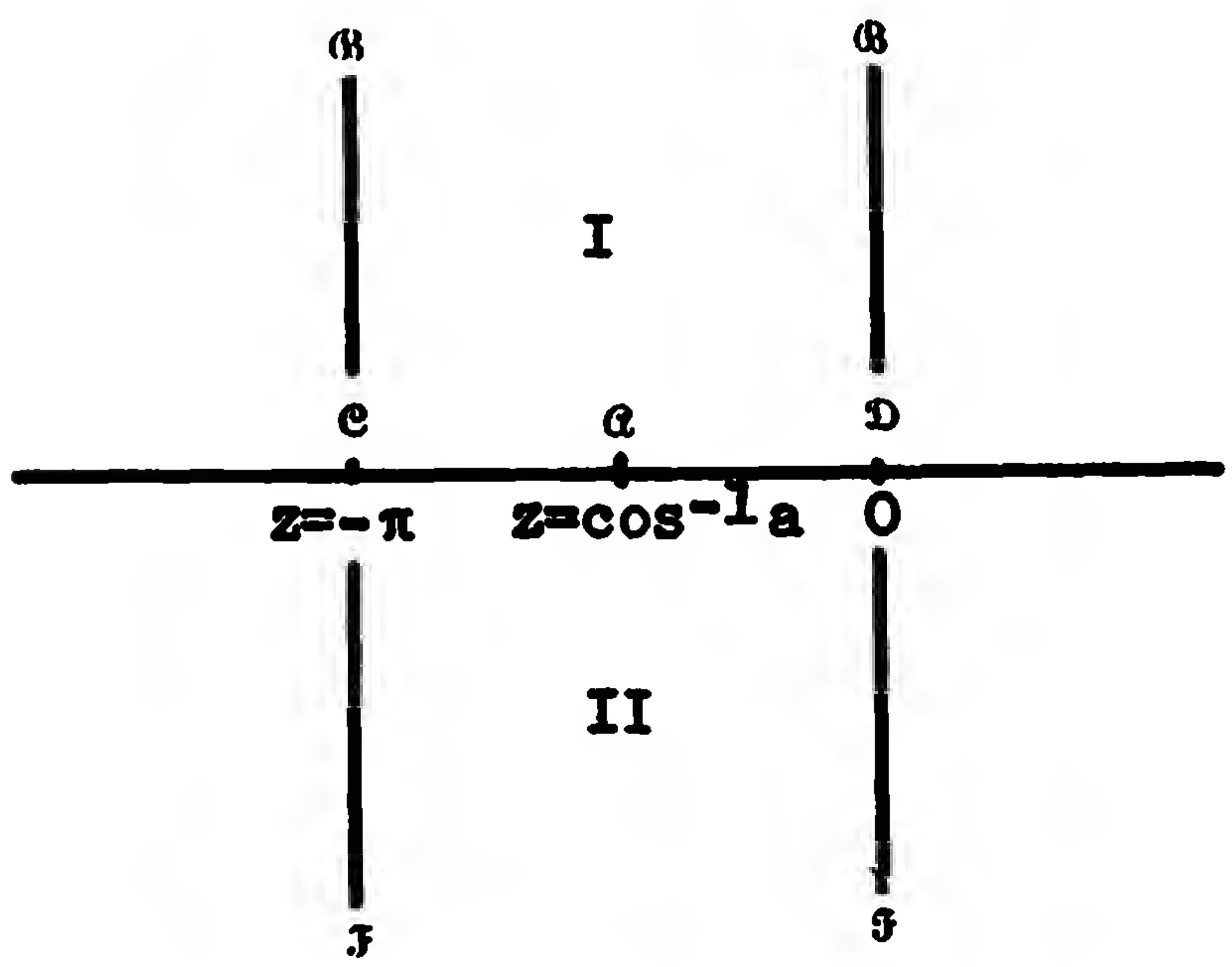
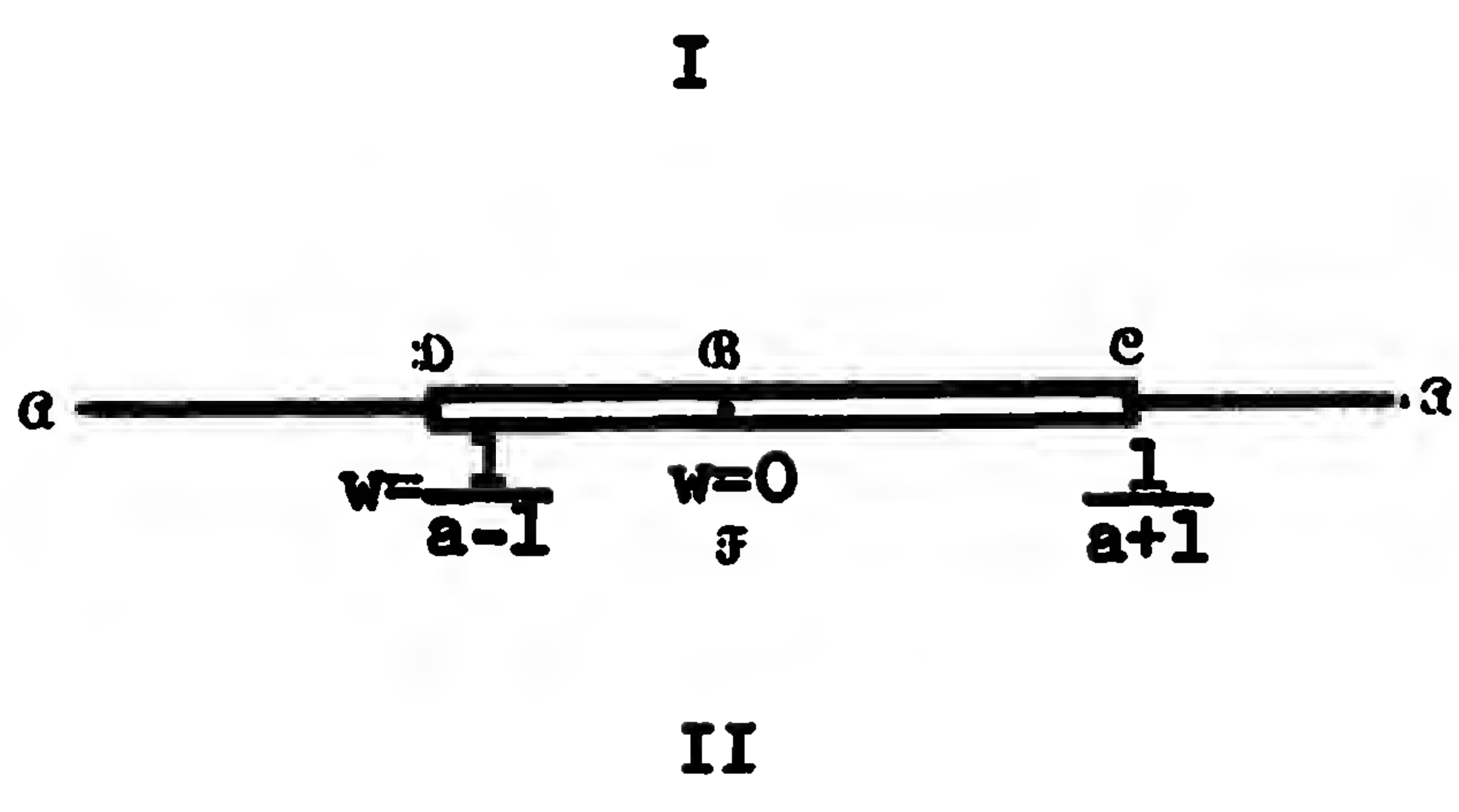
Critical points: $z = \pm\cos^{-1}a + 2k\pi$; $k\pi$; ∞ $[k=0,\pm 1,\ldots]$

(i) $a > 1$. $z_0 = i\zeta_0$, where $\cosh \zeta_0 = a$, $\zeta_0 > 0$.

z - plane	w - plane
	

(i)' $a < -1$. $-w = \dfrac{1}{|a| - \cos(z + \pi)}$, where $|a| > 1$; similar to (i).

(ii) $-1 < a < 1$.

z - plane	w - plane
	

10.8 $\boxed{w = \dfrac{\tanh z}{z}}$

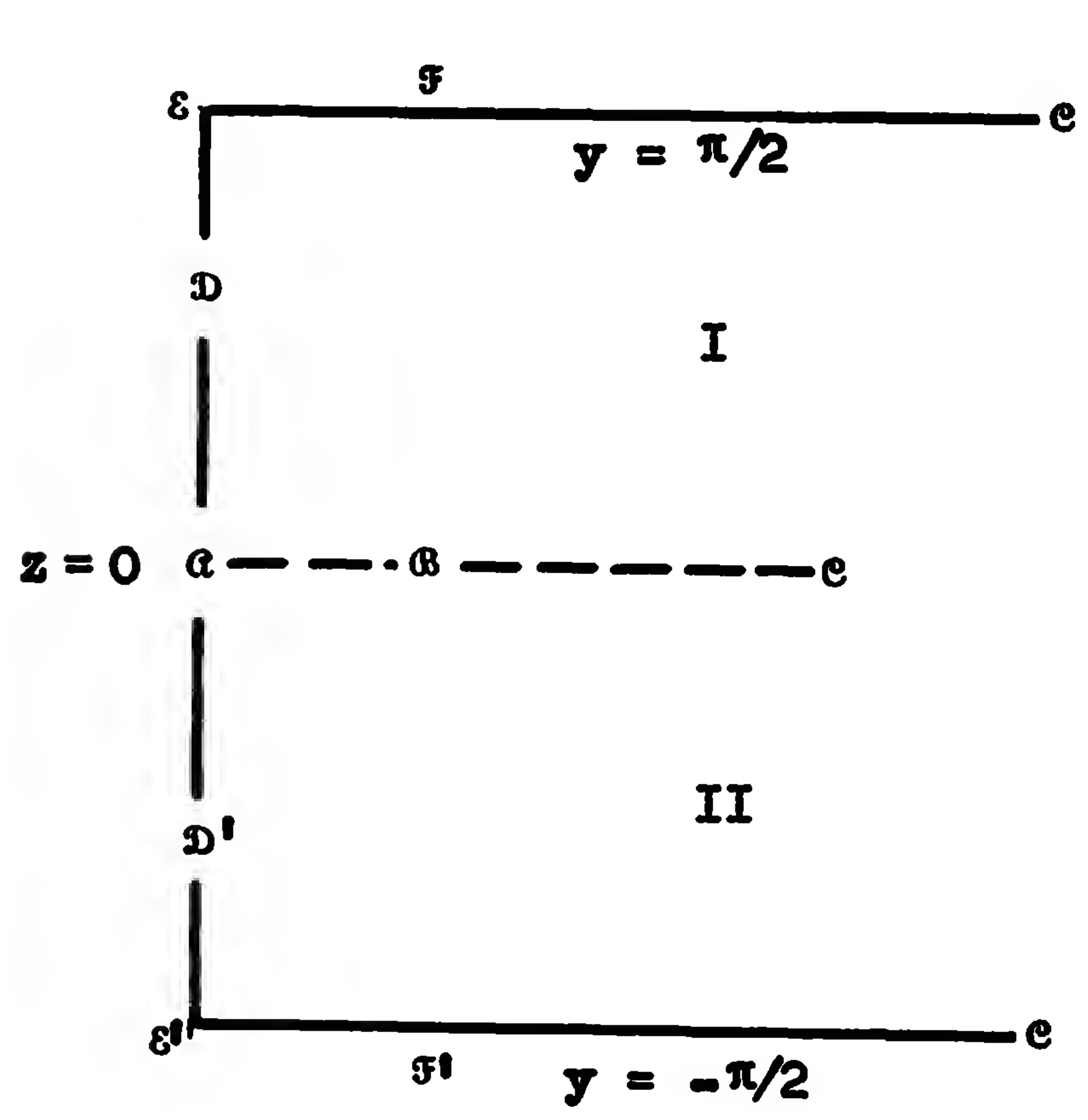
Critical points: $z = \infty$, and the solutions of $\sinh 2z = 2z$, namely

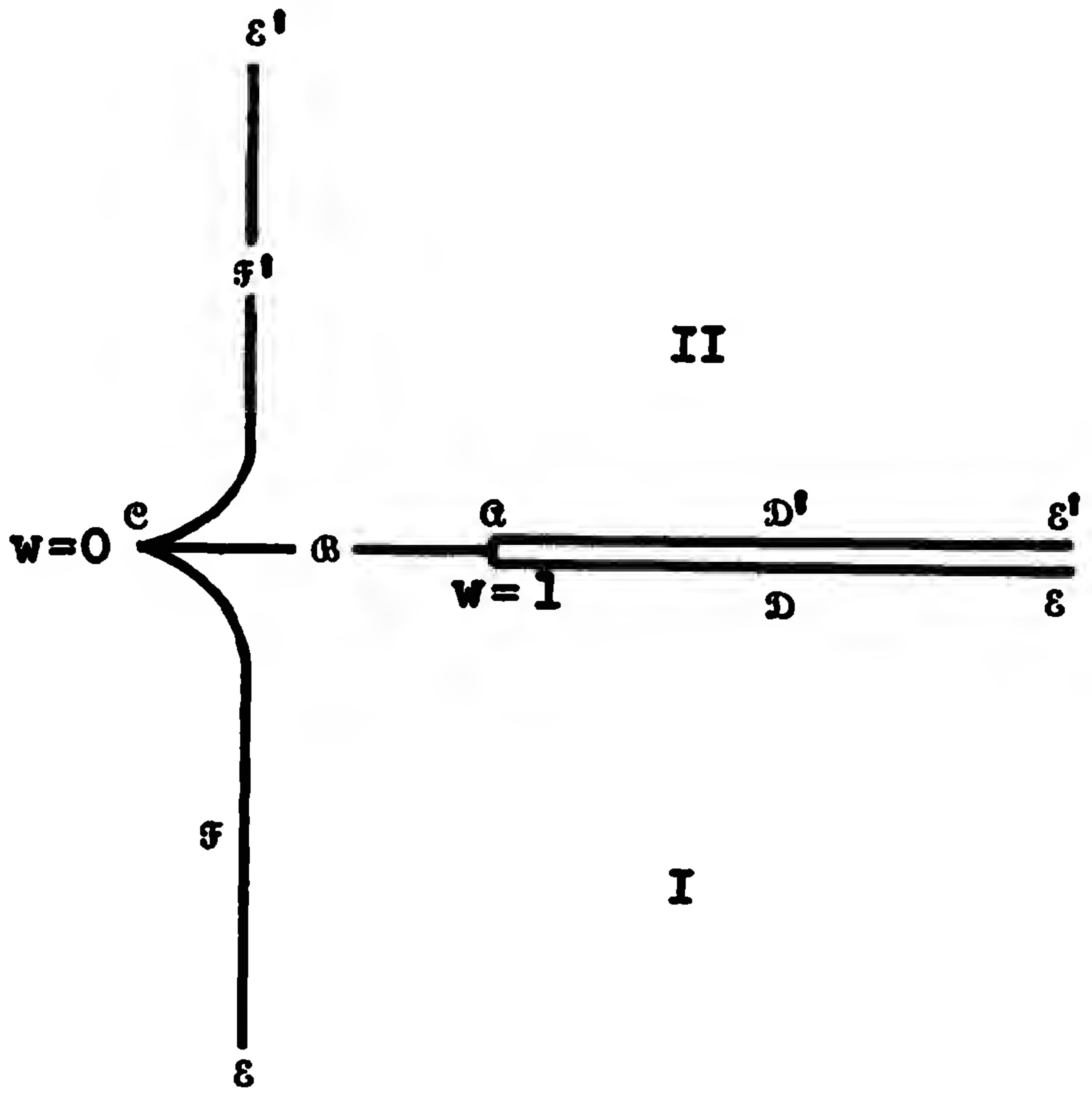
$z = 0$; $1.3843 \pm 3.7488i$, $1.6761 \pm 6.95i$; $1.8584 \pm 10.12i$;

$1.9916 + 13.277i$, etc., etc.

$k = 1, 2, 3, \ldots$; a real.

z - plane	w - plane
points $z = 0$; ± 1; $ai \pm \infty$; $\pm i\pi/2$	points $w = 1$; $\tanh 1$; 0; ∞
line-segment $x = 0$, $k\pi \le y < (k+\frac{1}{2})\pi$	half-line $0 \le u < \infty$, $v = 0$
line-segment $x = 0$, $(k + \frac{1}{2})\pi < y \le (k + 1)\pi$	half-line $-\infty < u \le 0$, $v = 0$
line-segment $x = 0$, $0 \le y < \pi/2$ line-segment $x = 0$, $0 \ge y > -\pi/2$	half-line $1 \le u < \infty$, $v = 0$





z - plane	w - plane
strip $0 < y < \frac{\pi}{2}$, $-\infty < x < \infty$	the same region as above, but cut from $w = 0$ to $w = 1$ only. asymptote of the above curve: $u = \frac{4}{\pi^2}$

$$\boxed{w = \frac{\coth z}{z}}$$

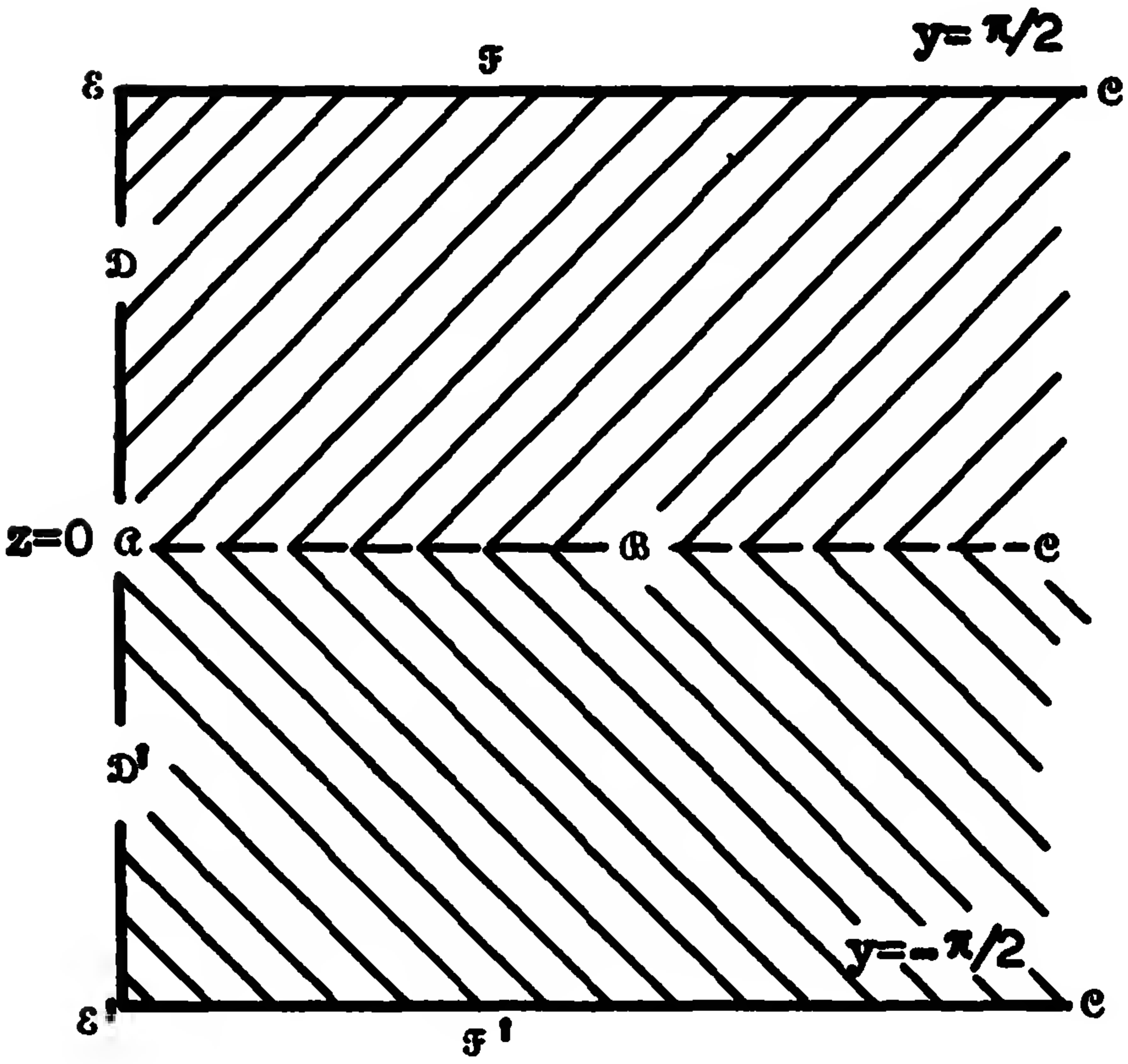
Critical points: $z = \infty$, and the solutions of $\sinh 2z = -2z$, namely

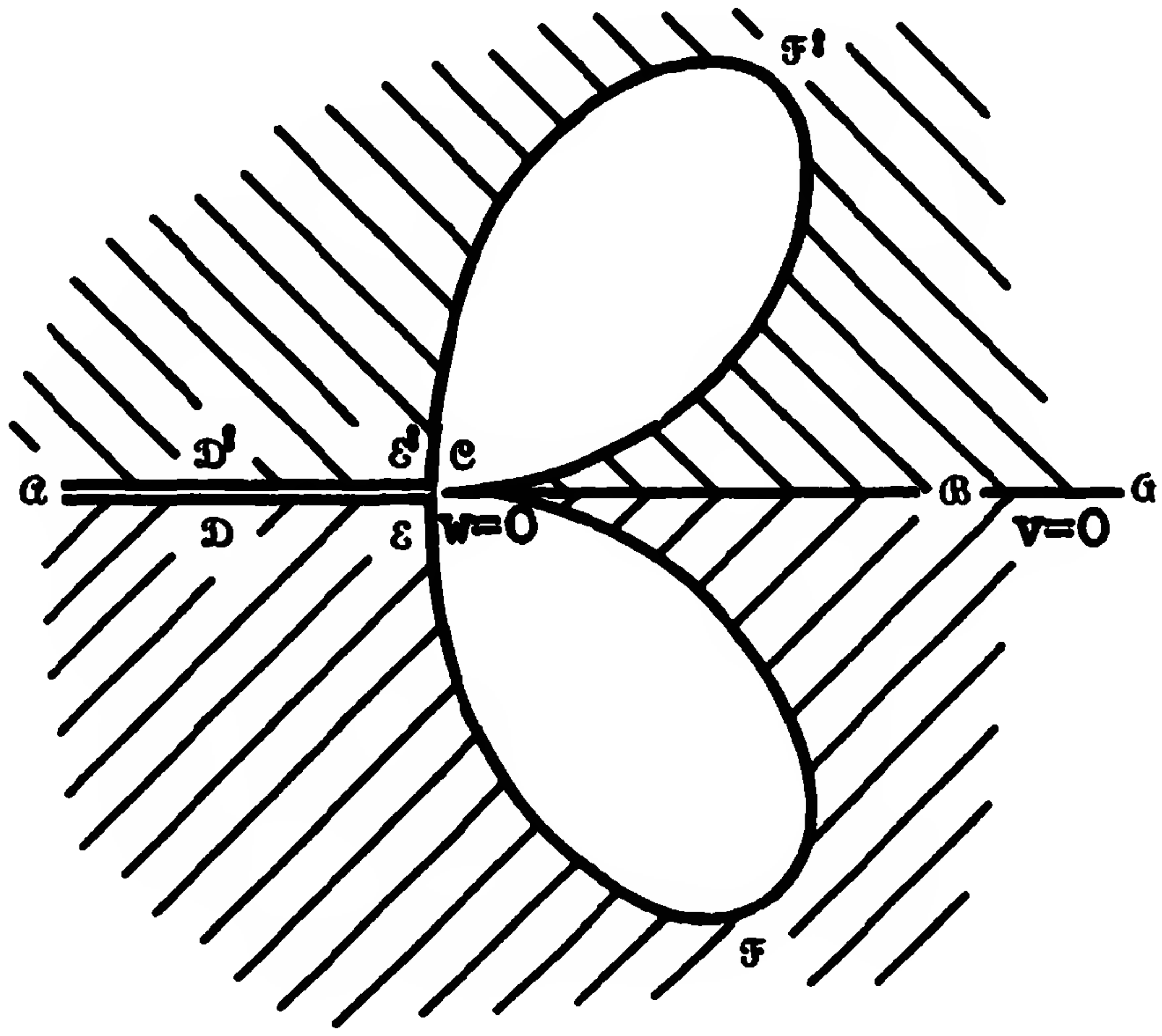
$z = 0;\ 1.1254 \pm 2.1062i;\ 1.5516 \pm 5.3563i;\ 1.7755 \pm 8.5367i;$

$1.9294 \pm 11.6992i;$ etc.

$k = 0, 1, 2, 3, \ldots;$ a real.

z - plane	w - plane
points: $z = 0;\ \pm 1;\ ai \pm \infty;\ \pm i\pi/2$	points: $w = \infty;\ \coth 1;\ 0;\ 0$
line-segment $x = 0,\ k\pi < y \leq (k+\frac{1}{2})\pi$	half-line $-\infty < u \leq 0,\ v = 0$
line-segment $x = 0,\ (k + \frac{1}{2})\pi \leq y < (k + 1)\pi$	half-line $0 \leq u < \infty,\ v = 0$



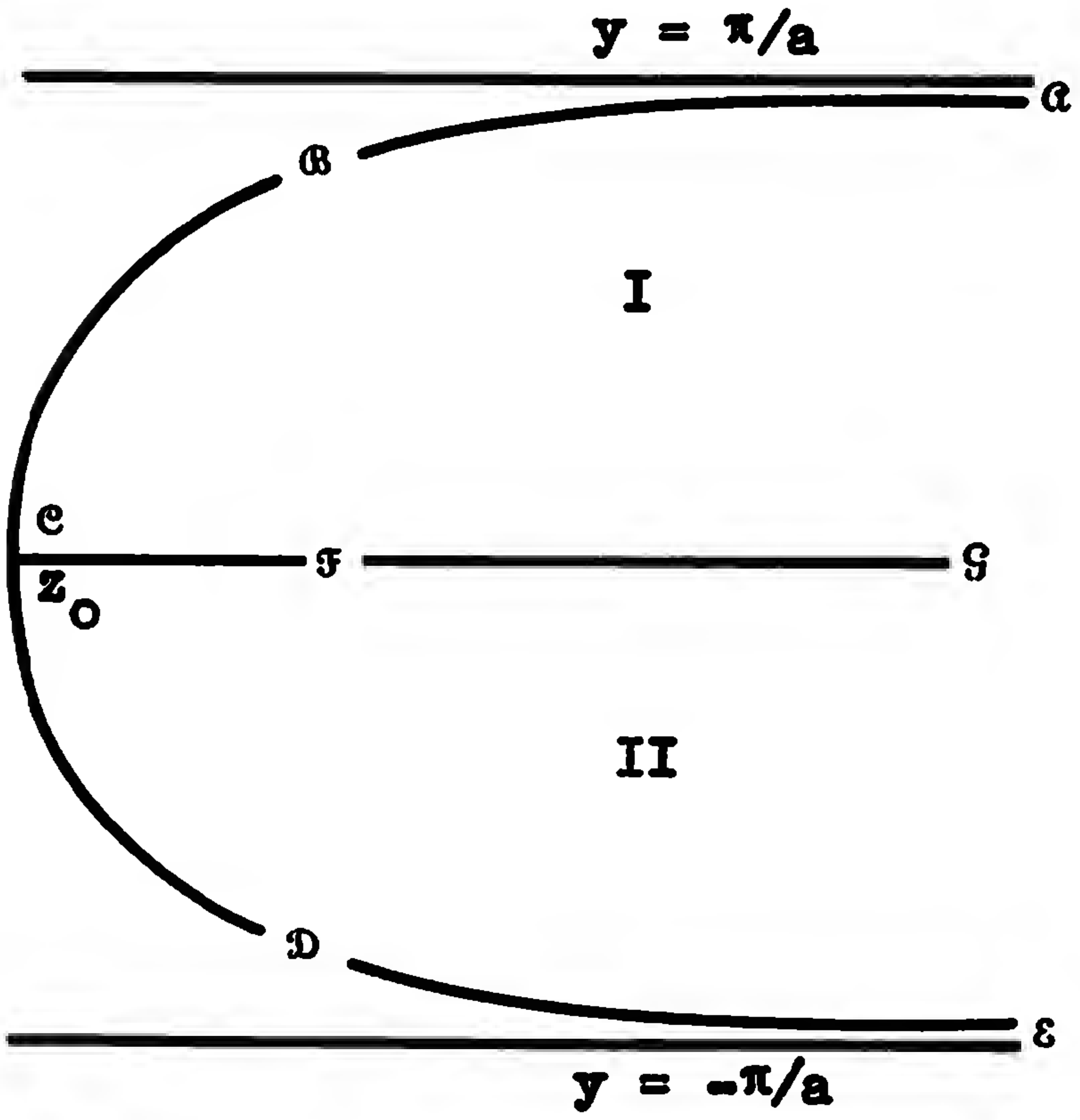
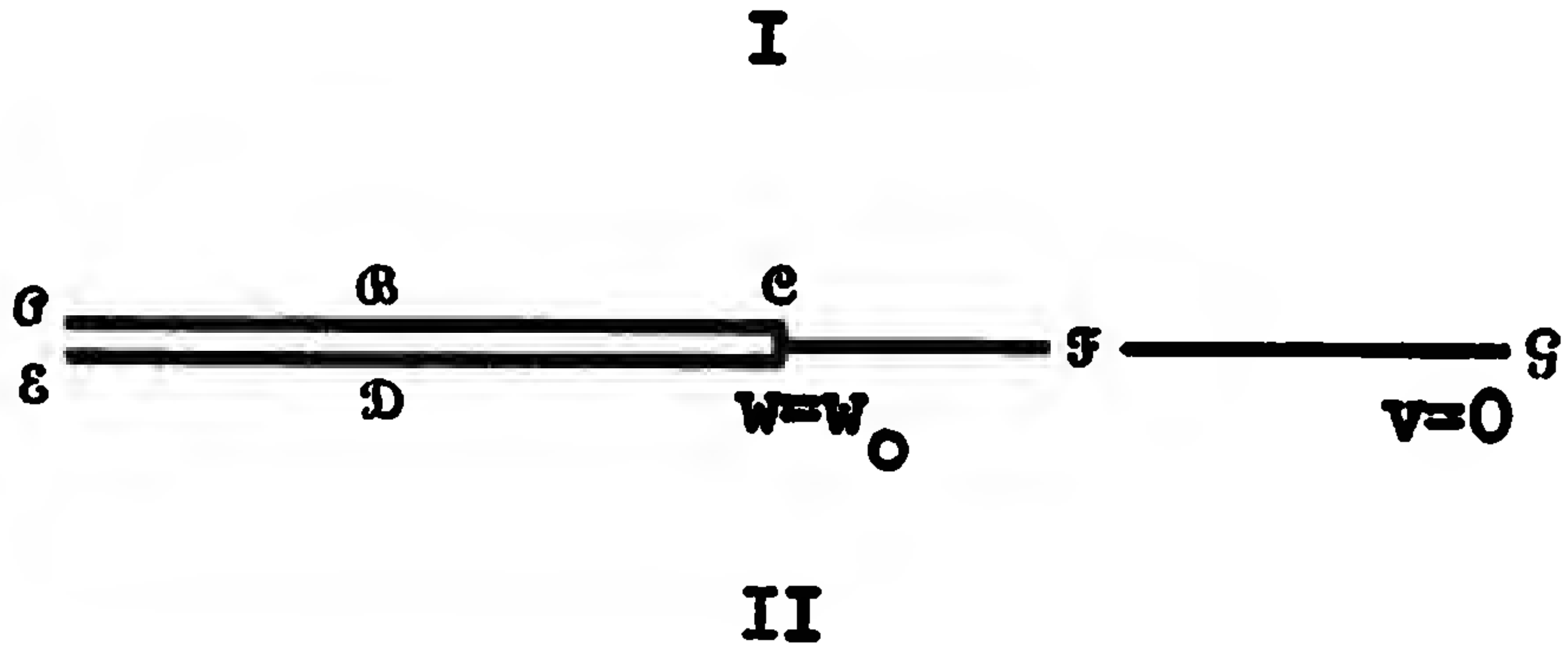


For details concerning $w = z^{-1} \tanh z$ and $w = z^{-1} \coth z$, see A.M.P. Note 18 of the Applied Mathematics Panel, NDRC, "Tables for solutions of the wave equation for rectangular and circular boundaries having finite impedance", by A. N. Lowan, P. M. Morse, H. Feshbach and E. Hourwitz. The values for the solutions of $\sinh 2z = \pm 2z$ are taken from this report.

11. COMPOSITE FUNCTIONS.

11.1 $\boxed{w = e^{az} - ce^{bz}}$; a, b, c real, $a > b > 0$, $c > 0$.

Critical points: $z = \infty$; $\dfrac{\log bc - \log a + 2k\pi i}{a - b}$, $k = 0, \pm 1, \pm 2, \ldots$

z - plane, $-\pi/a < y < \pi/a$	w - plane
	
point $z_o = \dfrac{\log (bc/a)}{a - b}$	point $w_o = -\dfrac{a-b}{b}\left(\dfrac{bc}{a}\right)^{a/(a-b)}$
half-line $z_o \leq x < \infty$, $y = 0$	half-line $w_o \leq u < \infty$, $v = 0$
curve $\dfrac{\sin ay}{\sin by} = ce^{(b-a)x}$ (i.e. $\mathcal{A}\,\mathcal{B}\,\mathcal{C}\,\mathcal{D}\,\mathcal{E}$), $-\pi/a < y < \pi/a$, with asymptotes $y = \pm\pi/a$; if $a = 2b = c = 2$, $\cos y = e^{-x}$	half-line $-\infty < u \leq w_o$; counted twice

Example: $\boxed{w = \log(\sqrt{z} - 1)}$, $z = (e^w + 1)^2$.

Combination of $z = \xi + 1$, $w = \zeta + i\pi$, and $\xi = e^{2\zeta} - 2e^{\zeta}$.

Critical points: $z = 0;\ 1;\ \infty$.

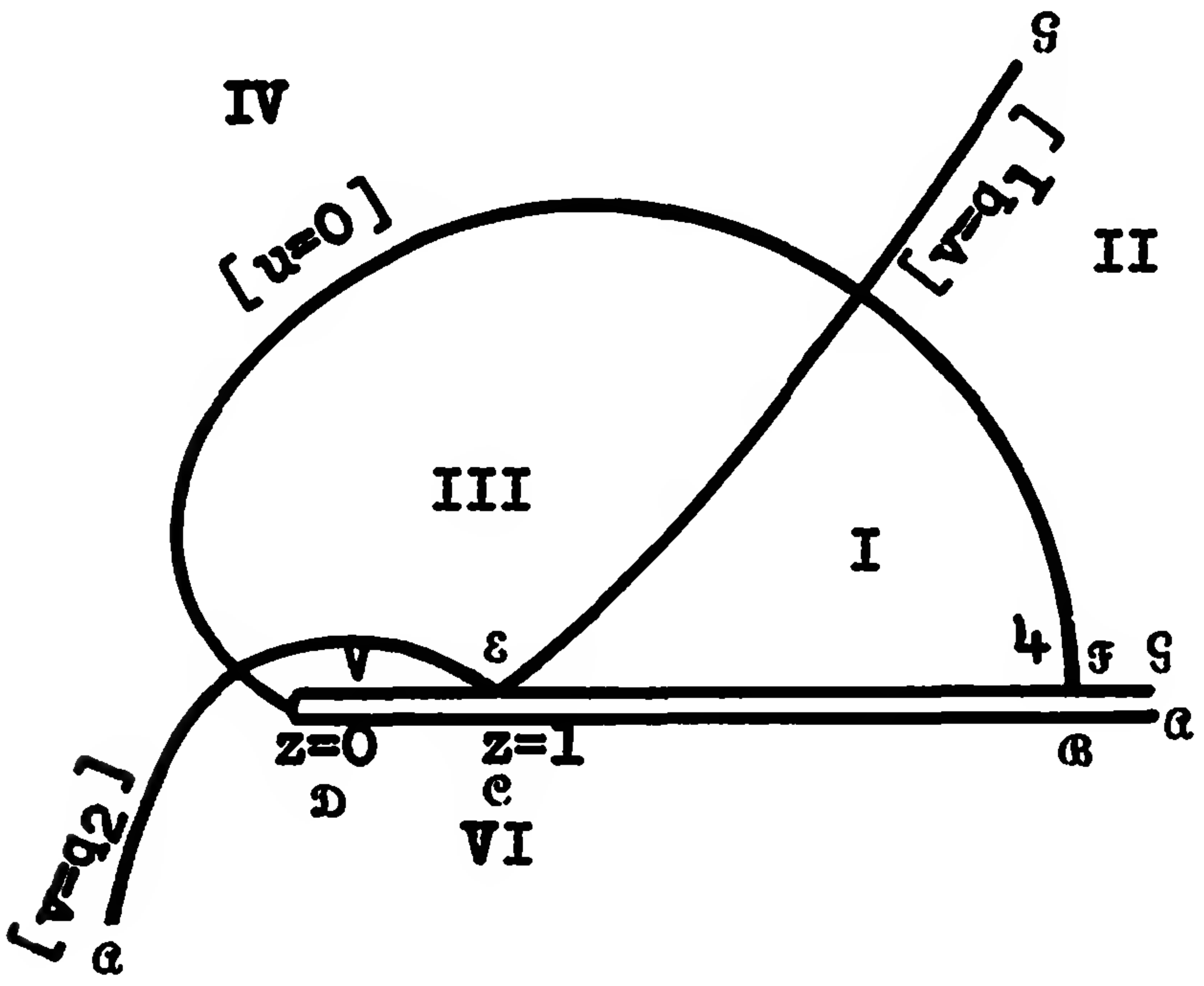
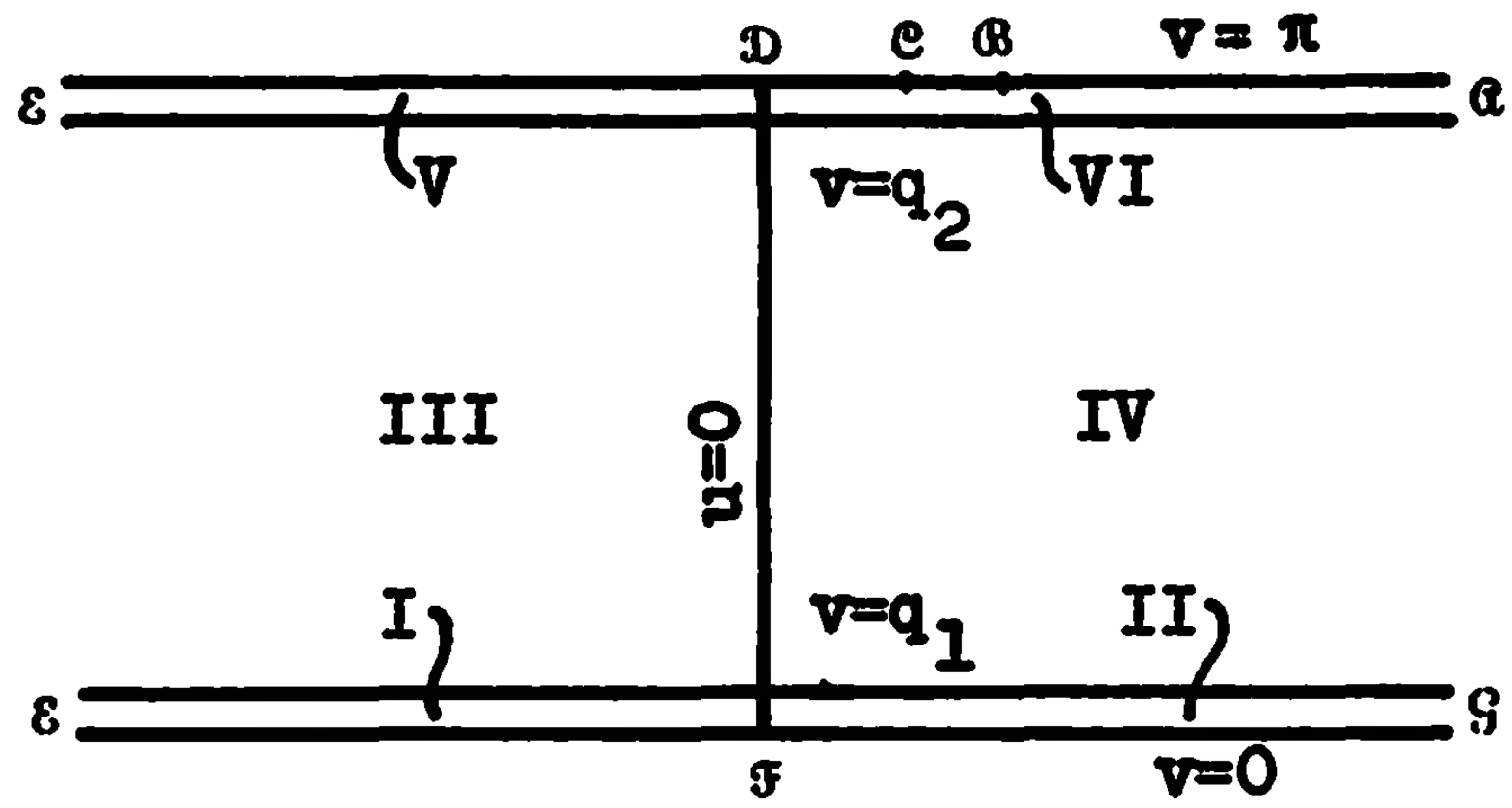
$k = 0, \pm 1, \pm 2, \ldots$

z - plane	w - plane
point $z_0 = (e^{w_0} + 1)^2$	points $w_0 + 2k\pi i$; $\log(2+e^{w_0}) + (2k+1)\pi i$
plane, cut along $y = 0$, $0 \leq x < \infty$	strip $2k\pi < v < (2k+1)\pi$
plane, cut along $y = 0$, $0 \geq x > -\infty$	interior of part $\lvert (2k+1)\pi - v \rvert < \frac{\pi}{2}$ of K.
half-plane $y > 0$	$\left\{\begin{array}{l}\text{upper half of interior of any part of K} \\ \text{part } \cos v > -e^{-u} \text{ of strip } 2k\pi < v < (2k+1)\pi\end{array}\right.$

K is the curve $\cos v = -e^{-u}$; consisting of an infinity of parts $(2k + \frac{1}{2})\pi < v < (2k + \frac{3}{2})\pi$, $0 \leq u < \infty$, symmetric with respect to $v = (2k + 1)\pi$.

points $z = 0;\ 2i;\ -1;\ (\sqrt{2}+1)(1+i);\ (\sqrt{2}-1)(i-1);\ 4$ (point $\mathfrak{F}$); $4(\mathfrak{B})$; $1(\mathfrak{E})$; $1(\mathfrak{C})$	points $w = i\pi;\ \frac{i\pi}{2};\ \frac{1}{2}\log 2 + \frac{3i\pi}{4};\ \frac{i\pi}{4};\ \frac{3i\pi}{4};\ 0;\ \log 3 + i\pi;\ \infty;\ \log 2 + i\pi$.

Lines parallel to the axes of w - plane.

<table>
<tr><th>z - plane</th><th>w - plane</th></tr>
<tr><td>
</td><td>

The four points marked on u=0 are: $w=0$ ($\mathfrak{F}$); iq_1; iq_2; $i\pi$ ($\mathfrak{D}$)</td></tr>
<tr><td>part $0 \leq v \leq \pi$ of limaçon [with pole $z_1 = 1 - e^{2p}$] $z - z_1 = 2e^{p+iv}(1 + e^p \cos v)$ [cardioid for $p = 0$]</td><td>segment of line $u = p$ (p real), $0 \leq v \leq \pi$.</td></tr>
<tr><td>part, bounded by $\mathfrak{E}$ ($z = 1$), of parabola $\Re(ze^{-2iq}) + 2\sin^2 q = |z|$ (with focus $z = 0$, vertex $z = -(\sin^2 q)e^{2iq}$)</td><td>line $v = q$, $0 < q < \pi$.</td></tr>
</table>

$\boxed{w = \alpha e^{az} + \beta e^{bz}}$; $\frac{a}{b}$ real, $\frac{a}{b} > 1$; $\alpha\beta \neq 0$.

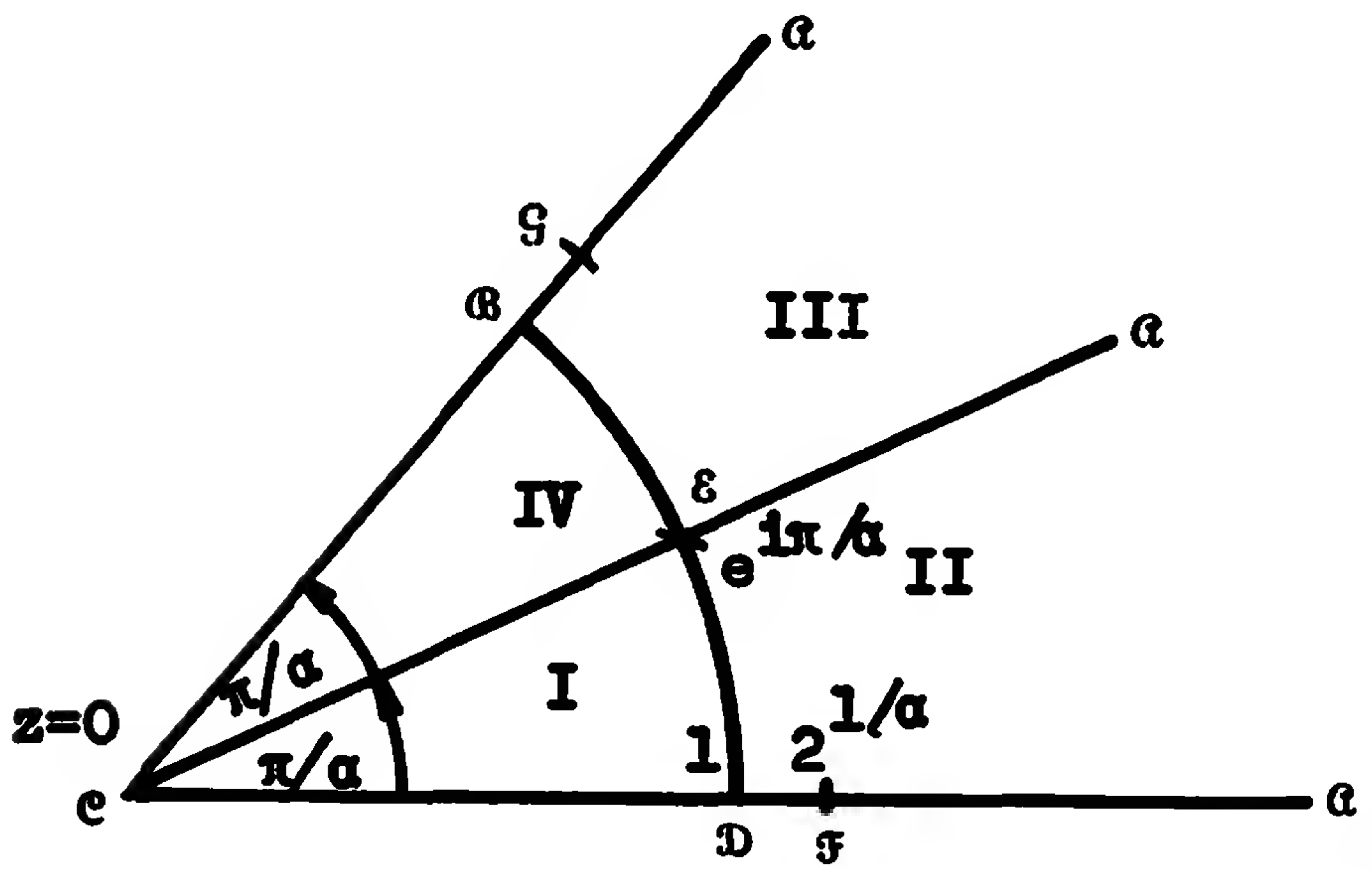
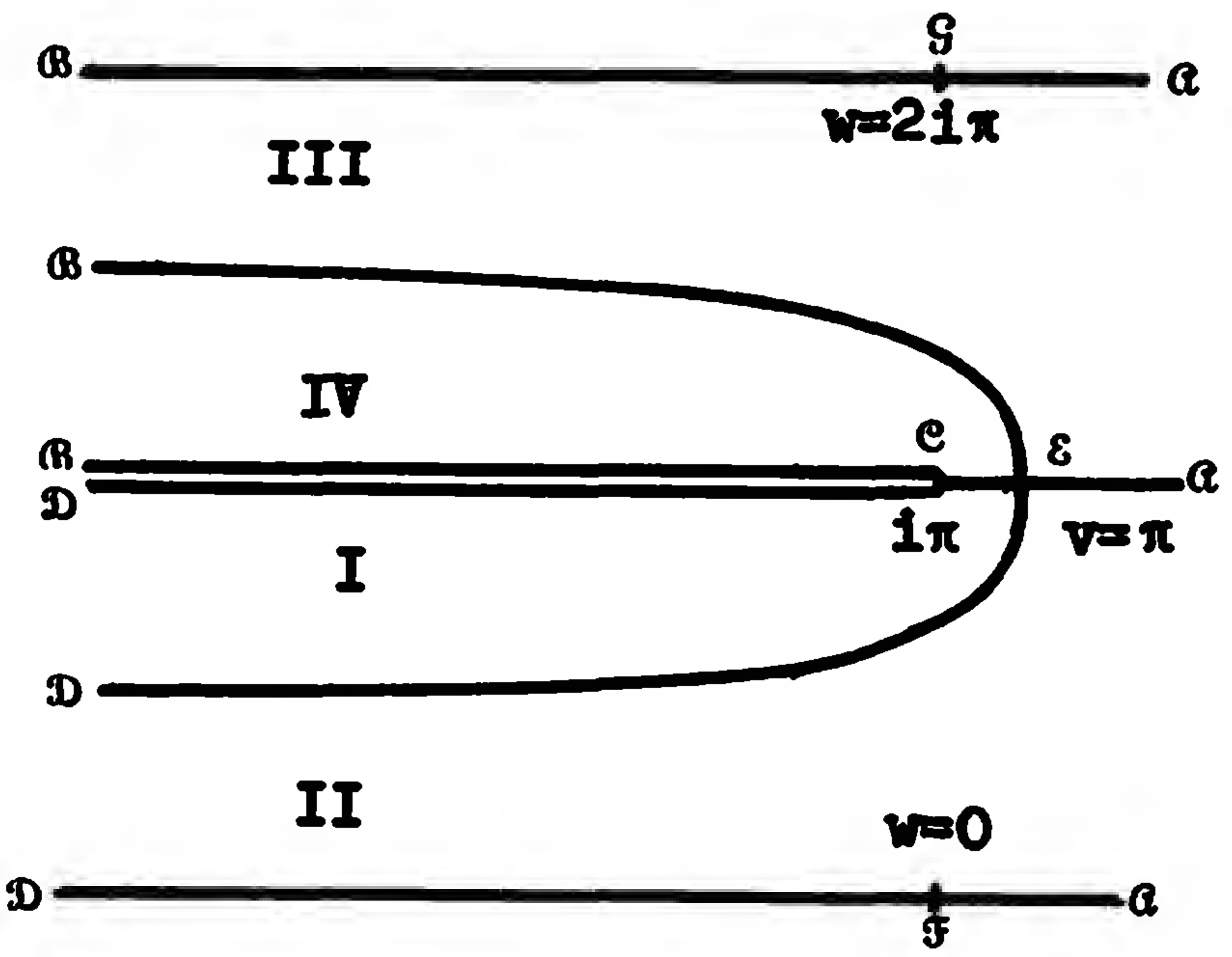
Combination of $w = \alpha e^{a\lambda}\xi$, $z = e^{-i \arg a}\zeta + \lambda$, and $\xi = e^{|a|\zeta} - e^{|b|\zeta}$,

where $\lambda = \log(-\frac{\beta}{\alpha})/(a-b)$.

11.2 $\boxed{w = \log(z^{\alpha} - 1)}$; $\alpha > 1$; $\boxed{z = (e^{w} + 1)^{1/\alpha}}$. For $\alpha = 2$ cf. §10.4

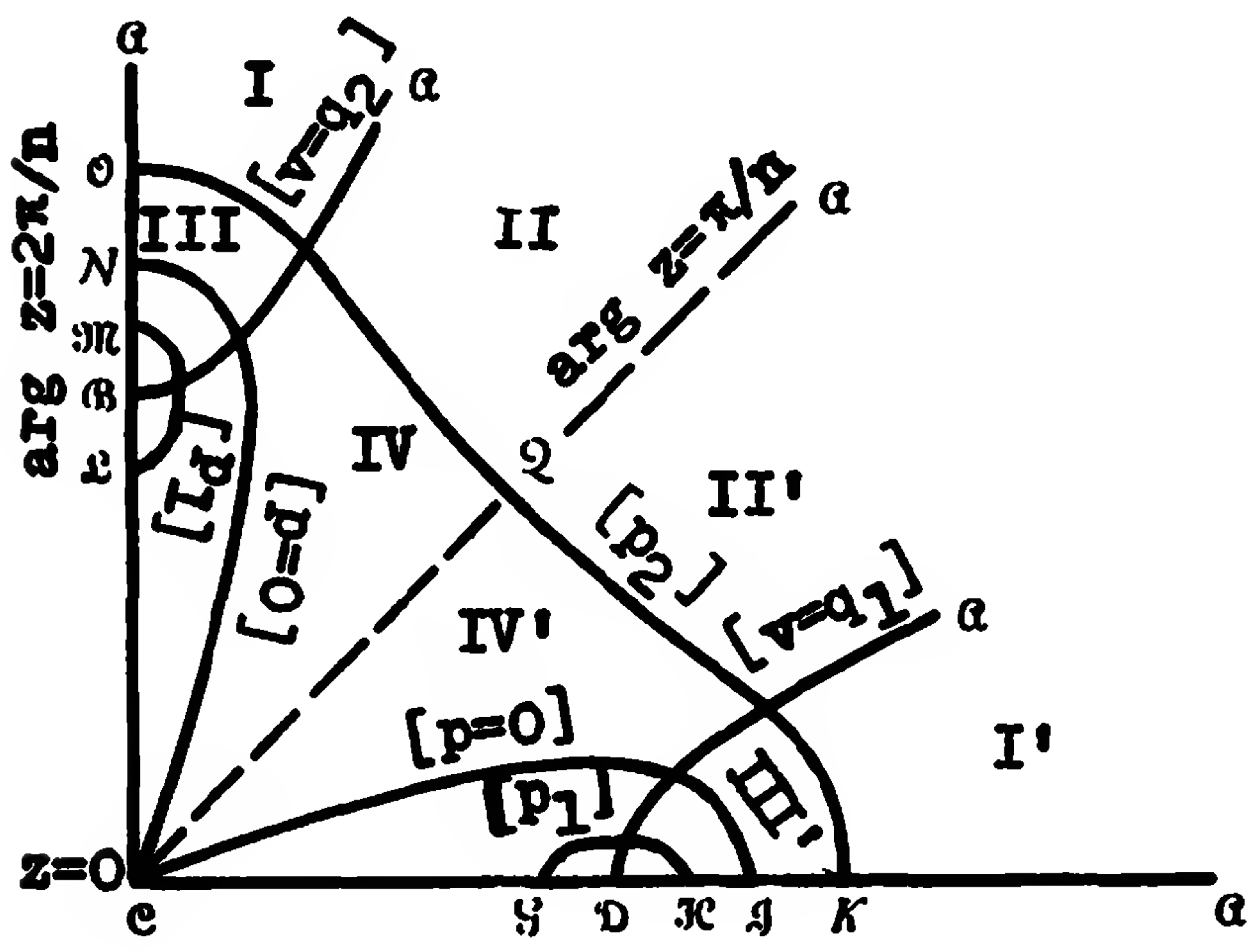
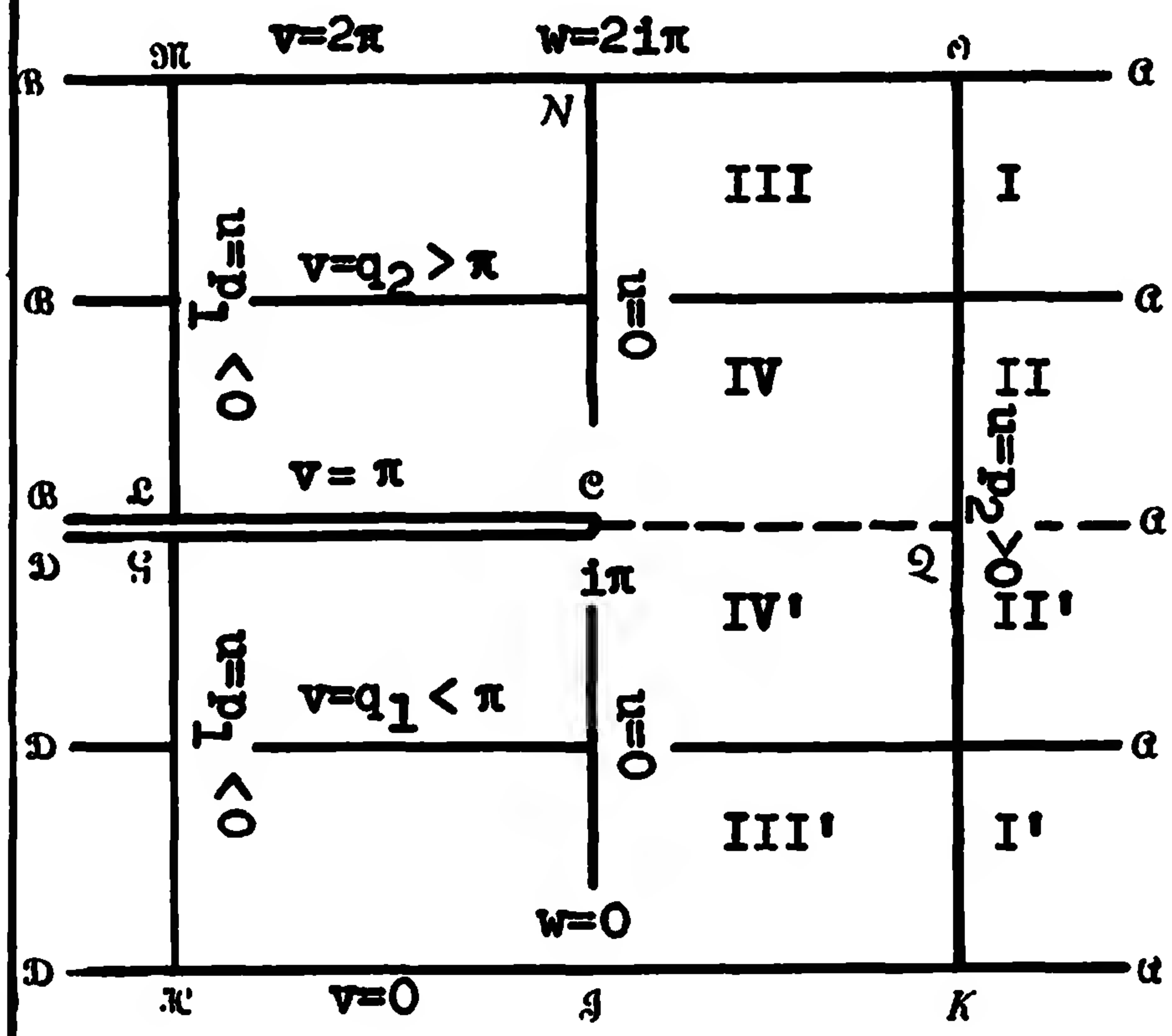
Critical points: $z = 0$; ∞; $e^{2k\pi i/\alpha}$.

$k = 0, \pm 1, \ldots$; $\ell = 0, \pm 1, \ldots$; p, q real.

z - plane	w - plane
point $z = z_0 e^{2k\pi i/\alpha}$	point $w = w_0 + 2\ell\pi i$
$0 \le \arg z \le 2\pi/\alpha$	$0 \le v \le 2\pi$
points 0; 1; $e^{i\pi/\alpha}$; $e^{2i\pi/\alpha}$; $2^{1/\alpha}$; $2^{1/\alpha} e^{2i\pi/\alpha}$ ($\mathfrak{G}$)	points $i\pi$; ∞; $\log 2 + i\pi$; ∞; 0; $2i\pi$.
	
arc $0 < \varphi < 2\pi/\alpha$ of circle $z = e^{i\varphi}$	curve $2\cos v = -e^{u}$ $[u \le \log 2, \lvert v - \pi \rvert < \pi/2]$.
half-line $\arg z = \pi/\alpha$	half-line $v = \pi$, $u \ge 0$.

<u>Lines parallel to axes of w - plane; α is an integer n.</u>

In this figure, $\alpha = n = 4$.

z - plane; $z = re^{i\varphi}$	w - plane
	
part, lying in the interior of angle $0 < \arg z < \frac{\pi}{n}$, of generalised Cassinian $\lvert z^n - 1 \rvert = e^p$	line segment $u = p,\ 0 < v < \pi$
part, lying in angle $\frac{\pi}{n} < \arg z < \frac{2\pi}{n}$	line segment $u = p,\ \pi < v < 2\pi$
part $0 < \varphi < \frac{q}{n}$ of generalised hyperbola $r^n = \dfrac{\sin q}{\sin(q - n\varphi)}$	line $v = q\ \ (0 < q < \pi)$
part $\frac{q}{n} < \varphi < \frac{2\pi}{n}$ of $r^n = \dfrac{\sin q}{\sin(q - n\varphi)}$	line $v = q\ \ (\pi < q < 2\pi)$

For the generalised Cassinians and hyperbolae, see §6.2.

The latter curves pass through $z_k = e^{2k\pi i/n}$ $(k = 0, 1, \ldots, n)$.

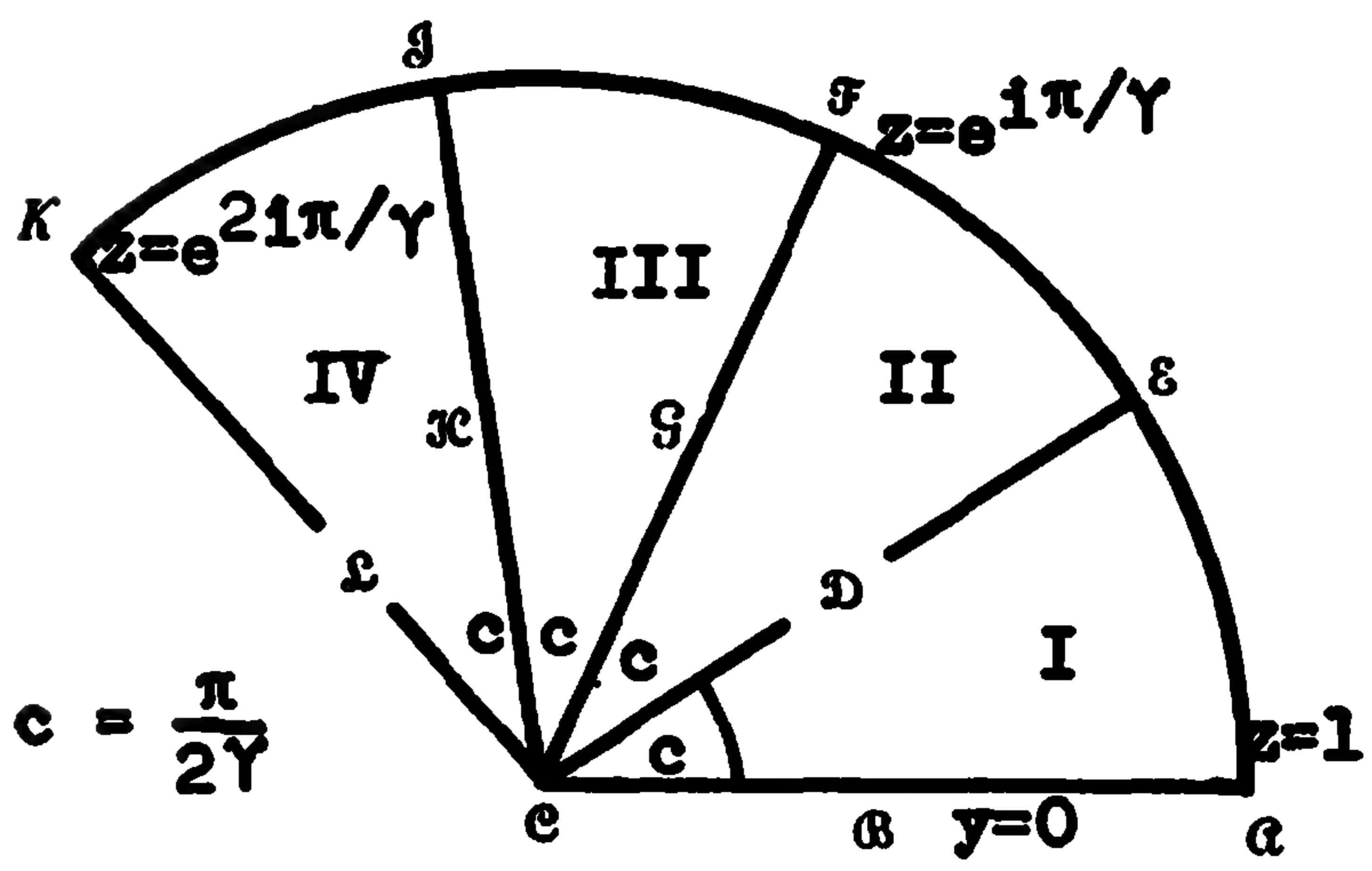
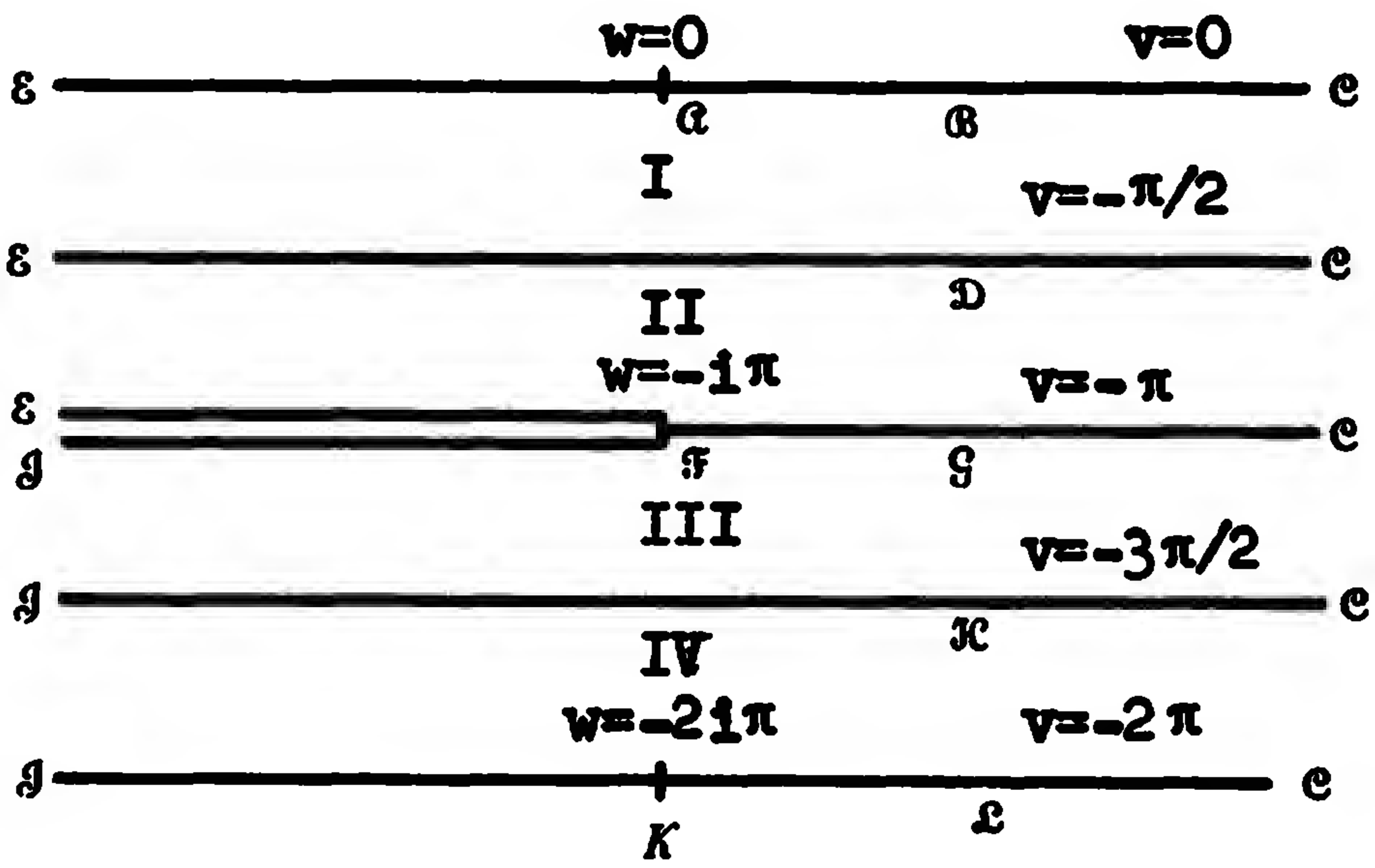
$\boxed{w = \log(z^{\alpha} + a)}$, $\alpha > 1$, $a \neq 0$.

Combination of $z = (-a)^{1/\alpha}\zeta$, $w = \xi + \log(-a)$, and $\xi = \log(\zeta^{\alpha} - 1)$.

11.3 $\boxed{w = \log \dfrac{z^{\gamma} + z^{-\gamma}}{2}}$, $\gamma > 0$; $\boxed{z = (e^{w} \pm \sqrt{e^{2w} - 1})^{1/\gamma}}$.

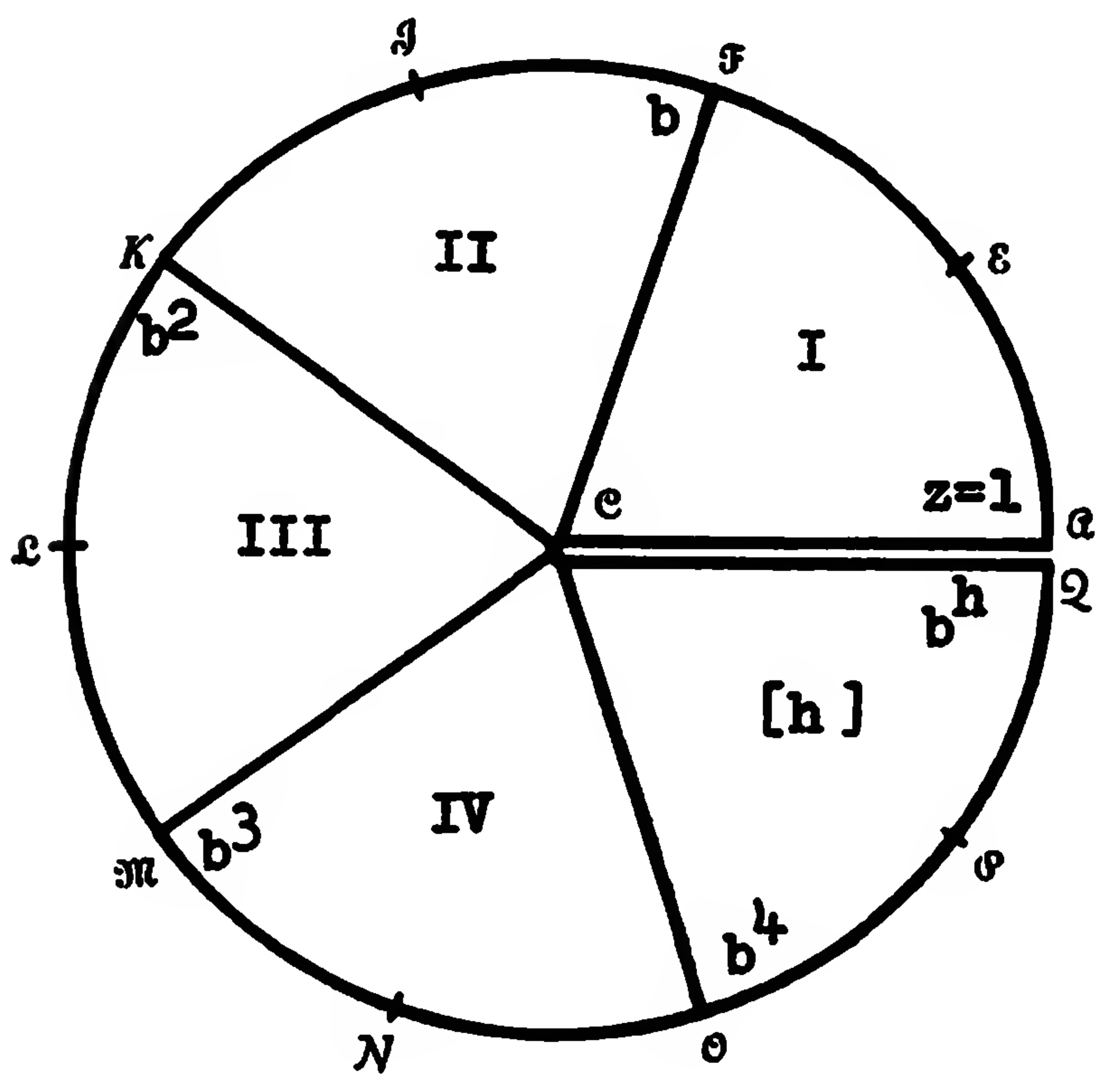
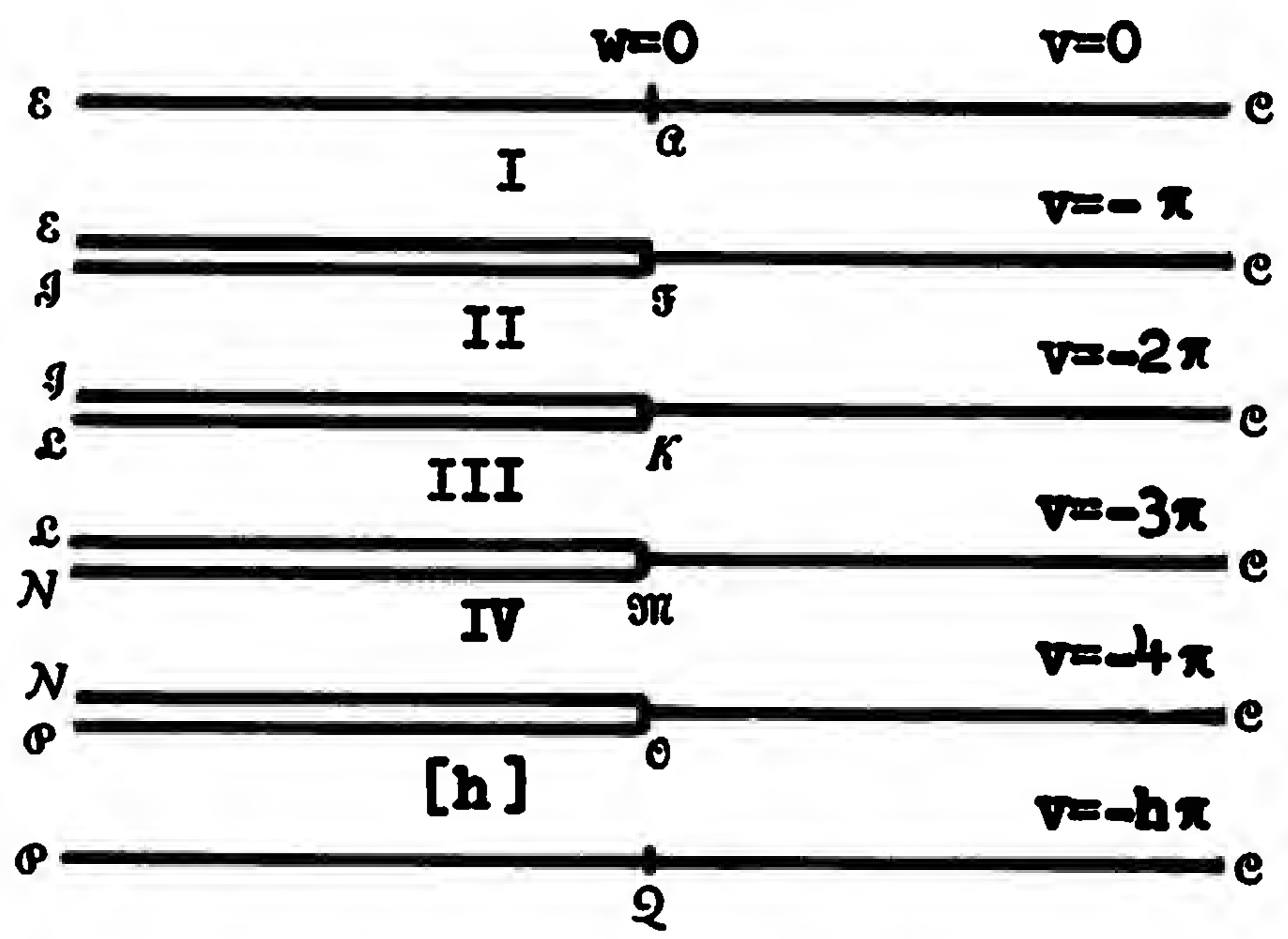
Critical points: $z = 0$; ∞; $e^{k\pi i/\gamma}$.

$k = 0, \pm 1, \pm 2, \ldots$

z - plane	w - plane
sector area $0 < \arg z < \frac{\pi}{\gamma}$, $\lvert z\rvert < 1$	strip $2k\pi > v > (2k-1)\pi$
sector area $\frac{\pi}{\gamma} < \arg z < \frac{2\pi}{\gamma}$, $\lvert z\rvert < 1$	strip $(2k-1)\pi > v > (2k-2)\pi$.
region $0 < \arg z < \frac{\pi}{2\gamma}$, $\lvert z\rvert > 1$	strip $2k\pi < v < (2k + \frac{1}{2})\pi$.
	
sector area $0 < \arg z < \frac{h\pi}{\gamma}$, $\lvert z\rvert < 1$; $(h = 1, 2, 3, \ldots)$	strip $2k\pi > v > (2k-h)\pi$, with $h-1$ slits
points $z_0 e^{2h\pi i/\gamma}$; $z_0^{-1} e^{2h\pi i/\gamma}$ $(h = 0, \pm 1, \ldots)$	points $w_0 + 2k\pi i$; $w_0 + 2k\pi i$.

z - plane, $0 < \arg z < 2\pi/\gamma$	w - plane, $0 \geq v > -2\pi$
points $e^{i\pi/(3\gamma)}$; $e^{2i\pi/(3\gamma)}$; $e^{i\pi/\gamma}$; $e^{4i\pi/(3\gamma)}$; $e^{5i\pi/(3\gamma)}$; $e^{\frac{i\pi}{2\gamma}}\left(\frac{\sqrt{5}-1}{2}\right)^{1/\gamma}$; $e^{\frac{3i\pi}{2\gamma}}\left(\frac{\sqrt{5}-1}{2}\right)^{1/\gamma}$ $e^{\frac{i\pi}{2\gamma}}(\sqrt{2}-1)^{1/\gamma}$; $e^{\frac{3i\pi}{2\gamma}}(\sqrt{2}-1)^{1/\gamma}$	points $-\log 2$; $-\log 2 - i\pi$; $i\pi$; $-\log 2 - i\pi$; $-\log 2$; $-\log 2 - \frac{i\pi}{2}$; $-\log 2 - \frac{3i\pi}{2}$; $-\frac{i\pi}{2}$; $-\frac{3i\pi}{2}$.

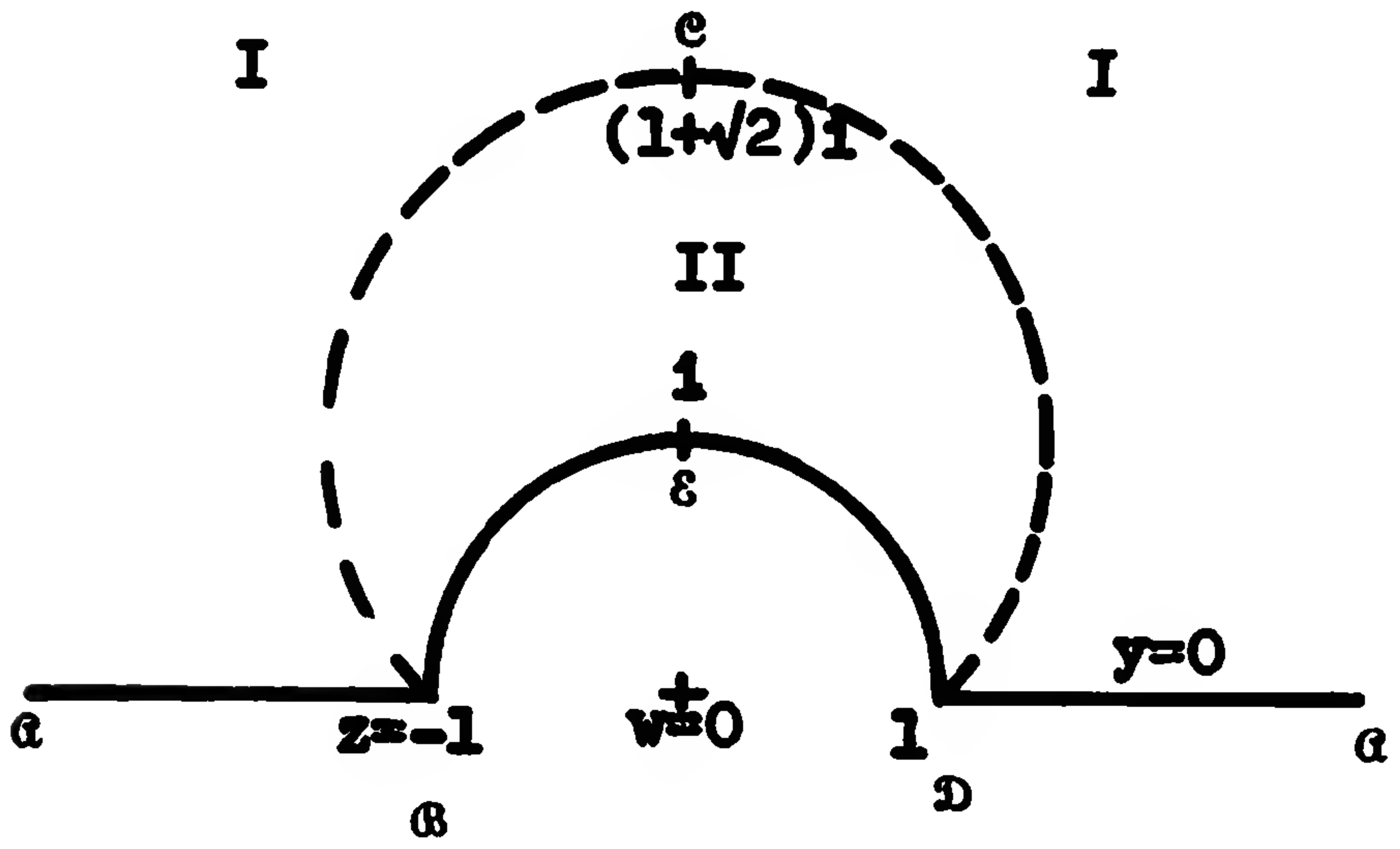
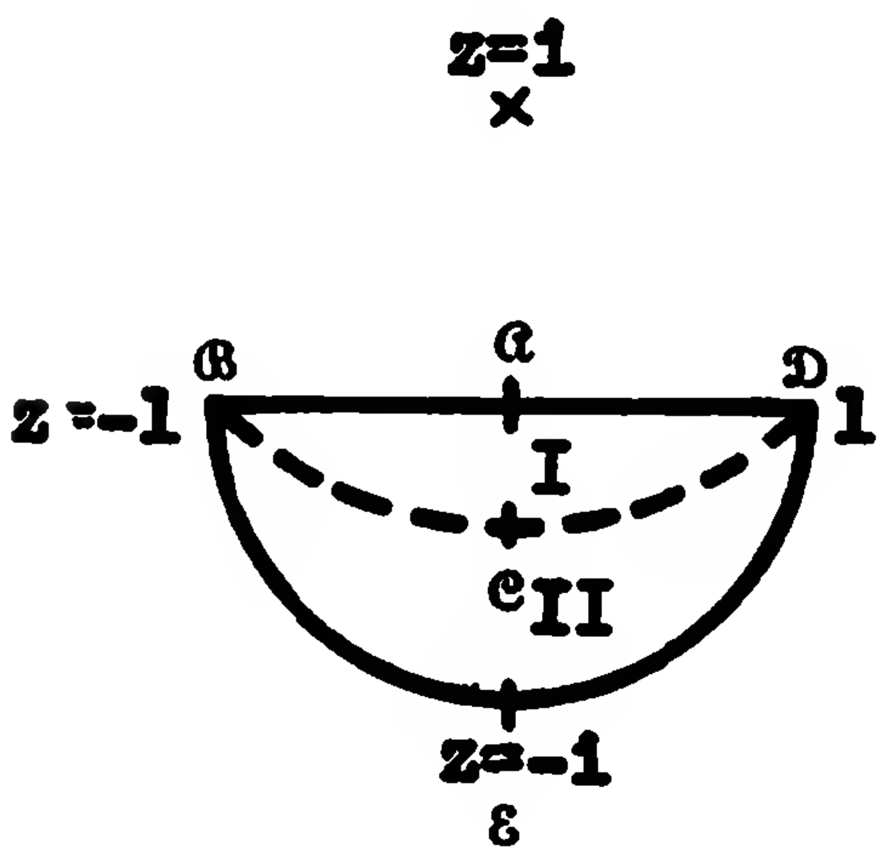
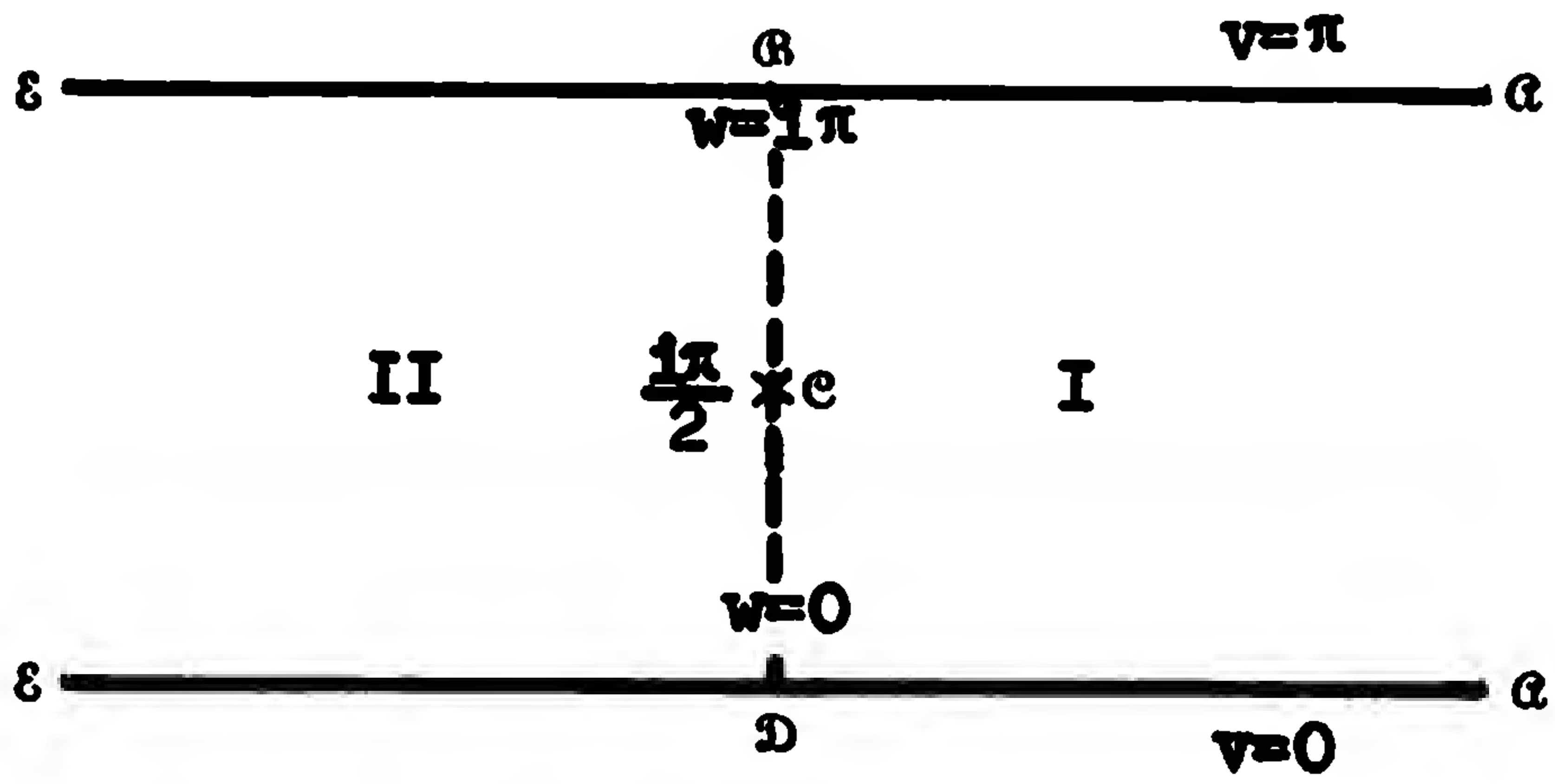
Case when 2γ is a positive integer h.

z - plane; $b = e^{i\pi/\gamma}$	w - plane; $0 \geq v \geq -h\pi$
	
cut interior of circle	strip, with h-1 slits

Example (i) $\boxed{w = \log \frac{z^2+1}{2z}}$, $\boxed{z = e^w \pm \sqrt{(e^{2w}-1)}}$ (i.e. $h = 2\gamma = 2$).

See previous figure; strip with one slit.

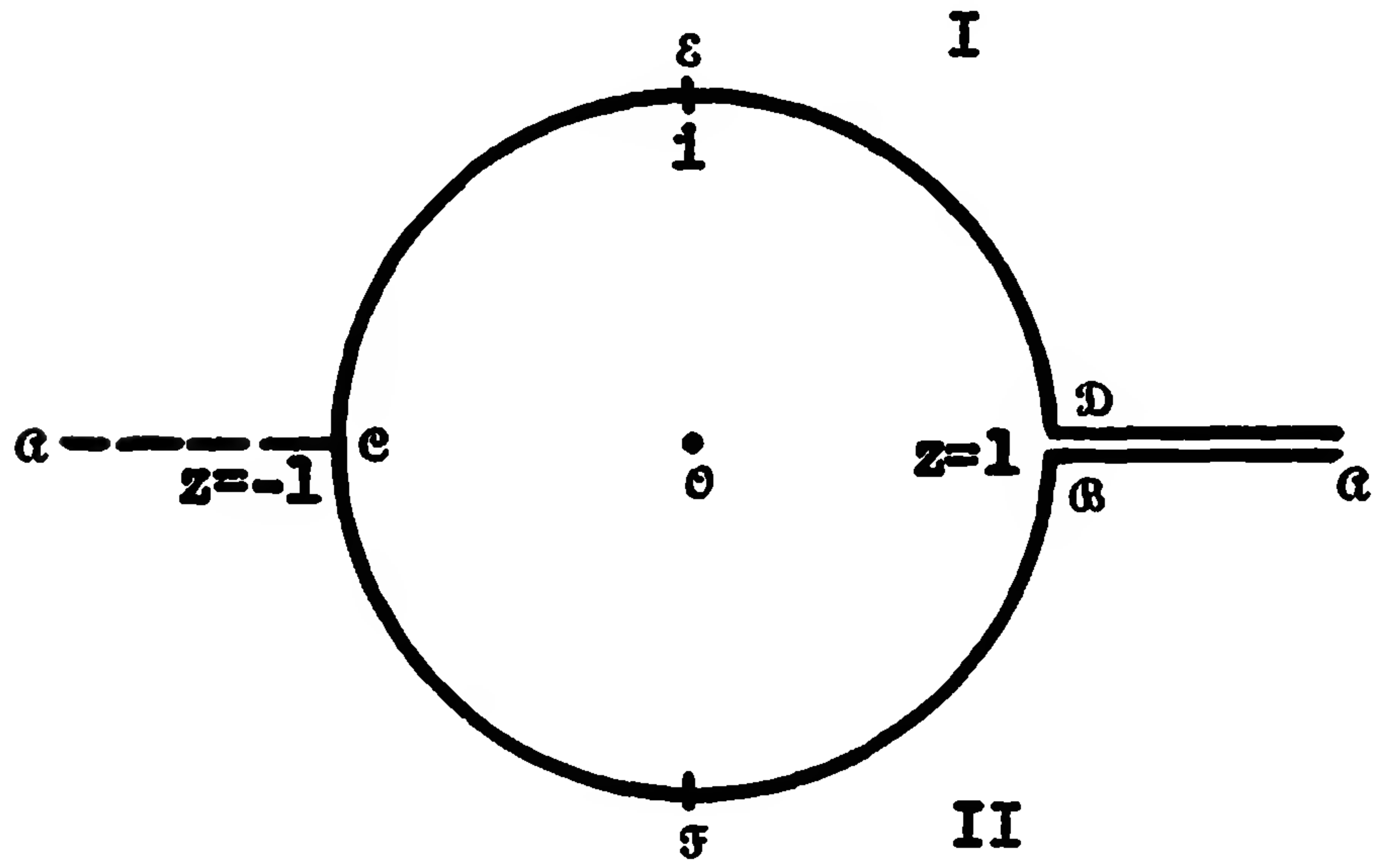
Alternative method

z - plane	w - plane, $0 \leq v \leq \pi$
 OR 	
points $z = 1;\ i(1 \pm \sqrt{2})$ (i.e., 𝔈); -1	points $w = 0;\ i\pi/2;\ i\pi$
arc 𝔇 𝔈 𝔅 of circle $\lvert z-i \rvert = \sqrt{2}$	line-segment $u = 0,\ 0 \leq v \leq \pi$
half-line $x = 0,\ 1 < y < \infty$ line-segment $x = 0,\ -1 < y < 0$	line $v = \frac{\pi}{2},\ -\infty < u < \infty$.

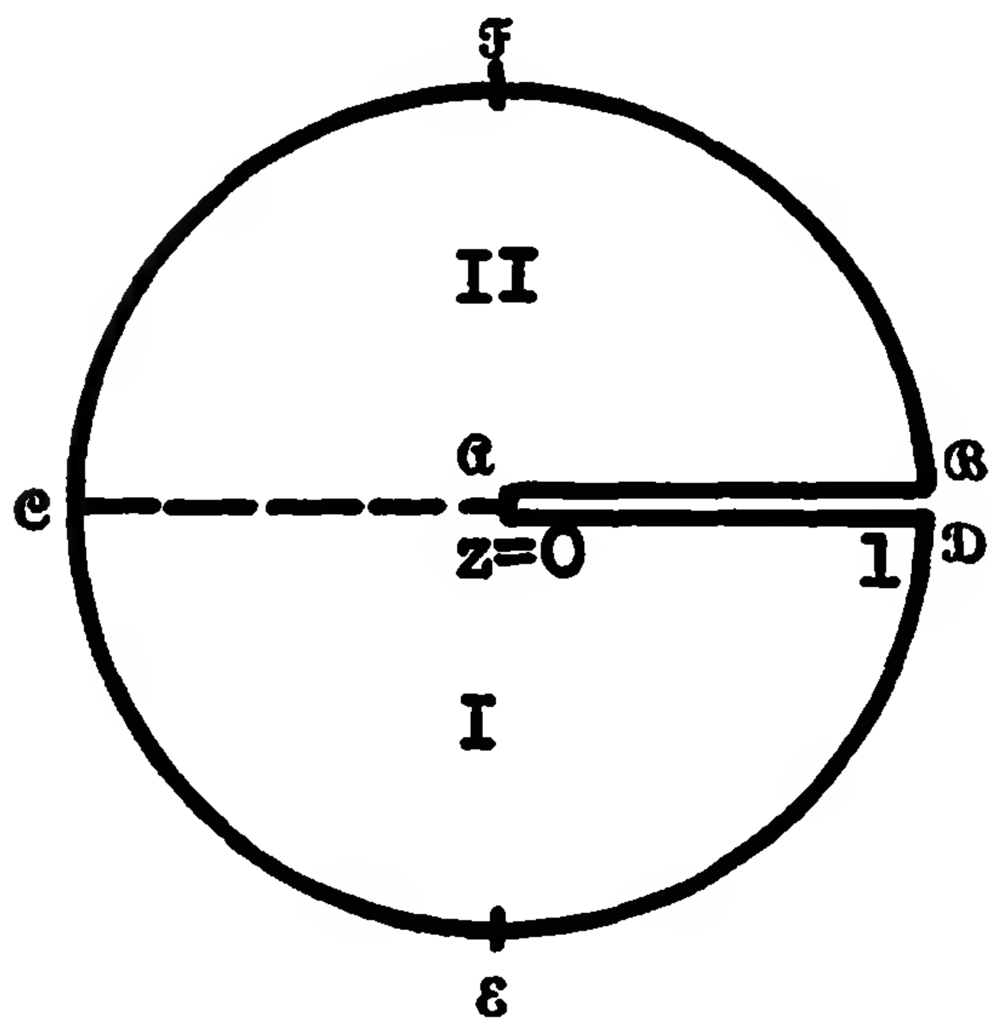
Example (ii) $\boxed{w = \log \frac{z+1}{2} - \frac{1}{2}\log z}$; $\boxed{z = (e^{w} \pm \sqrt{e^{2w}-1})^2}$

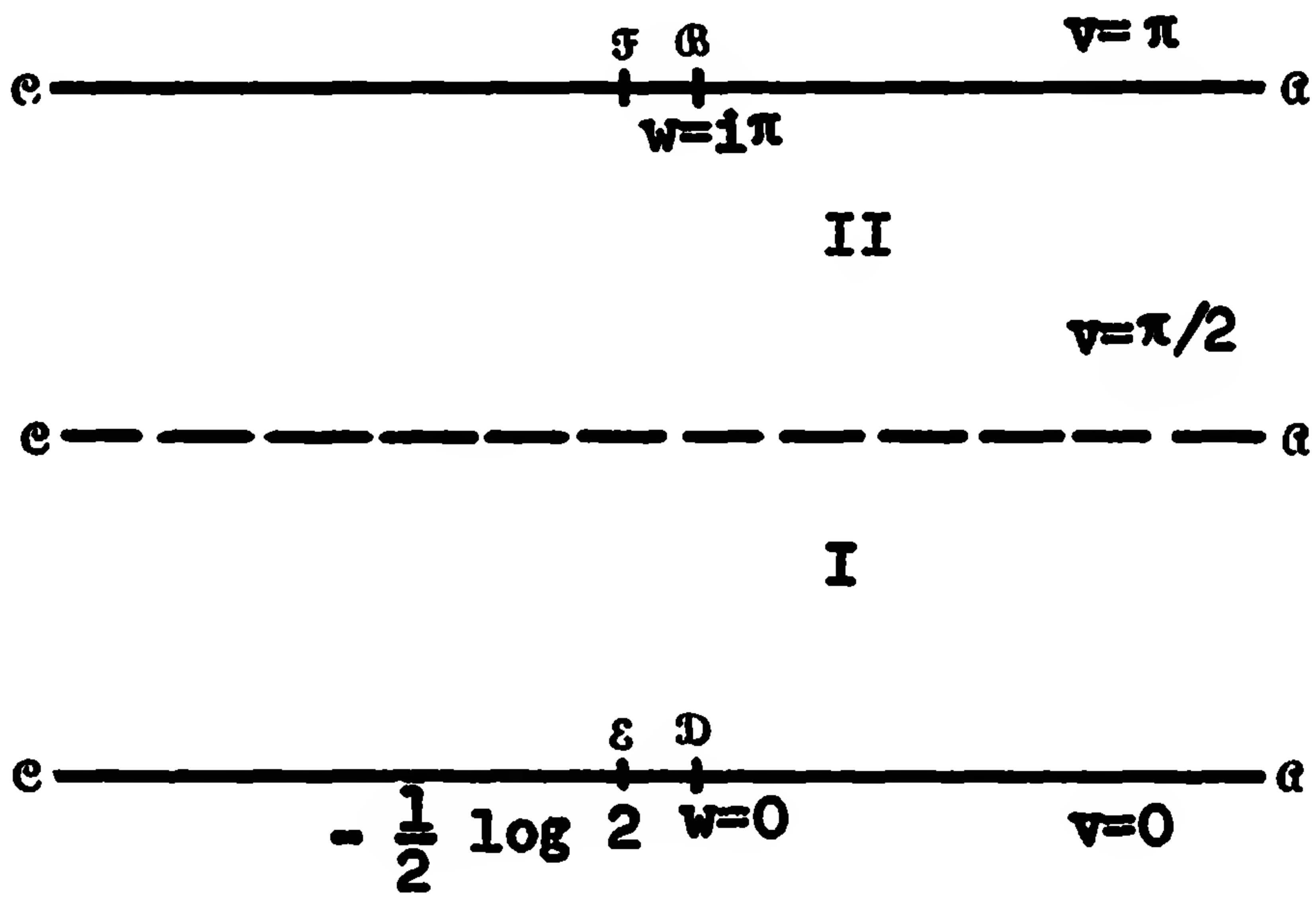
(i.e., $h = 2\gamma = 1$).

z - plane	w - plane, $0 \le v \le \pi$



OR



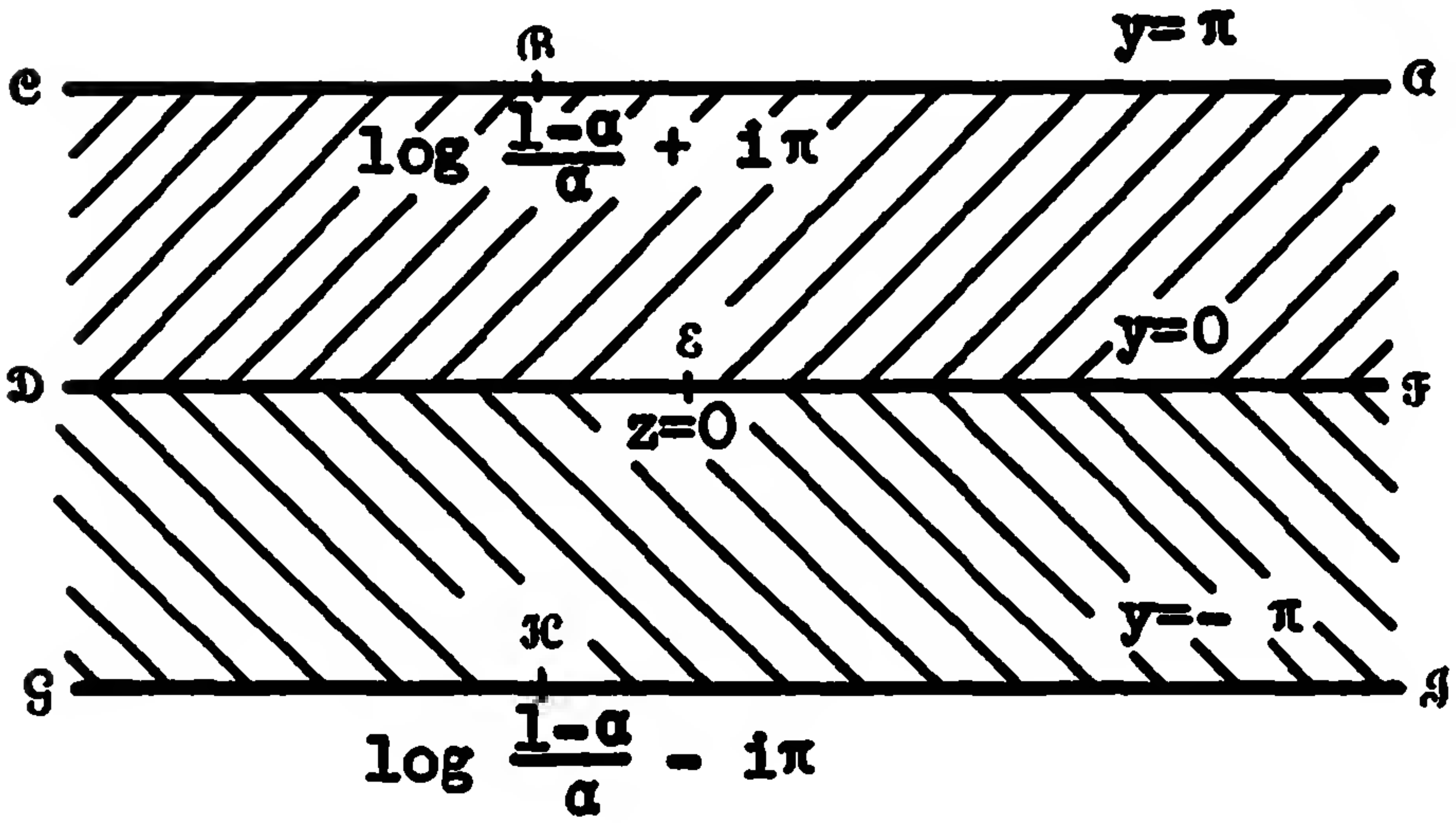


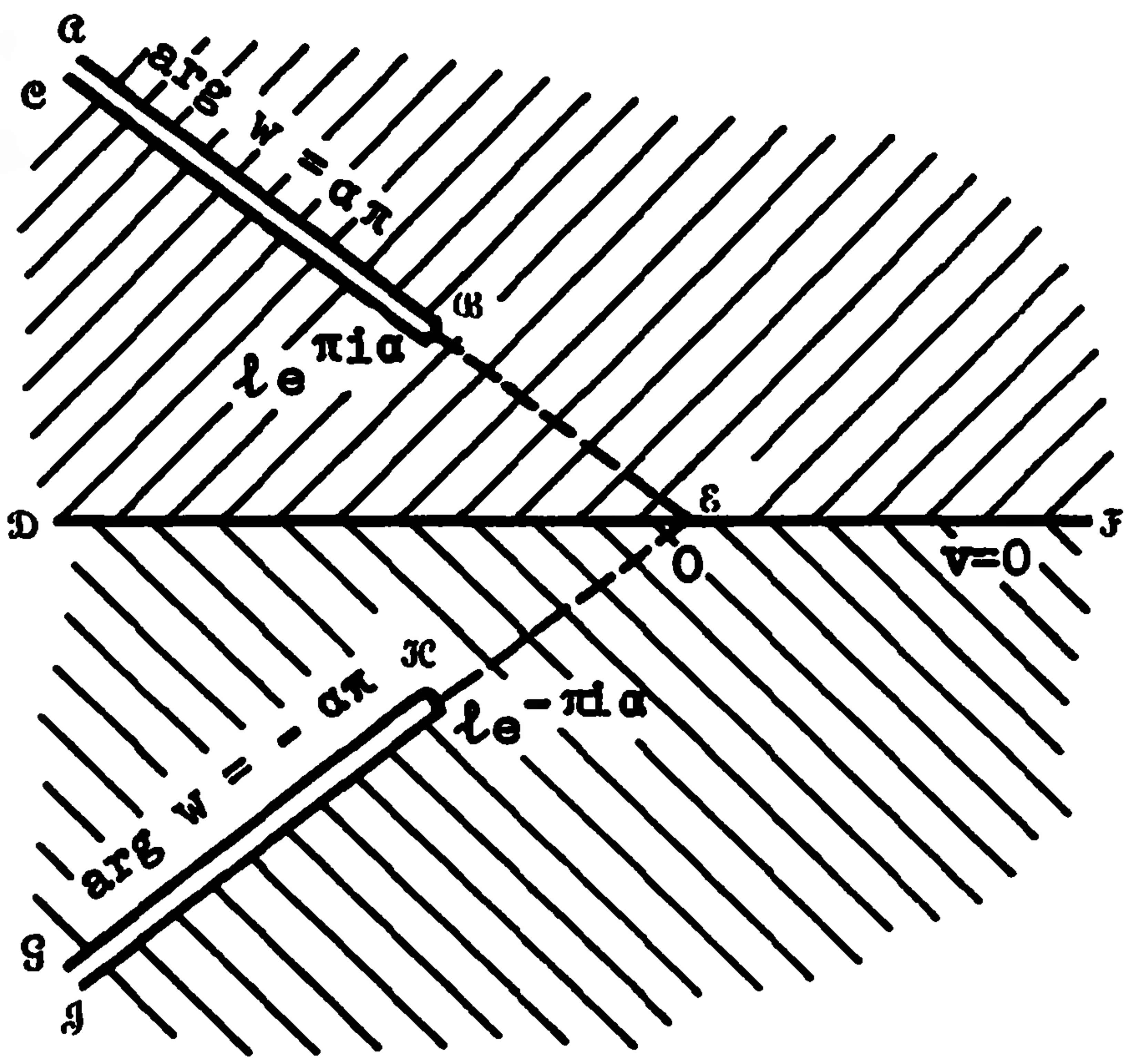
11.4 $\boxed{w = e^{\alpha z} - e^{(\alpha-1)z}}$; $0 < \alpha < 1$. Cf. §6.2, p. 46.

Critical points: $z = \infty$; $\log \frac{1-\alpha}{\alpha} + (2k+1)\pi i$ $(k = 0, \pm 1, \ldots)$.

$$\ell = \alpha^{-\alpha}(1-\alpha)^{\alpha-1}.$$

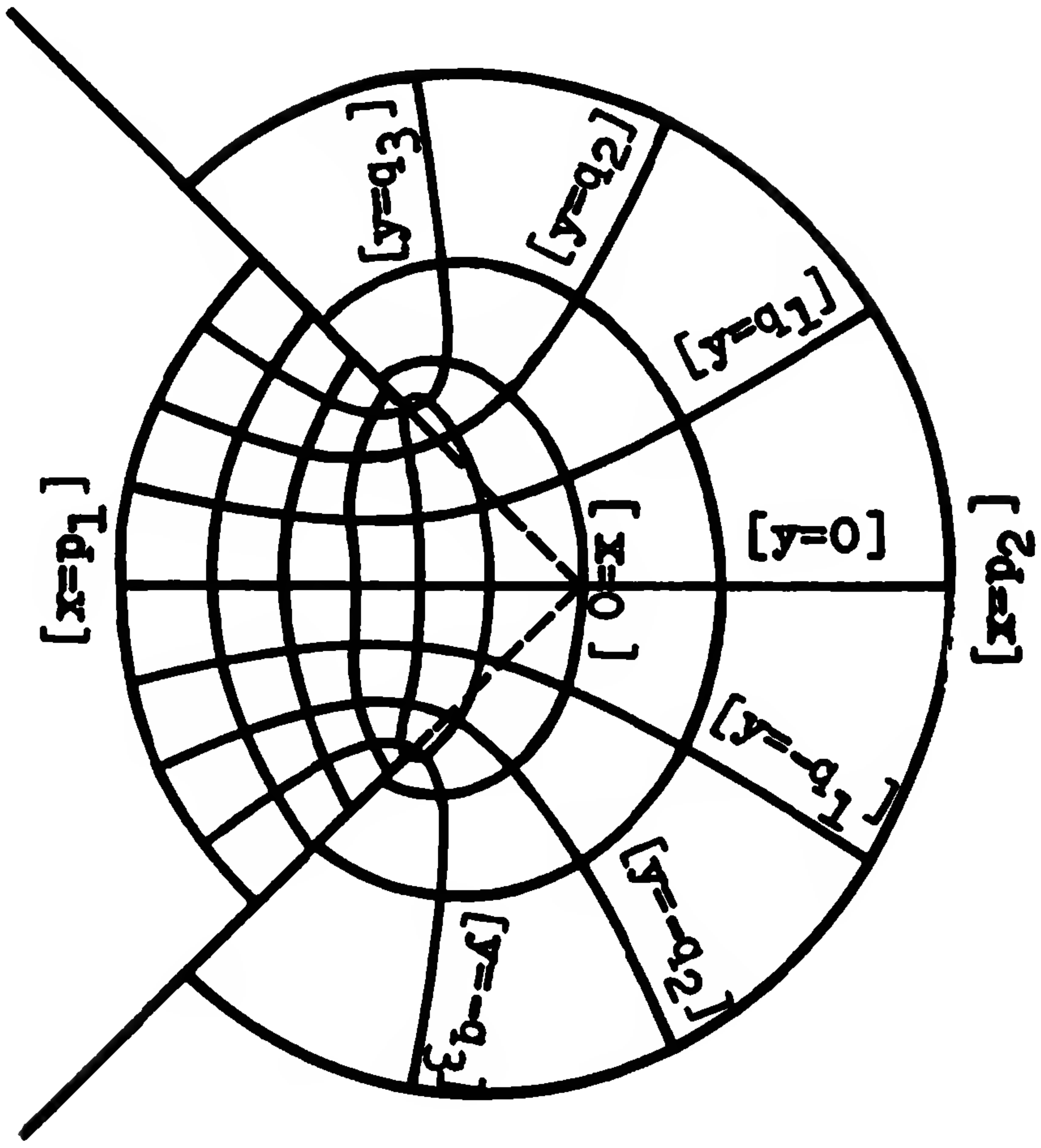
z - plane; $-\pi \leq y \leq \pi$	w - plane
half-line $y = \pi$, $\log \frac{1-\alpha}{\alpha} \leq x < \infty$ half-line $y = \pi$, $\log \frac{1-\alpha}{\alpha} \geq x > -\infty$	line segment $\arg w = \alpha\pi$, $\ell \leq \lvert w \rvert < \infty$
half-line $y = -\pi$, $\log \frac{1-\alpha}{\alpha} \leq x < \infty$ half-line $y = -\pi$, $\log \frac{1-\alpha}{\alpha} \geq x > -\infty$	line segment $\arg w = -\alpha\pi$, $\ell \leq \lvert w \rvert < \infty$
line $y = 0$, $-\infty < x < \infty$	line $v = 0$, $-\infty < u < \infty$
points $z = 0$; $\log \frac{1-\alpha}{\alpha} + i\pi$; $\log \frac{1-\alpha}{\alpha} - i\pi$	points $w = 0$; $\ell e^{i\pi\alpha}$; $\ell e^{-i\pi\alpha}$





<u>Curves in the w - plane</u>, corresponding to the lines $x = p$, or $y = q$, respectively; $p_1 < 0 < p_2$; $0 < q_1 < q_2 < q_3 < \pi$.

w - plane



Reprinted by permission from "On two-dimensional fluid motion through spouts composed of two plane walls" by R. A. Harris, Annals of Math., 2nd series, vol. II, p. 73 etc., Princeton University Press. Details are given in that article.

$\boxed{w = ae^{fz} + be^{gz}}$; $\frac{f}{g}$ real, $\frac{f}{g} < 0$; $ab \neq 0$.

Combination of

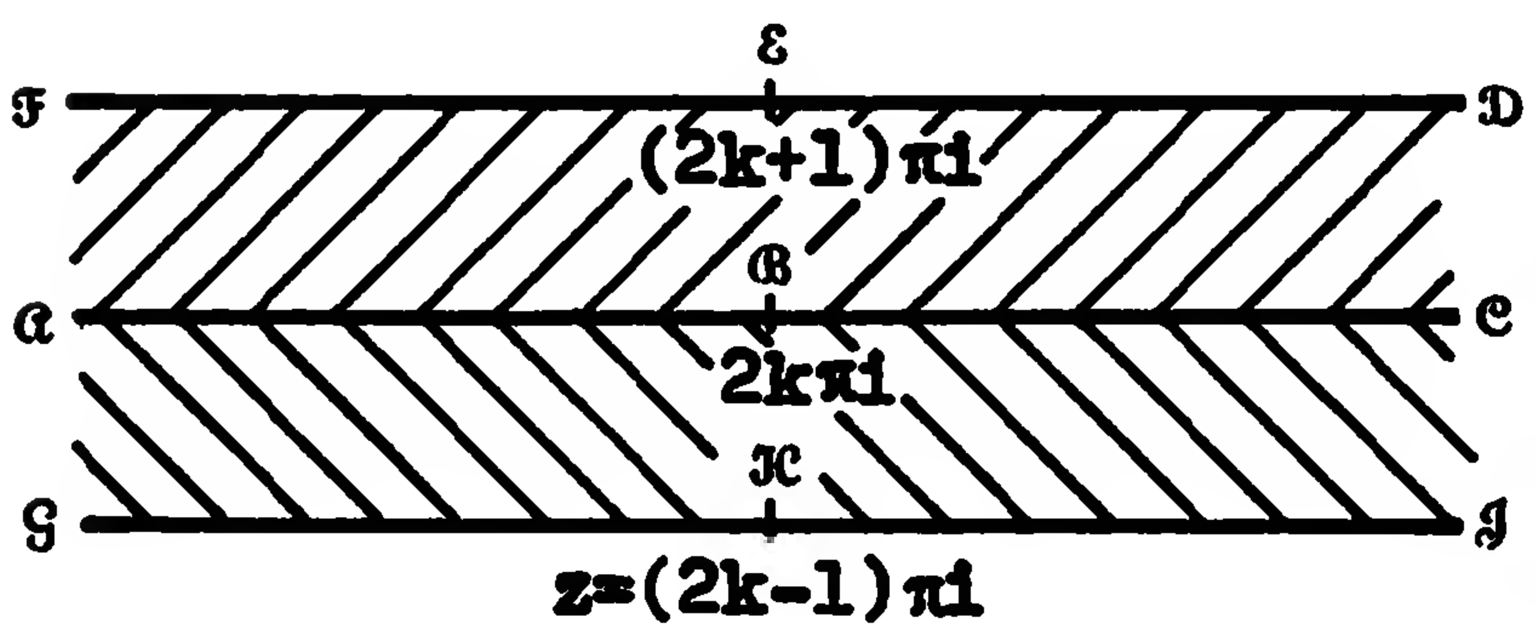
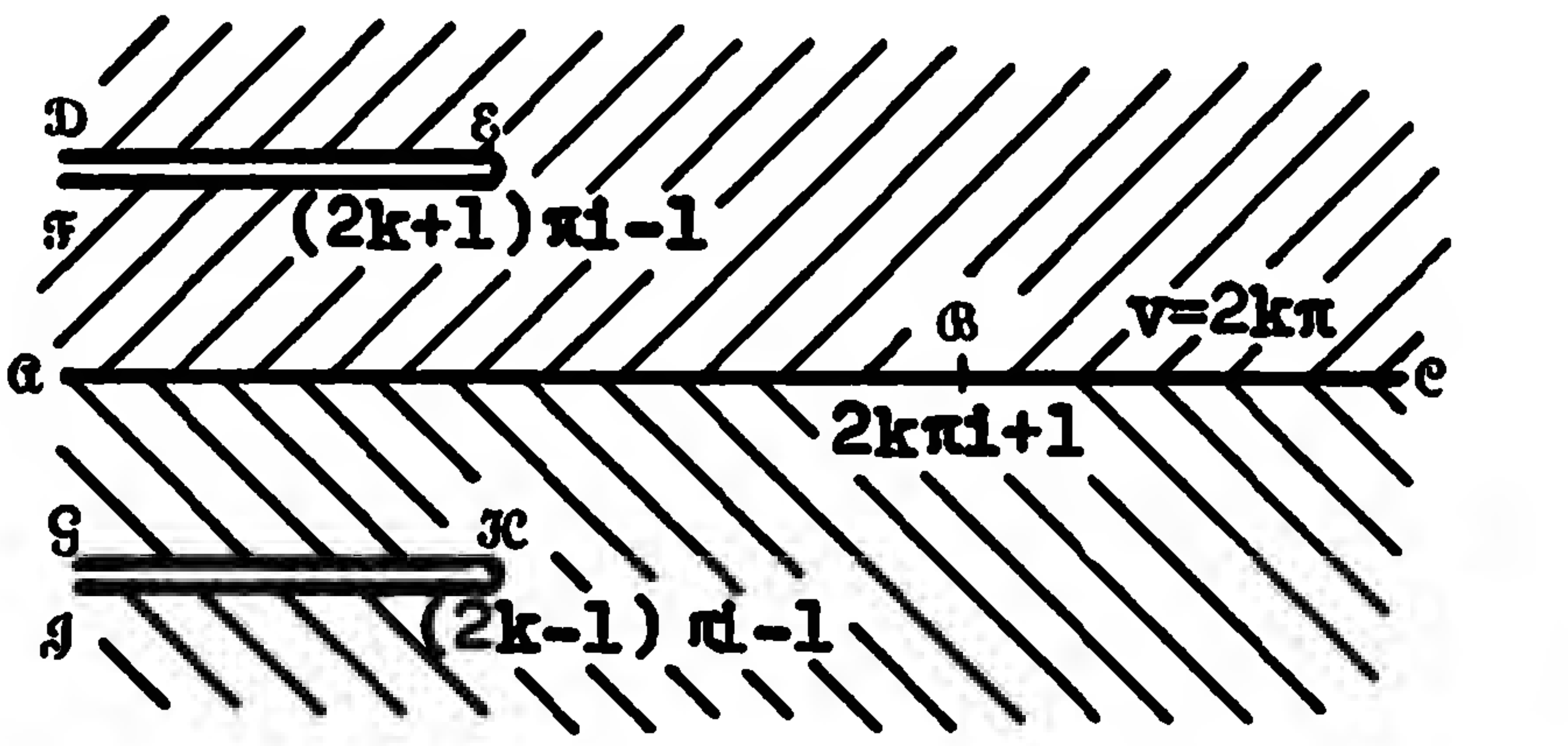
$$w = ae^{-fc}\xi, \quad z = \frac{\zeta}{f-g} - c, \quad \xi = e^{\alpha\zeta} - e^{(\alpha-1)\zeta},$$

where $c = \dfrac{\log(-b/a)}{g - f}$; $\quad \alpha = \dfrac{f}{f - g}$, $\quad 0 < \alpha < 1$.

11.5 $\boxed{w = z + e^z}$

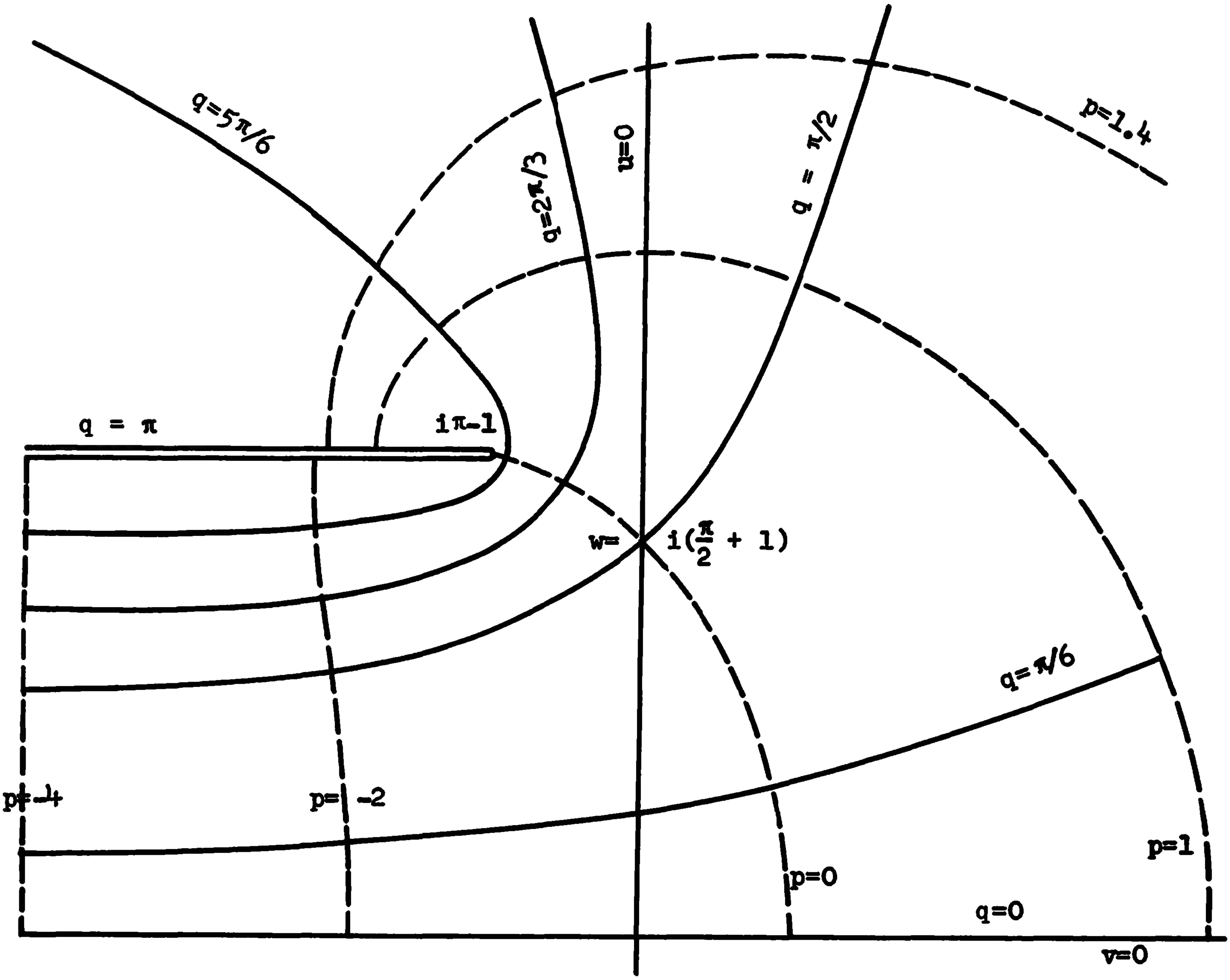
Critical points: $z = \infty$; $(2k+1)\pi i$.

$k = 0, \pm 1, \pm 2, \ldots$; p, q real.

z - plane	w - plane
line $x = p$	curve $v = \cos^{-1} \dfrac{u-p}{e^p} + \left\{ e^{2p} - (u-p)^2 \right\}^{1/2}$
line $y = q$	curve $u = \log \dfrac{v-q}{\sin q} + (v-q)\cot q$
line $y = 2k\pi$	line $v = 2k\pi$
line $y = (2k+1)\pi$	half-line $v = (2k+1)\pi$, $-\infty < u \leq -1$, counted twice
$z = z_o$; $z_o + 2k\pi i$	$w_o = w(z_o)$; $w_o + 2k\pi i$
	
strip $(2k-1)\pi < y < 2k\pi$	half-plane $y < 2k\pi$, slit along $-\infty < u \leq -1$, $v = (2k-1)\pi$
strip $(2k-1)\pi < y < (2k+1)\pi$	whole plane slit along $-\infty < u \leq -1$, $v = (2k-1)\pi$ and $v = (2k+1)\pi$.

$\boxed{w = z + e^z}$

Curves in the w - plane corresponding to x = p and y = q.



$\boxed{w = az + be^{cz}}$, abc ≠ 0.

Combination of $w = \frac{a}{c}\xi - (\frac{a}{c}\log\frac{bc}{a})$, $\zeta = cz + (\log\frac{bc}{a})$

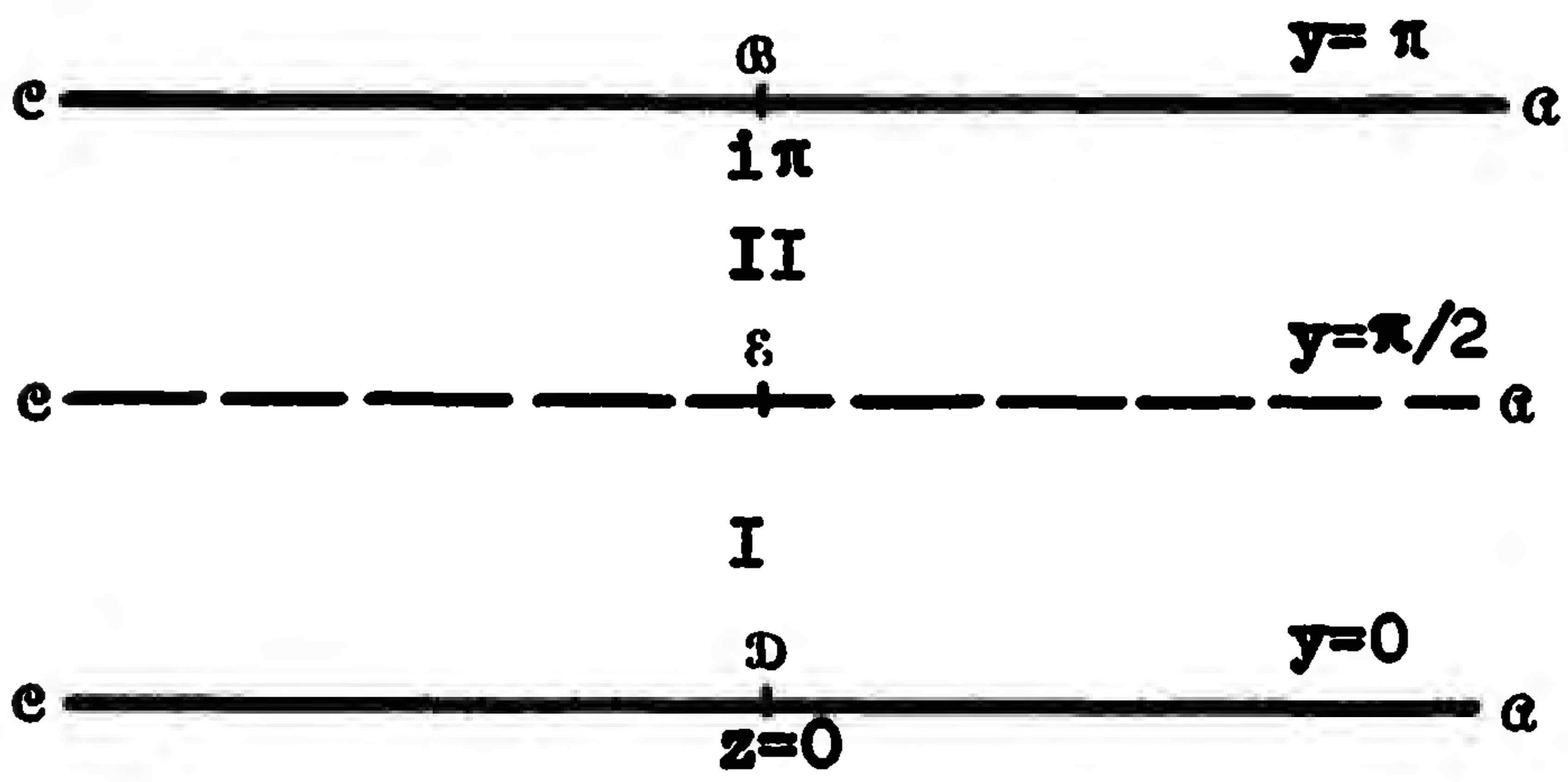
and $\xi = \zeta + e^{\zeta}$.

11.6 $\boxed{w = \log\left\{e^{z} + (e^{2z}-1)^{1/2}\right\}}$, $\boxed{z = \log\cosh w}$

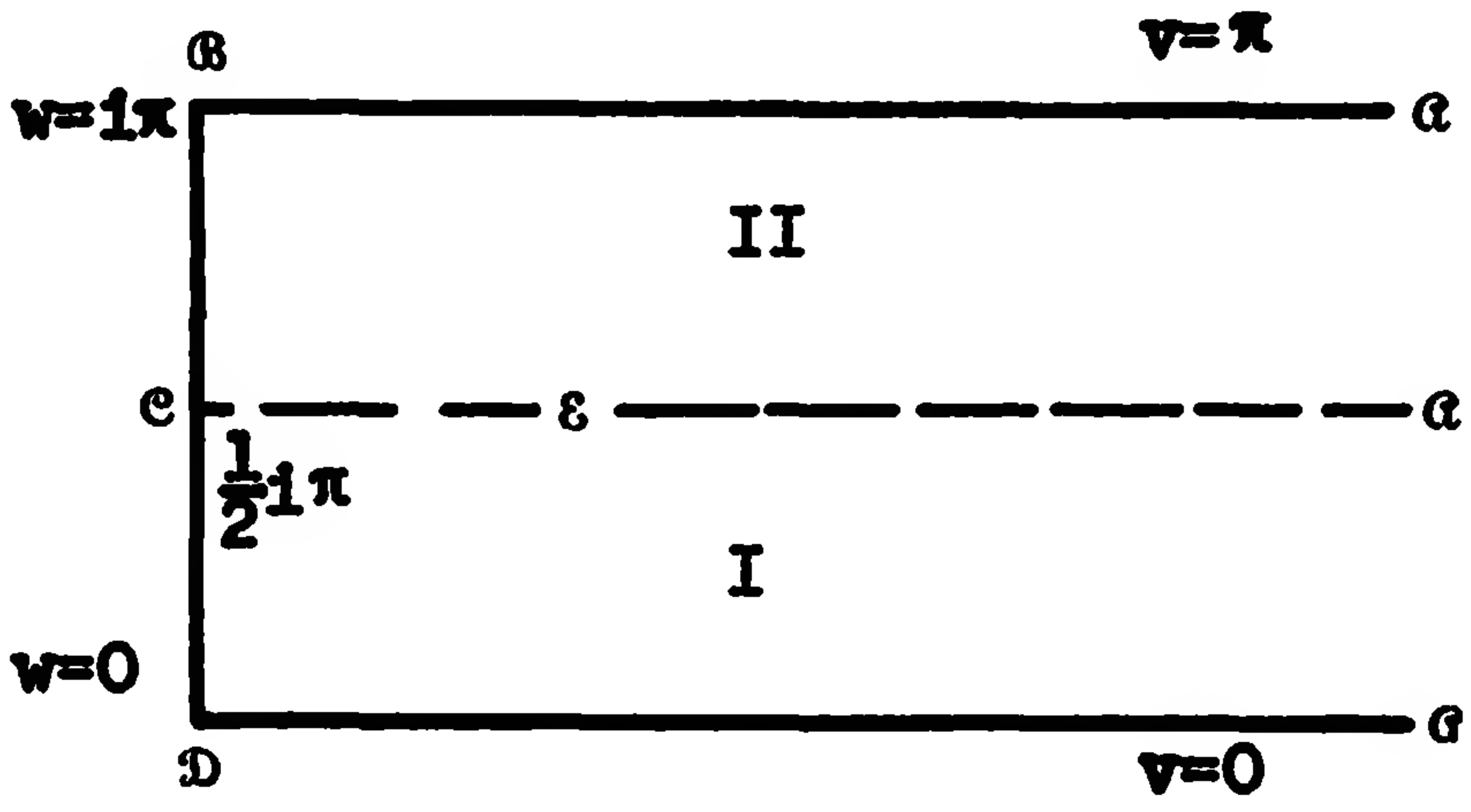
Combination of $w = \log\zeta$ and $z = \log\dfrac{\zeta^2+1}{2\zeta}$; cf. §11.3(i), p. 112.

Strip on semi-infinite strip.

z - plane	w - plane



points $z = 0;\ \frac{i\pi}{2};\ i\pi$

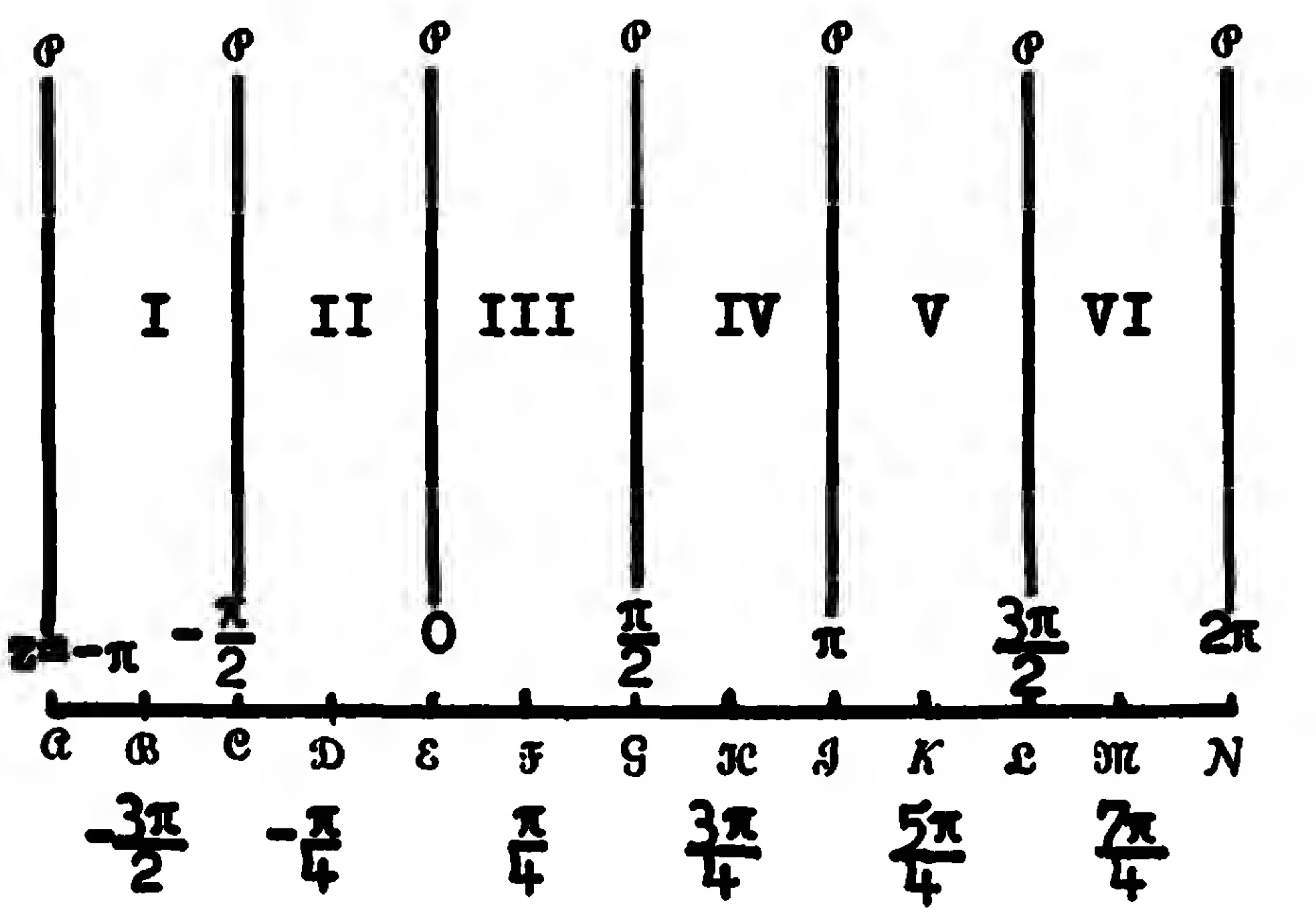


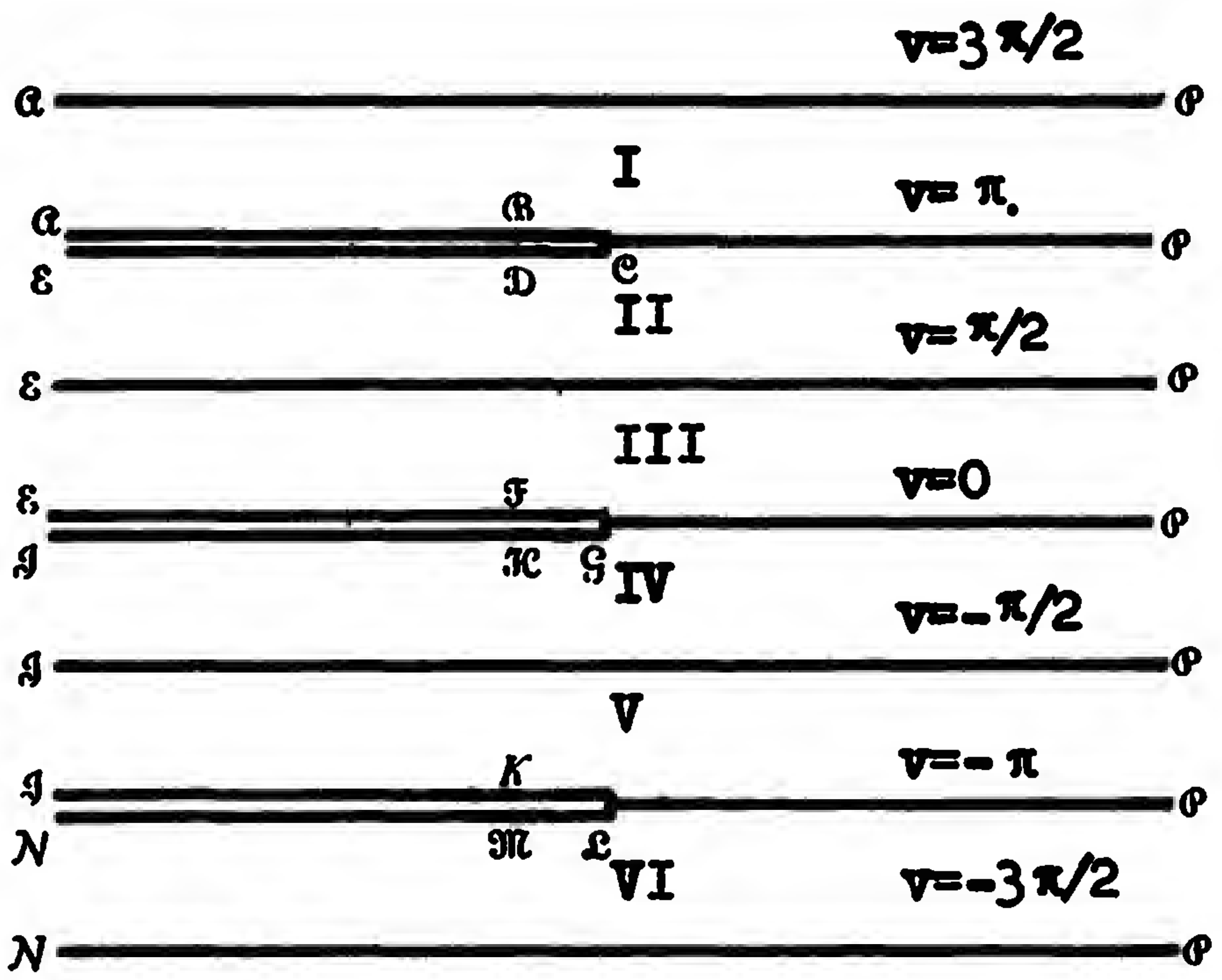
points $w = 0;\ \log(1+\sqrt{2}) + \frac{i\pi}{2};\ i\pi$

$\boxed{w = \log\sin z}$ $= \log\cosh(iz - \frac{i\pi}{2})$, $z = \sin^{-1} e^{w}$.

Critical points: $z = \infty;\ k\pi/2$ $(k = 0, \pm 1, \pm 2, \ldots)$.

z - plane	w - plane



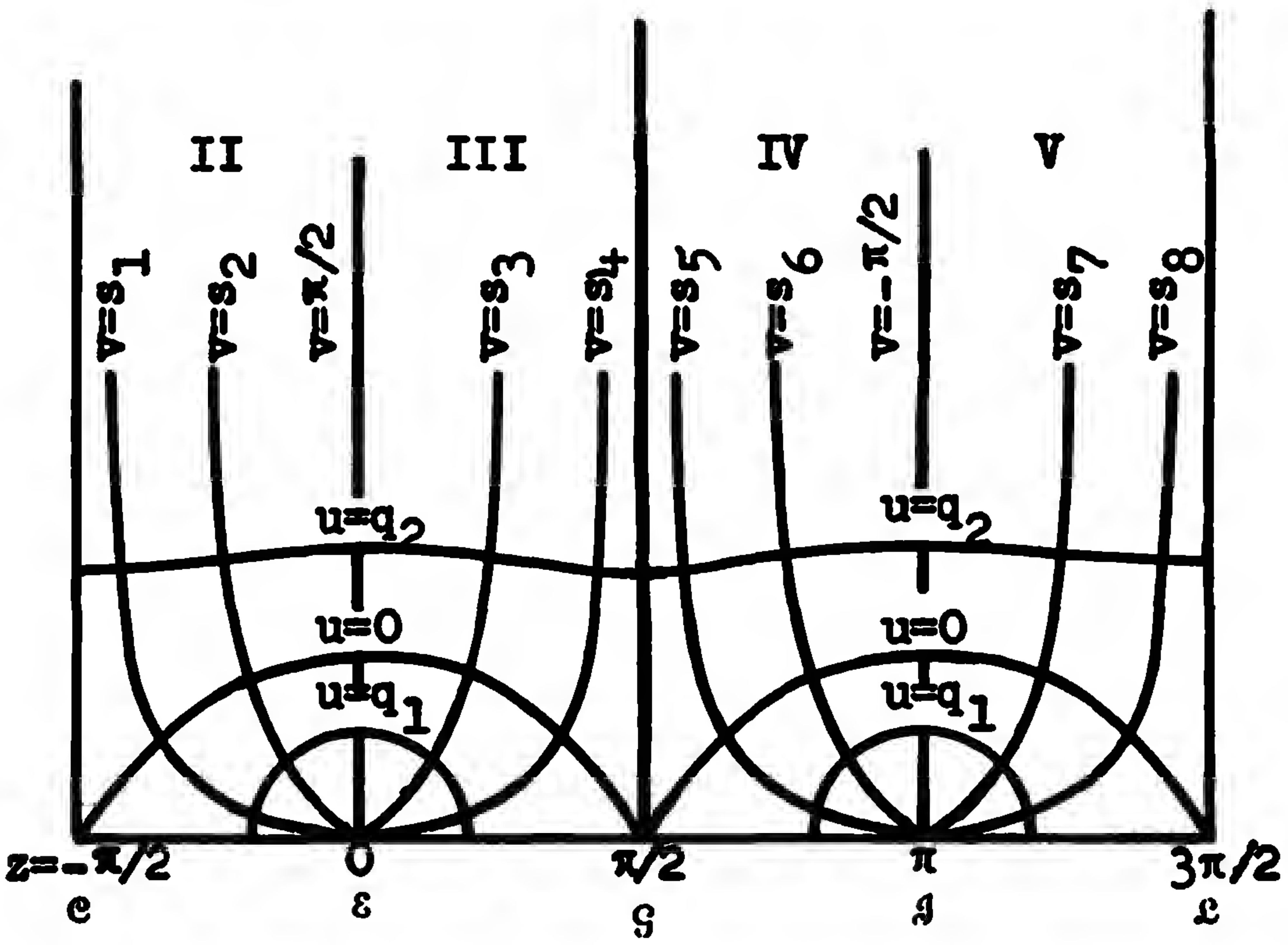


$\mathcal{L}$, $\mathcal{G}$, $\mathcal{C}$ lie on the line $u = 0$;
$\mathcal{M}$, K, $\mathcal{H}$, $\mathcal{F}$, $\mathcal{D}$, $\mathcal{B}$ on $u = -\frac{1}{2}\log 2$.

z - plane	w - plane
semi-infinite strip $-m\pi < x < n\pi$ (m, n non-negative integers), $y > 0$	strip $(m + \frac{1}{2})\pi > v > (-n + \frac{1}{2})\pi$, with m + n slits

Curves in the z - plane mapped on lines parallel to the axes of the w - plane.

z - plane



Reprinted by permission from the THEORY OF FUNCTIONS AS APPLIED TO ENGINEERING PROB¨MS by R. Rothe, F. Ollendorff and K. Pohlhausen, published by the Technology Press of the Massachusetts Institute of Technology.
The essential curves of fig. 43 are copied in this diagram.

q, s real constants; $-\infty < q_1 < 0 < q_2 < \infty$; $\pi > s_1 > s_2 > \frac{1}{2}\pi > s_3 > s_4 > 0 > s_5 > s_6 > -\frac{1}{2}\pi > s_7 > s_8 > -\pi$.

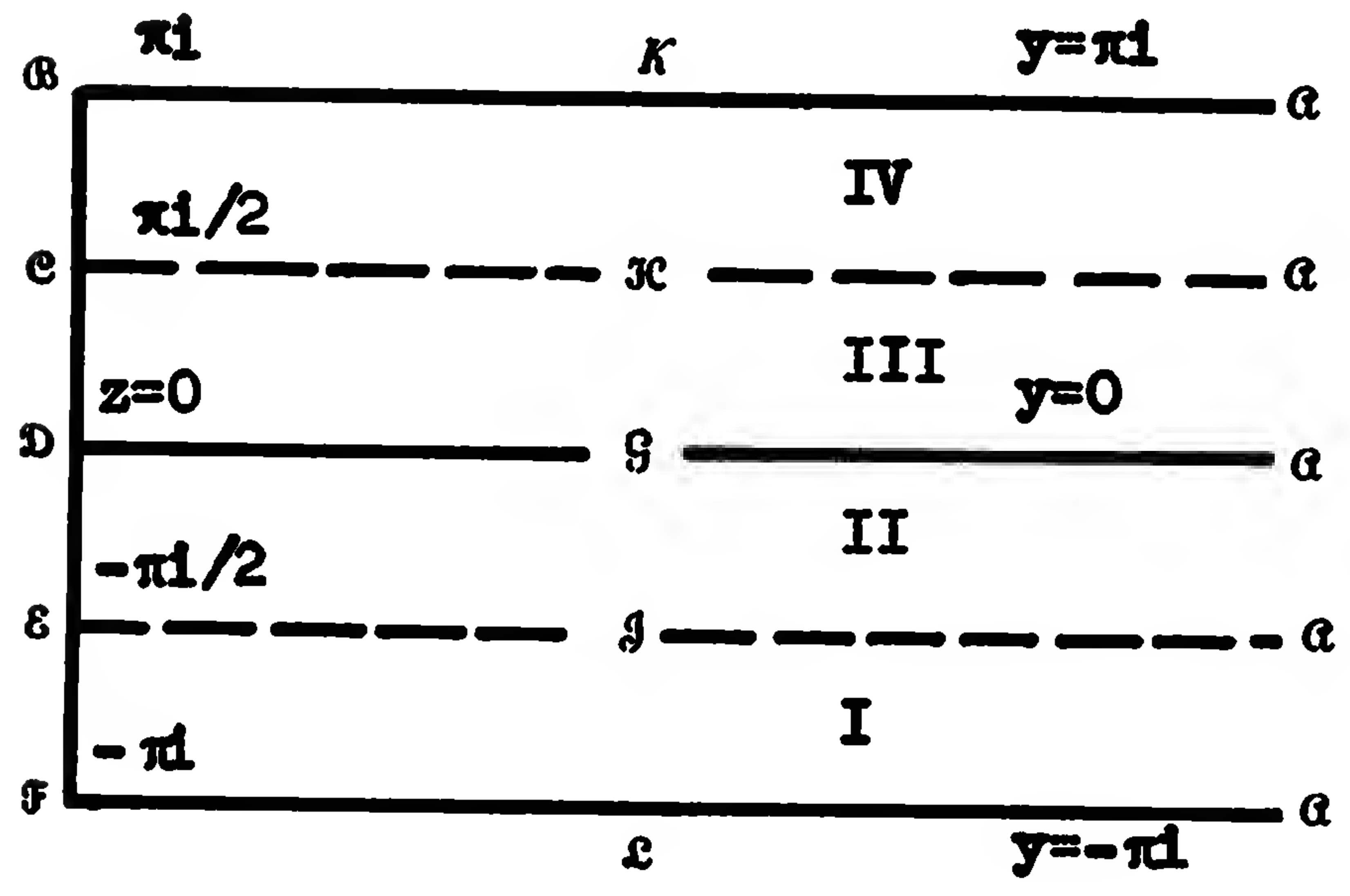
On the curves u = constant, $\pi > v > \frac{1}{2}\pi$ in II, $\frac{1}{2}\pi > v > 0$ in III, $0 > v > -\frac{1}{2}\pi$ in IV, $-\frac{1}{2}\pi > v > -\pi$ in V.

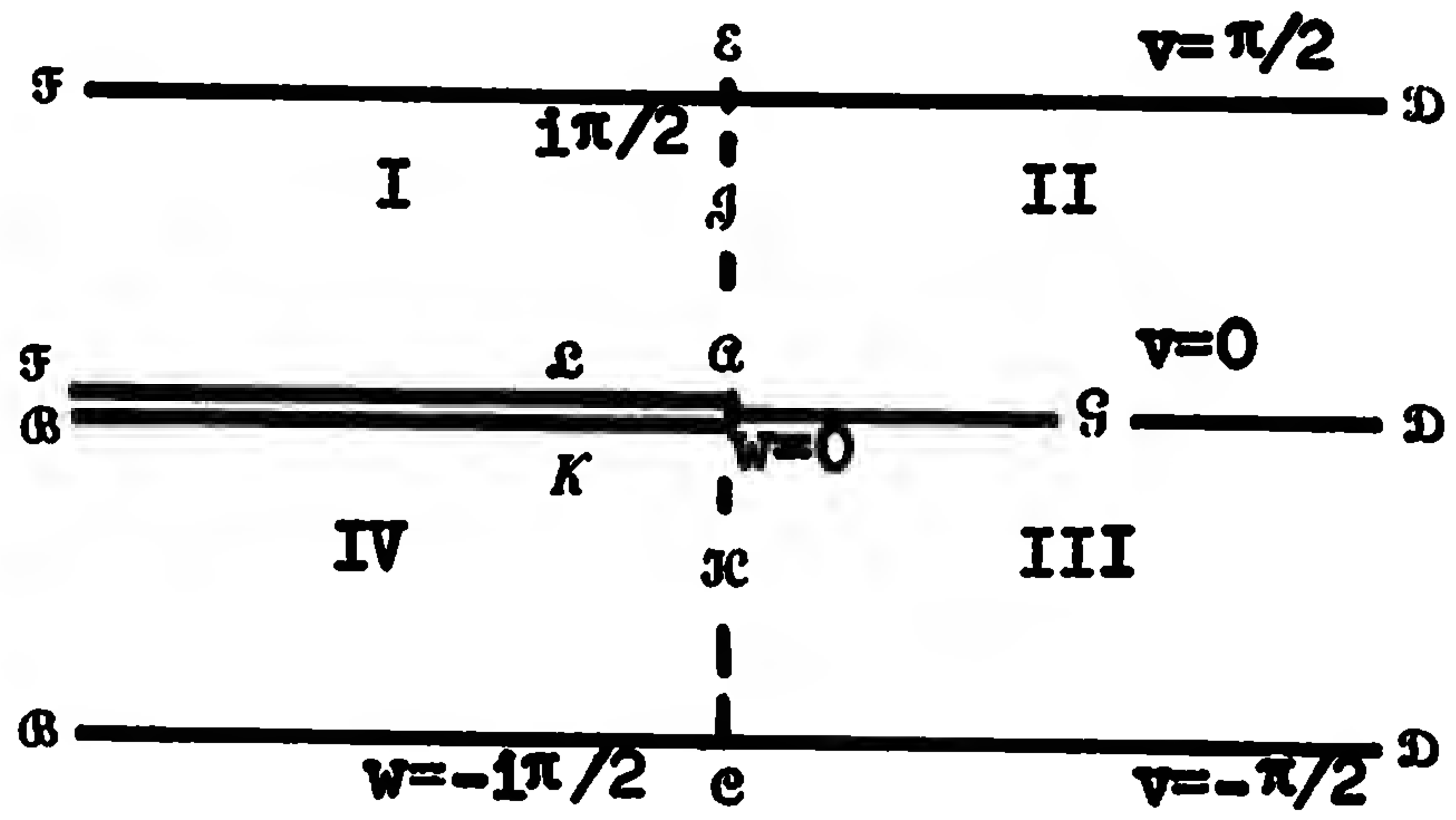
On the curves v = constant, $-\infty < u < \infty$.

z - plane	w - plane
part of $\cosh 2y - \cos 2x = 2e^{2q}$	part of line $u = q$
part of $\tanh y = \tan s \tan x$	line $v = s$ $(s \neq k\pi)$.

11.7 $\boxed{w = \log \coth \frac{z}{2}}$, $z = \log \coth \frac{w}{2}$; the transformation is involutory.

Critical points: $z = \infty$; $k\pi i$ $(k = 0, \pm 1, \ldots)$.

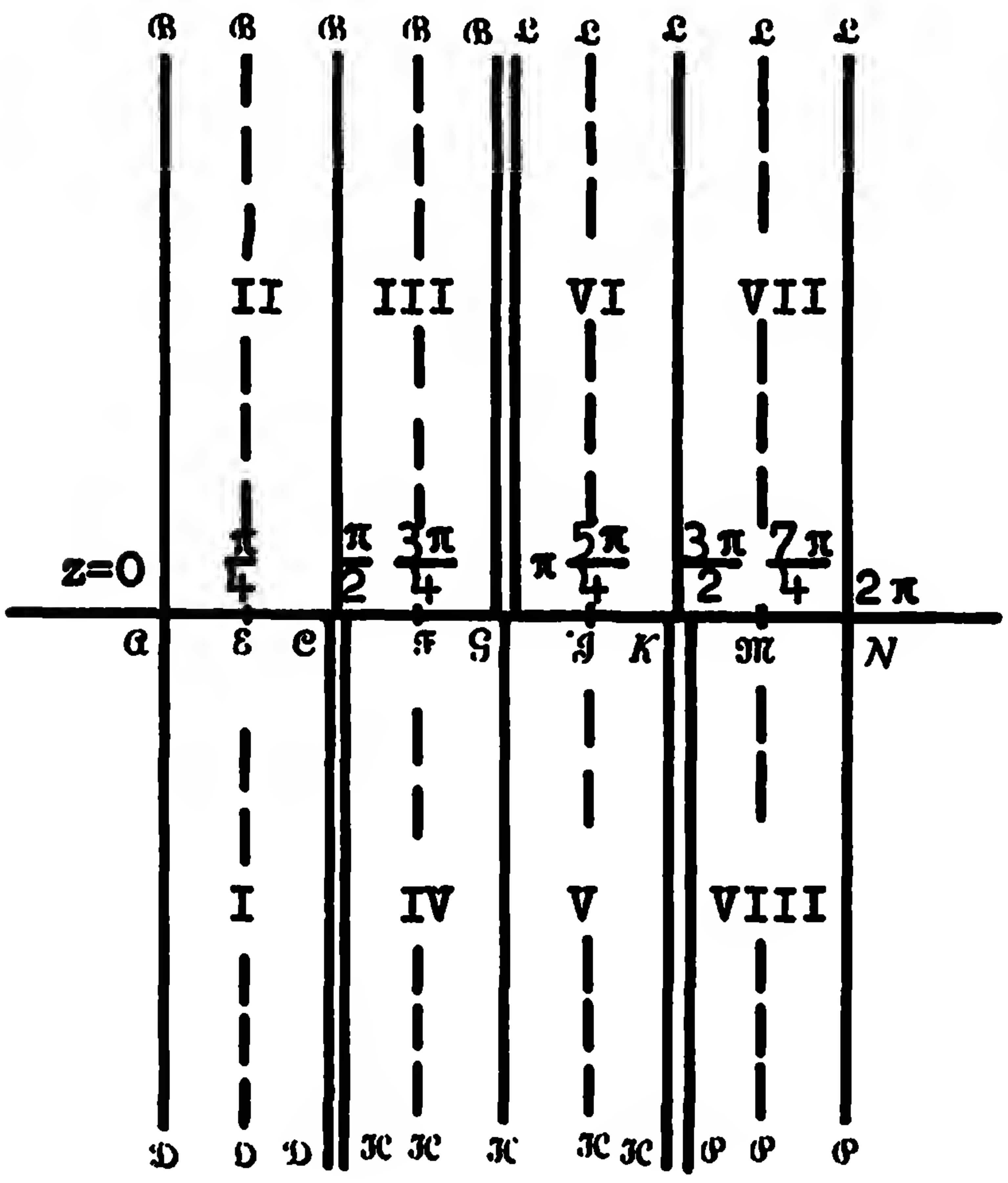
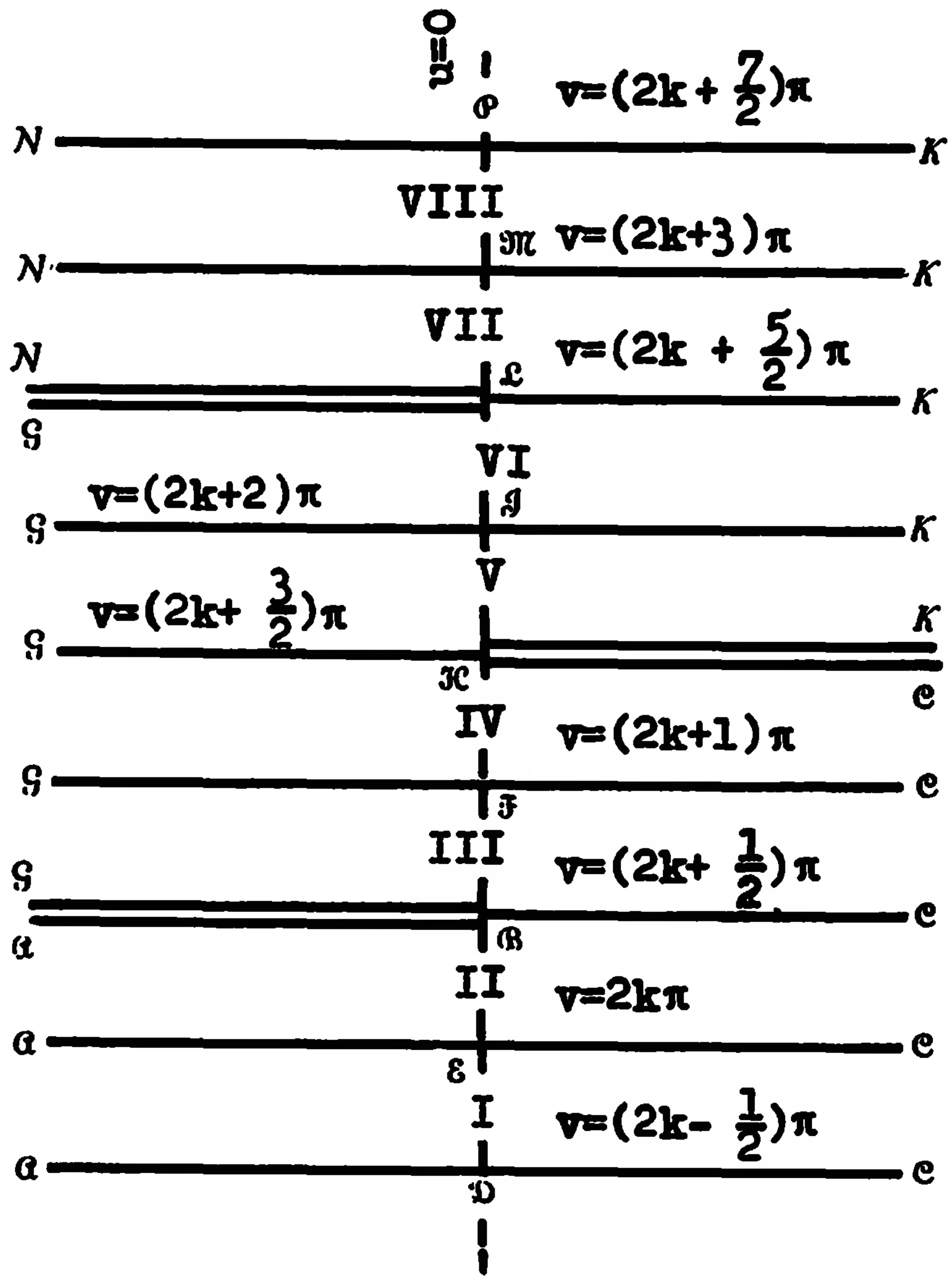
<table>
<tr><th>z - plane</th><th>w - plane</th></tr>
<tr><td>


</td><td>


</td></tr>
<tr><td>points $z = k\pi i$; $\log(1 + \sqrt{2})$;
$\frac{\pi i}{2}$; $-\frac{\pi i}{2}$</td><td>points $w = \infty$; $\log(1 + \sqrt{2})$;
$-\frac{\pi i}{2}$; $\frac{\pi i}{2}$.</td></tr>
</table>

$\boxed{w = \log \tan z}$ $= \log \coth\left(iz + \frac{i\pi}{2}\right) - \frac{i\pi}{2}$, $\boxed{z = \tan^{-1} e^w}$

$k = 0, \pm 1, \pm 2, \ldots$.

z - plane	w - plane
	
points $z = \frac{\pi}{4} - \frac{i}{2} \log(\sqrt{2} + 1)$; $\frac{\pi}{4} + \frac{i}{2} \log(\sqrt{2} + 1)$.	points $w = -\frac{\pi}{4} i + 2k\pi i$; $\frac{\pi}{4} i + 2k\pi i$.

Curves corresponding to lines parallel to the axes of the w - plane

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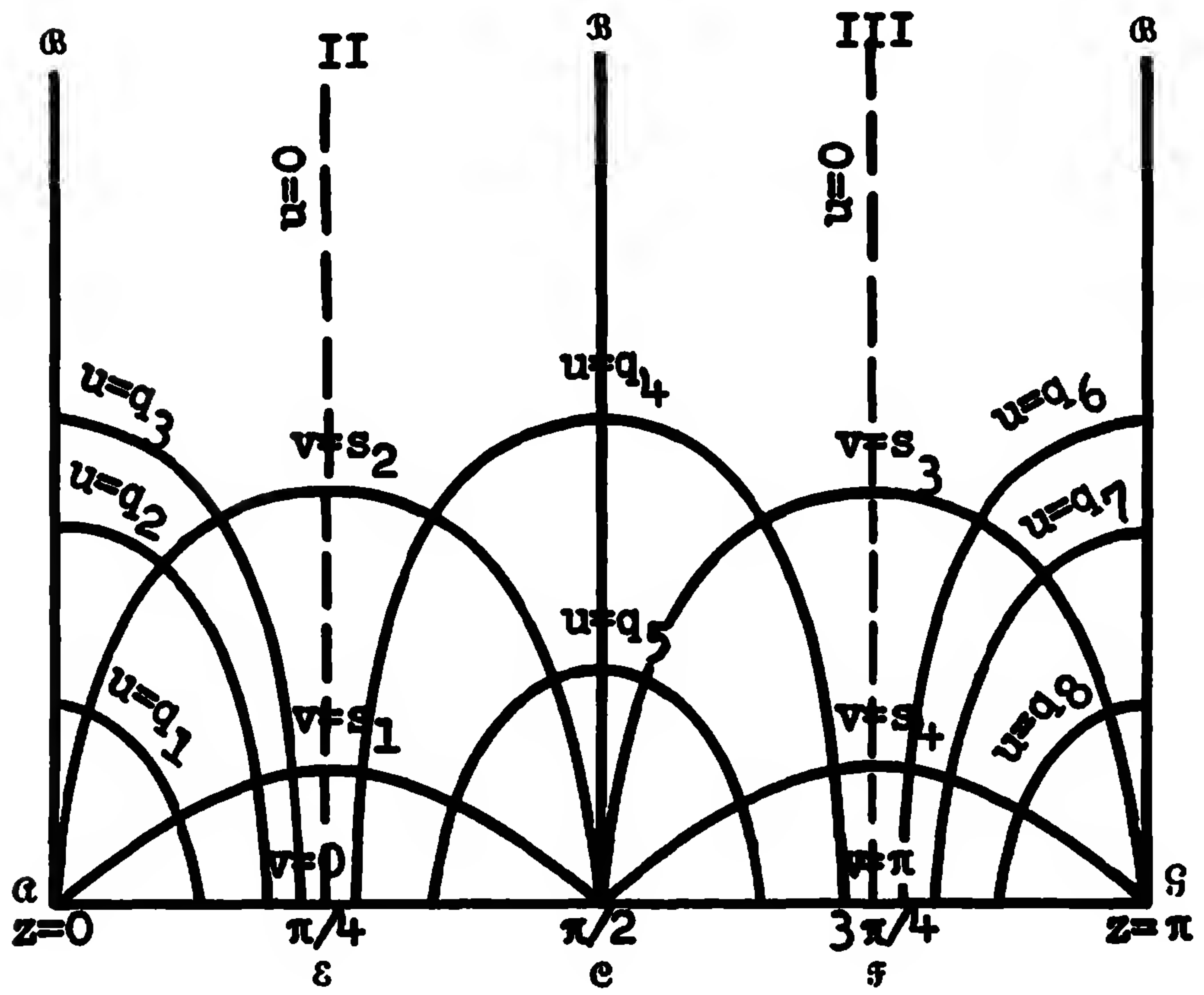
z - plane



Acknowledgment: As in §11.6. The essential curves of fig. 44 in Rothe-Ollendorff-Pohlhausen are copied in this diagram.

q, s are real constants; $-\infty < q_1 < q_2 < q_3 < 0 < q_4 < q_5 < \infty$,

$$-\infty < q_8 < q_7 < q_6 < 0,$$

$$0 < s_1 < s_2 < \tfrac{1}{2}\pi < s_3 < s_4 < \pi.$$

On the curves u = constant, $0 \le v \le \frac{1}{2}\pi$ in the half-strip $\mathfrak{B}\,\mathfrak{A}\,\mathfrak{E}\,\mathfrak{C}\,\mathfrak{B}$,

$$\tfrac{1}{2}\pi \le v \le \pi \text{ in } \mathfrak{B}\,\mathfrak{C}\,\mathfrak{F}\,\mathfrak{G}\,\mathfrak{B}\,.$$

On the curves v = constant, $-\infty < u < \infty$.

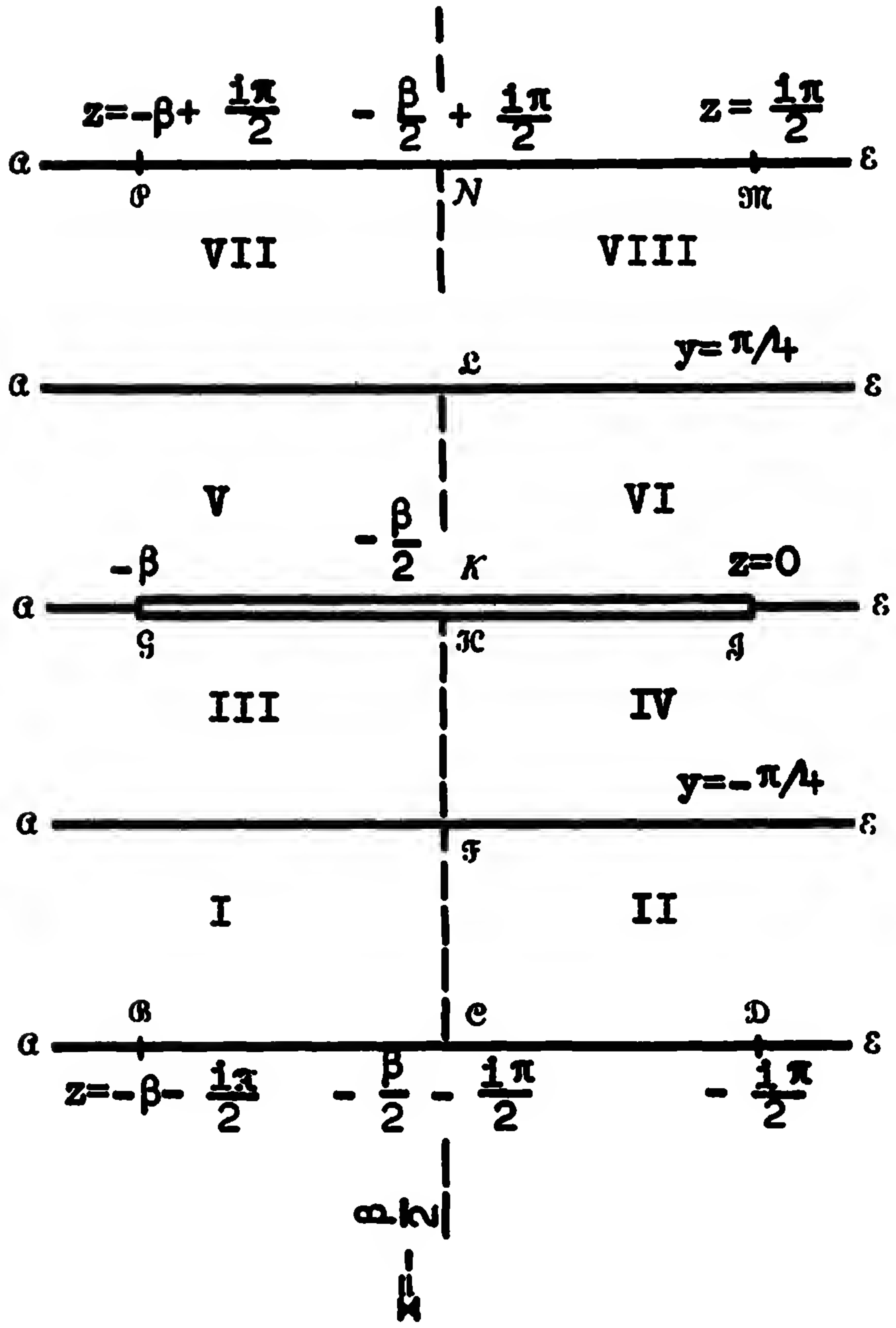
11.8 $$\boxed{w = \log \frac{\sinh(z+\beta)}{\sinh z} = \log(a + \coth z) - \log \sqrt{(a^2-1)}};$$

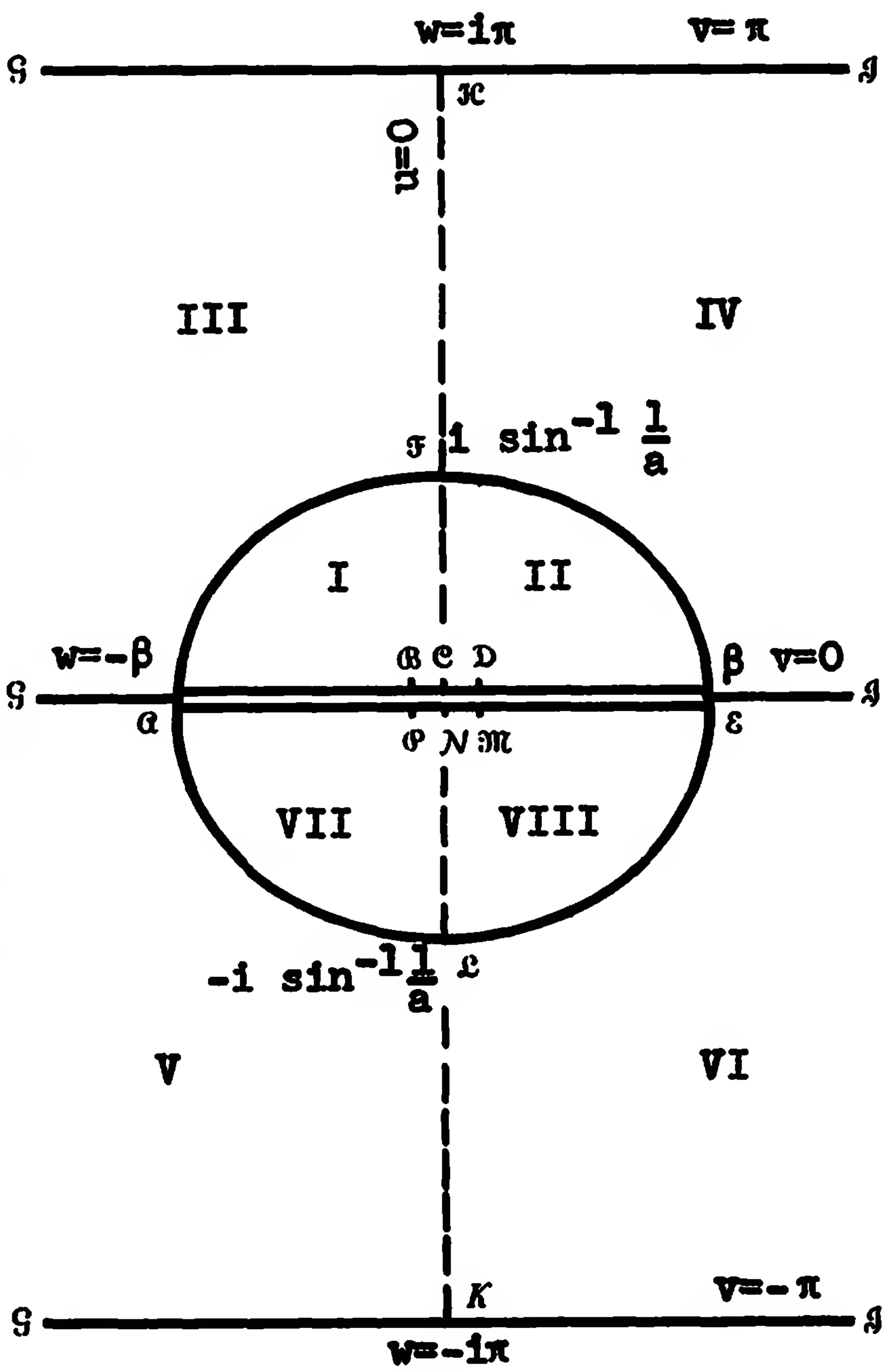
$\beta > 0, \quad a = \coth \beta > 1.$

$$\boxed{z = \frac{1}{2} \log \frac{\sinh \frac{w+\beta}{2}}{\sinh \frac{w-\beta}{2}} - \frac{1}{2}\beta}$$

Critical points: $z = k\pi i$; $-\beta + k\pi i$; ∞ $(k = 0, \pm 1, \pm 2, \ldots)$.

z - plane $(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2})$	w - plane $(-\pi \leq v \leq \pi)$
points $z = -\frac{\beta}{2} \pm \frac{i\pi}{2}$; $-\beta \pm \frac{i\pi}{2}$; $\pm\frac{i\pi}{2}$; $-\frac{\beta}{2}$; $-\frac{\beta}{2} + \frac{i\pi}{4}$; $-\frac{\beta}{2} - \frac{i\pi}{4}$; $-\frac{\beta}{2} - \frac{i}{2}\sin^{-1}\frac{1}{a}$; $-\frac{\beta}{2} + \frac{i}{2}\sin^{-1}\frac{1}{a}$	points $w = 0$; $-\log\cosh\beta$; $\log\cosh\beta$; $\pm i\pi$; $-i\sin^{-1}\frac{1}{a}$; $i\sin^{-1}\frac{1}{a}$; $\frac{i\pi}{2}$; $-\frac{i\pi}{2}$

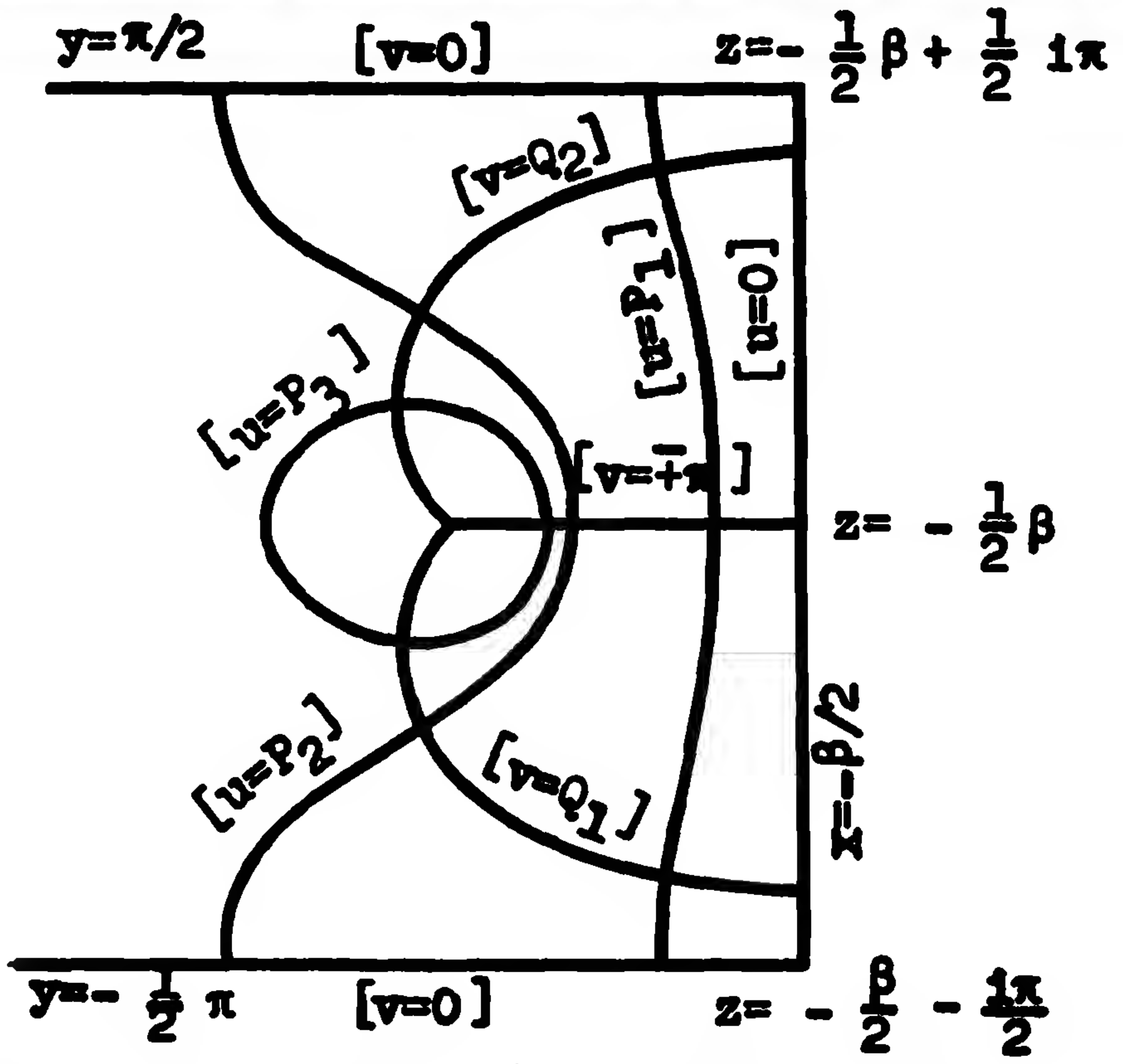




z - plane	w - plane
line $y = \frac{1}{2}\pi$, $-\infty < x < \infty$ line $y = -\frac{1}{2}\pi$, $-\infty < x < \infty$	line-segment $v=0$, $-\beta < u < \beta$
line $y = \frac{1}{4}\pi$, $-\infty < x < \infty$	part $v < 0$ of curve $\cosh u = \cosh \beta \cos v$
line $y = -\frac{1}{4}\pi$, $-\infty < x < \infty$	part $v > 0$ of curve $\operatorname{cosn} u = \cosh \beta \cos v$
line-segment $y = 0$, $-\beta < x < 0$	line $v = \pi$, $-\infty < u < \infty$ line $v = -\pi$, $-\infty < u < \infty$
line-segment $x = -\frac{1}{2}\beta$, $0 < y < \frac{1}{2}\pi$	half-line $u = 0$, $-\pi < v < 0$
line-segment $x = -\frac{1}{2}\beta$, $0 > y > -\frac{1}{2}\pi$	half-line $u = 0$, $\pi > v > 0$

<u>Curves in the z-plane which correspond to lines $u = P$, and $v = Q$.</u>

z - plane



<u>Acknowledgement</u>: As in §11.6. Essential curves of fig. 46 in Rothe-Ollendorff-Pohlhausen are copied in this diagram.

$Q_1 > 0$, $Q_2 < 0$; $0 > P_1 > P_2 > -\beta > P_3$. Only the part $u \leq 0$ of the curves $[v = Q]$ is shown.

$\boxed{w = \log \frac{ae^z + b}{ce^z + d}}$; $\frac{ad}{bc}$ real, $1 < \frac{ad}{bc} < \infty$.

Combination of $w = \beta + \log \frac{b}{d} + \xi$, $\zeta = \frac{z}{2} - \frac{1}{2} \log (- \frac{d}{c})$, and

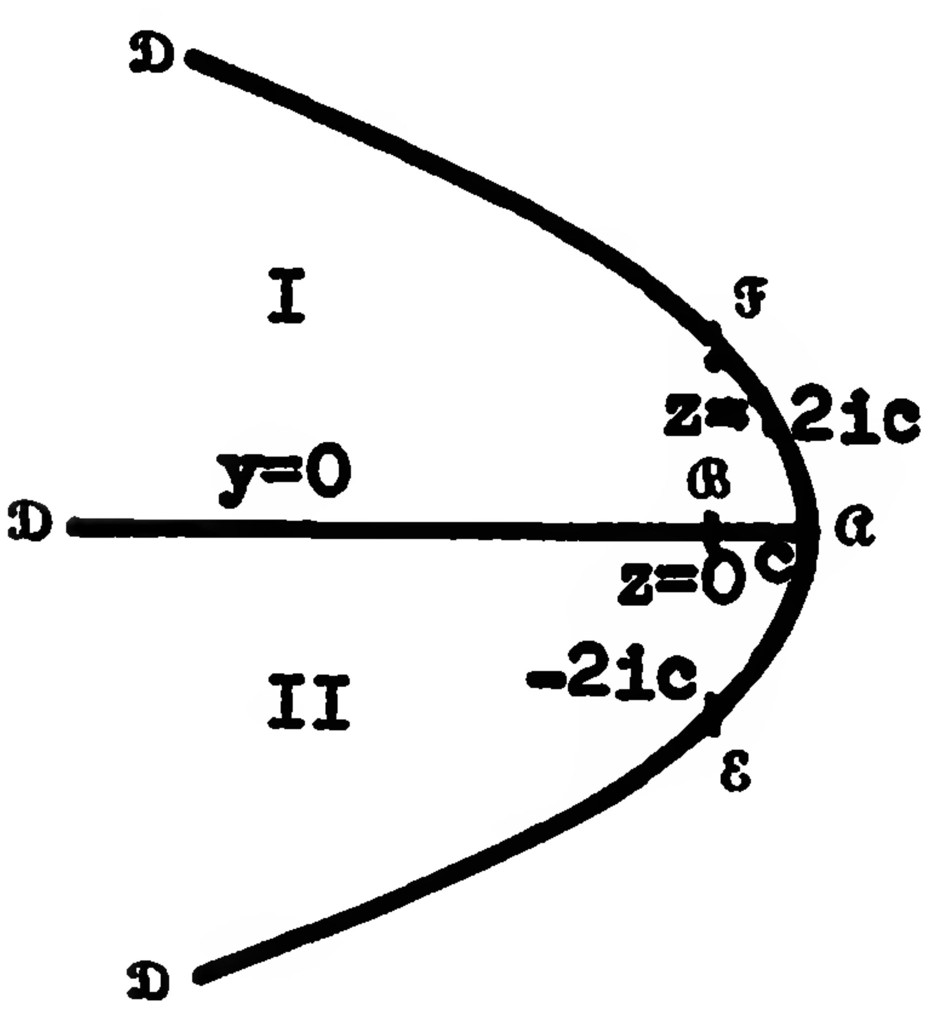
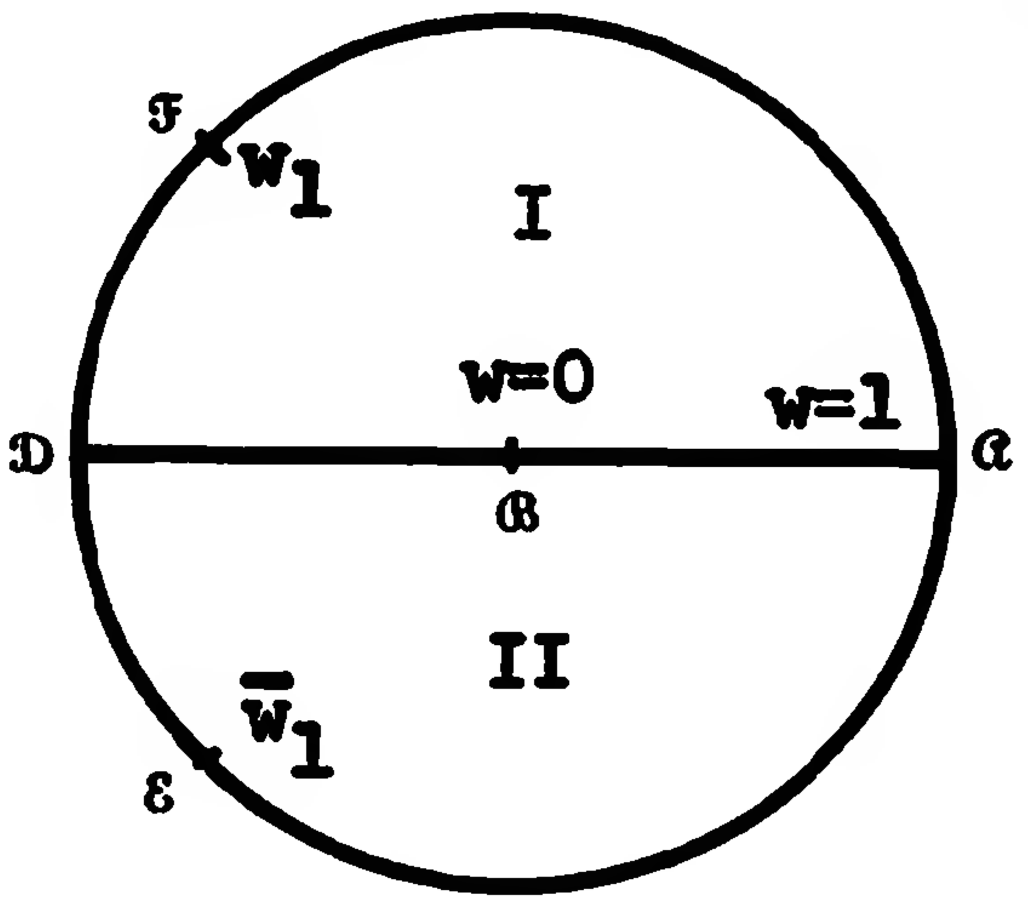
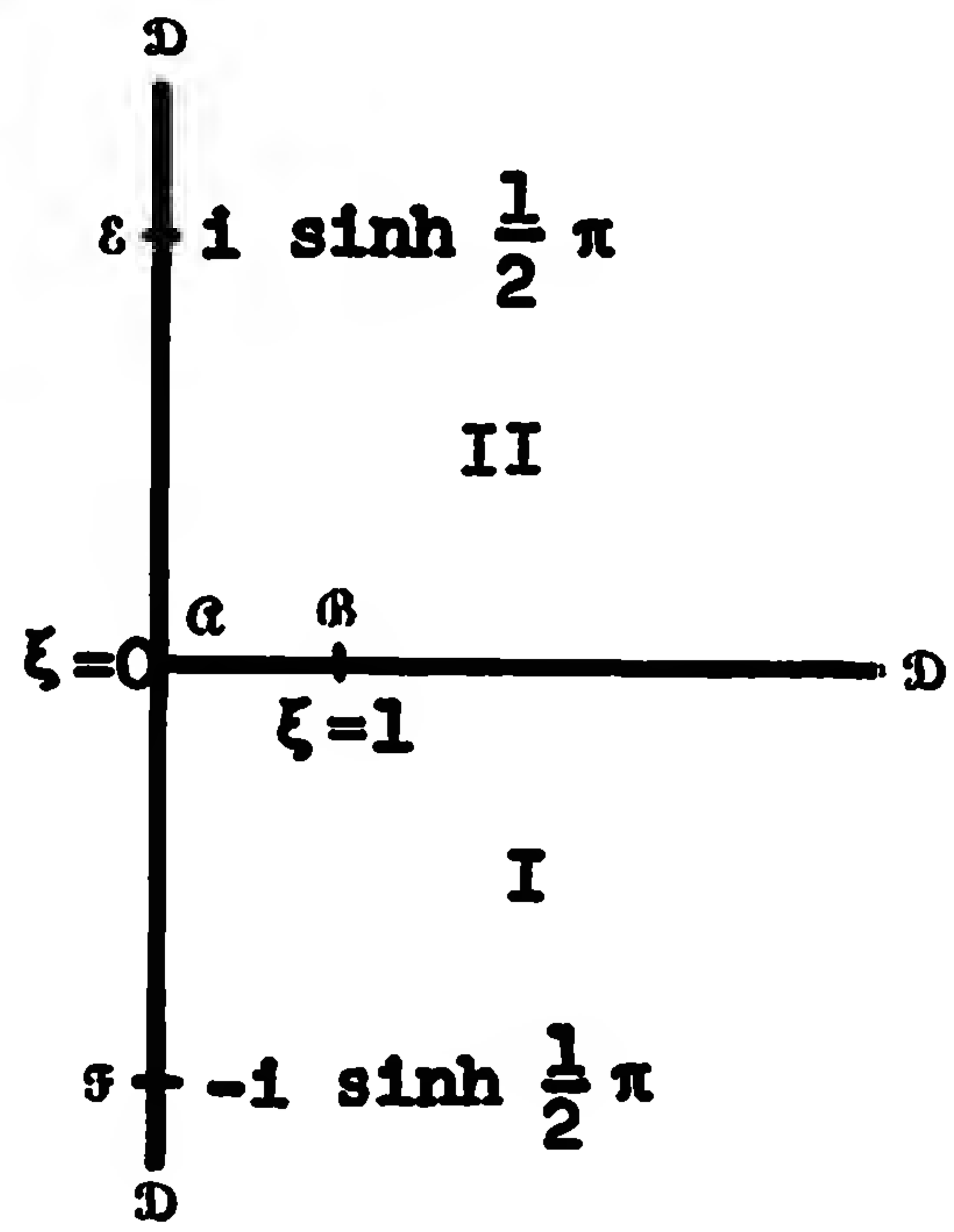
$$\xi = \log \frac{\sinh (\zeta + \beta)}{\sinh \zeta}, \quad \text{where } \beta = \frac{1}{2} \log \frac{ad}{bc}.$$

11.9 $\boxed{w = \tan^2(\frac{a}{2}\sqrt{z})}$, and $\boxed{\xi = \cos(a\sqrt{z})}$; $a \gtrless 0$.

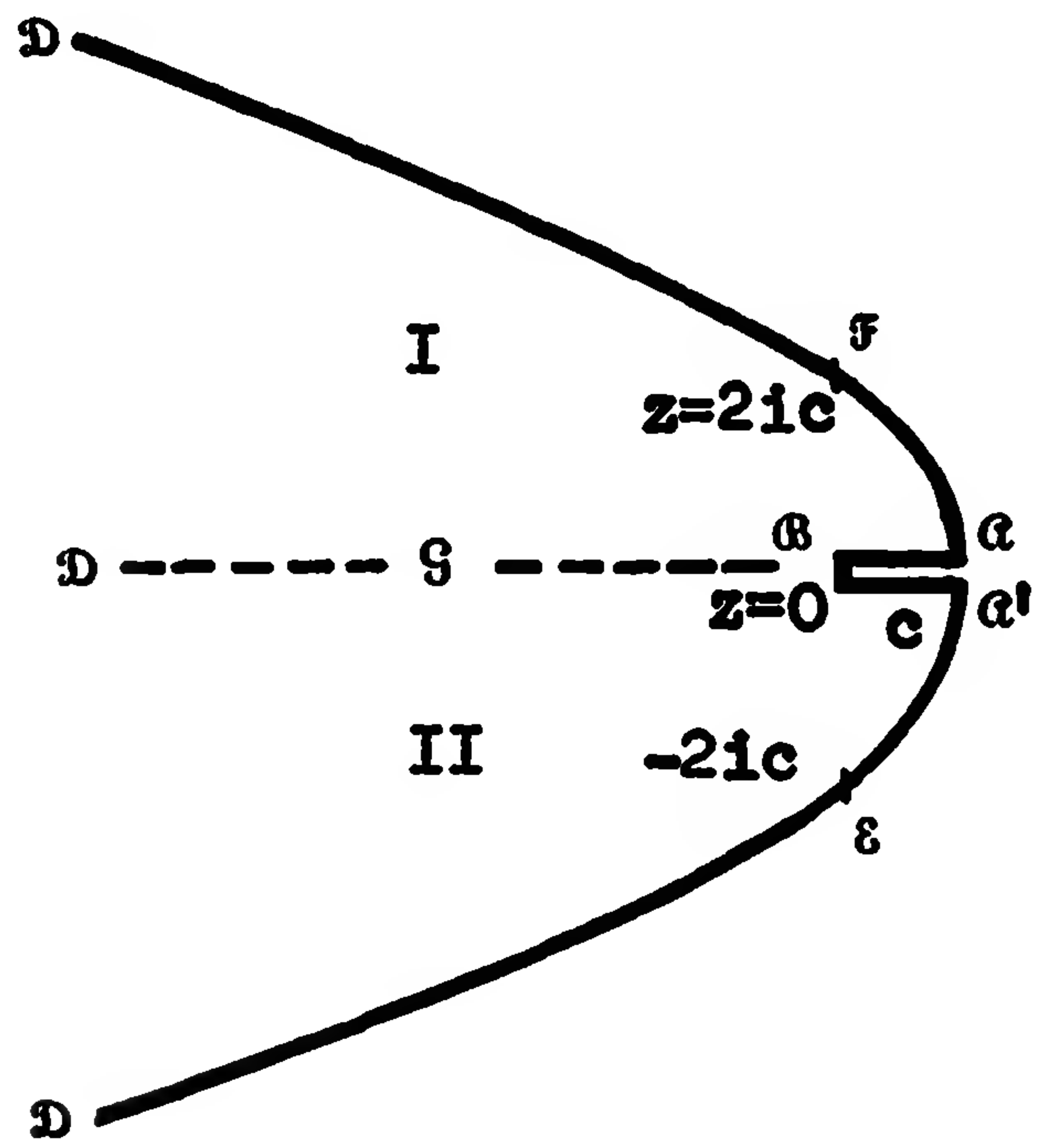
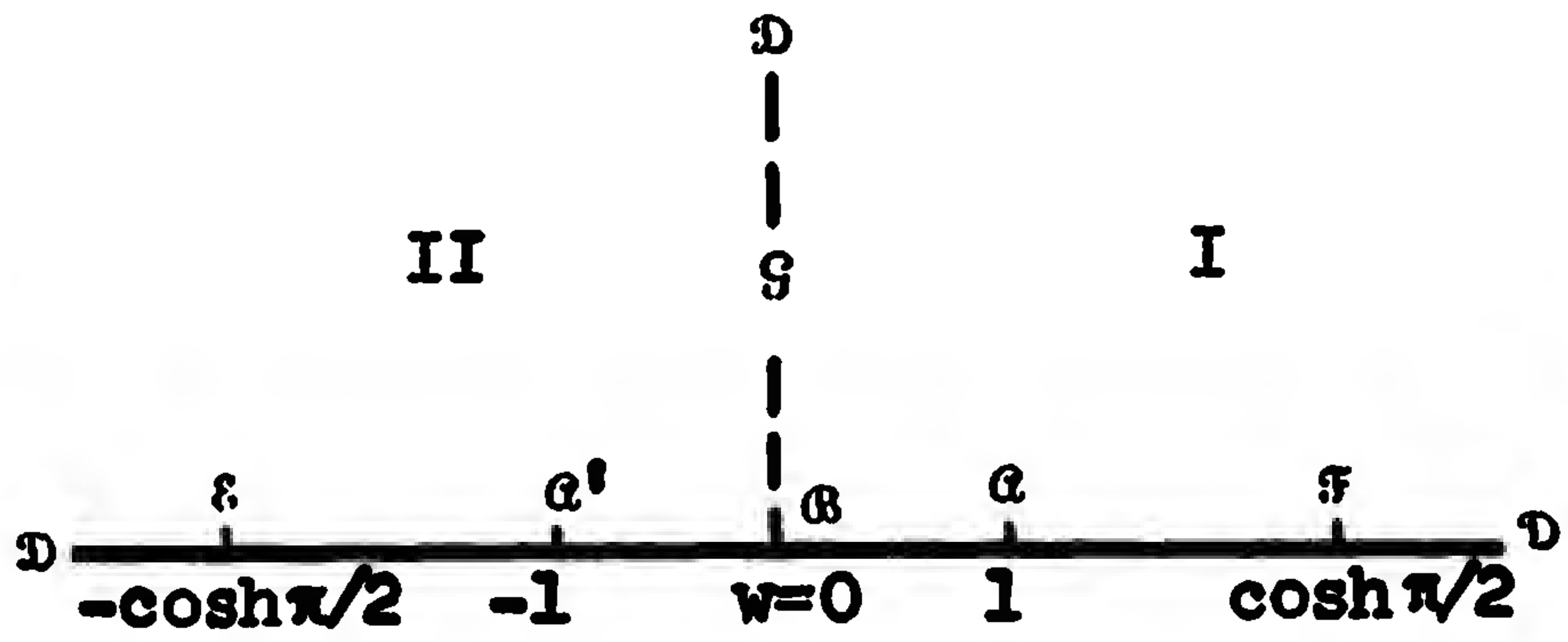
Critical points: $z = \infty$; $(k\pi/a)^2$ $(k = 0, \pm 1, \pm 2, \ldots)$.

$w = \frac{1-\xi}{1+\xi}$; $\xi = \frac{1-w}{1+w}$; $c = (\frac{\pi}{2a})^2$.

<u>Interior of parabola on that of circle or on half-plane.</u>

<table>
<tr><th>z - plane</th><th>w- plane</th><th>ξ - plane</th></tr>
<tr><td>z=0; c; 2ic; -2ic</td><td>w=0; 1; $\frac{1+i\sinh\pi/2}{1-i\sinh\pi/2} = w_1$; $\bar{w}_1$</td><td>ξ=1; 0; $-i\sinh\frac{\pi}{2}$; $i\sinh\frac{\pi}{2}$</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>half-line y=0, $c \geq x > -\infty$
interior of parabola $y^2 = 4c(c-x)$ [focus z=0, vertex z=c]</td><td>line-segment v=0, $1 \geq u > -1$
interior of circle $|w| = 1$</td><td>half-line $\Im(\xi)=0$, $0 \leq \xi < \infty$.
half-plane $\Re(\xi) > 0$.</td></tr>
</table>

$\boxed{w = \sin(a\sqrt{z})}$, $a > 0$. Critical points: $z = 0, \infty, (\frac{2k+1}{2a}\pi)^2$.

z - plane	w - plane
points $z = 0$; c; $(\frac{2k+1}{2a}\pi)^2$; $2ic$; $-2ic$	points $w = 0$; ± 1; ± 1; $\cosh \pi/2$; $\pm \cosh \pi/2$
	
Interior, cut from $z = 0$ to $z = c = (\pi/2a)^2$, of parabola $y^2 = 4c(c-x)$	$\begin{cases} \text{half-plane } v > 0 \\ \text{half-plane } v < 0 \end{cases}$

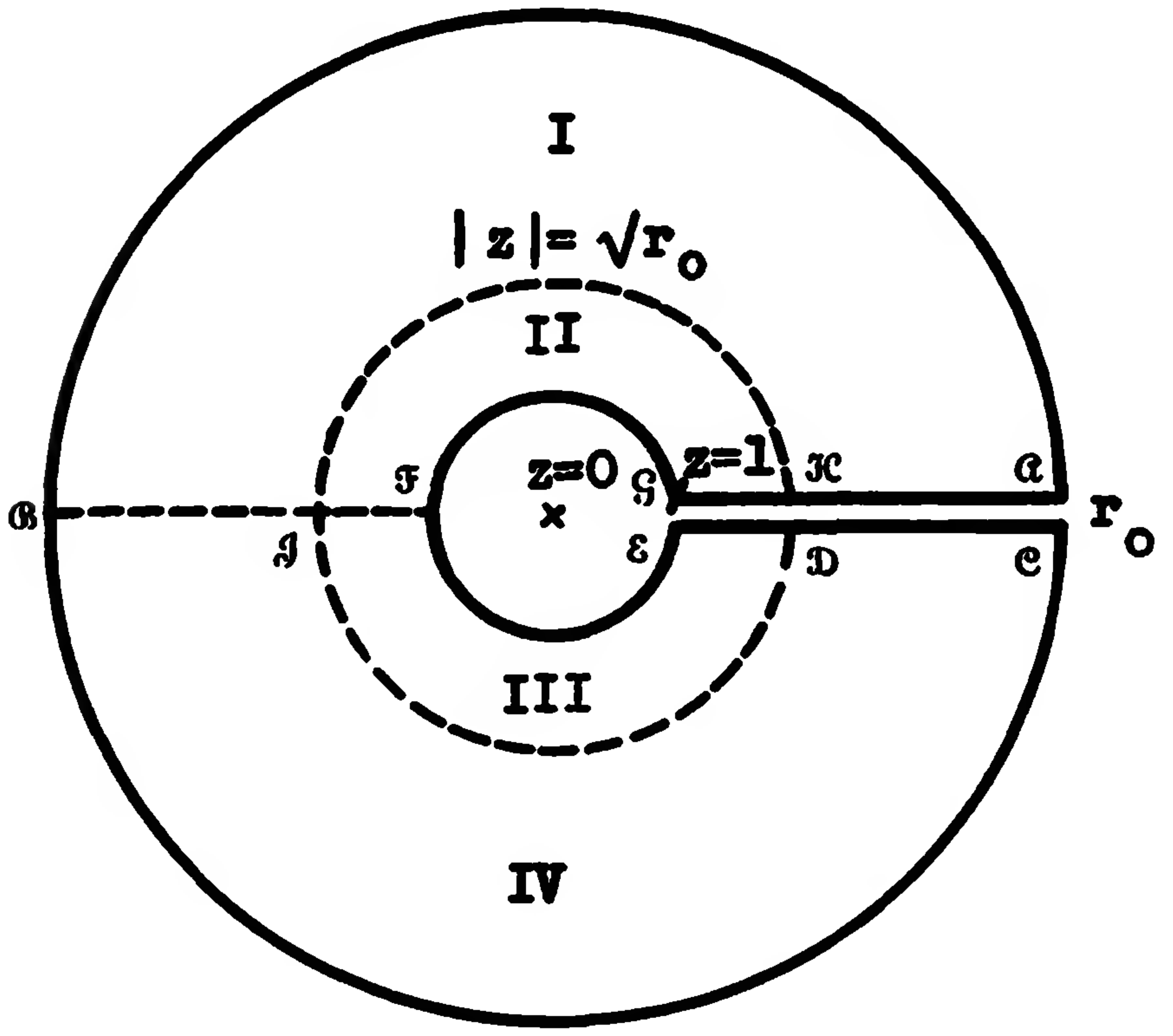
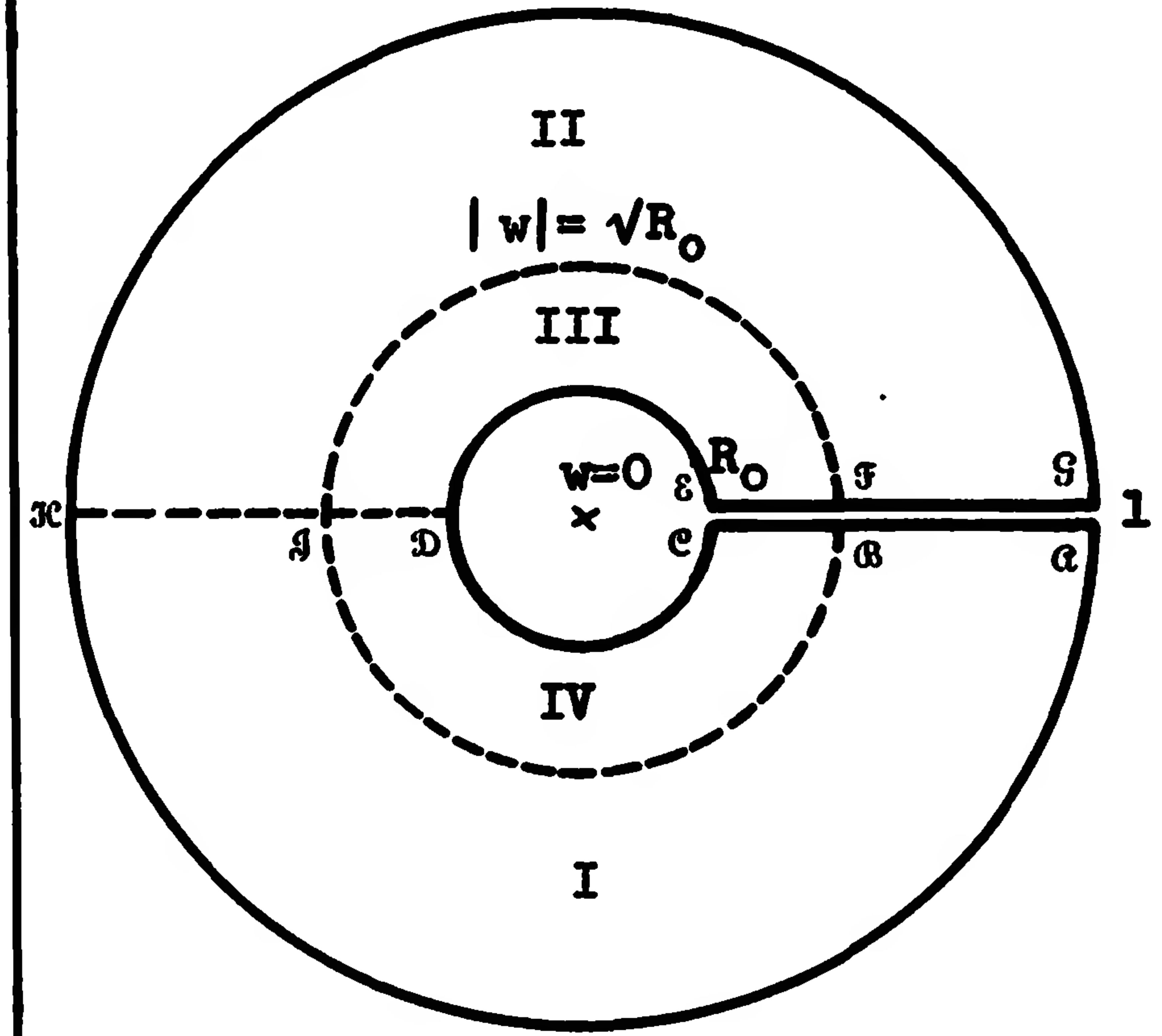
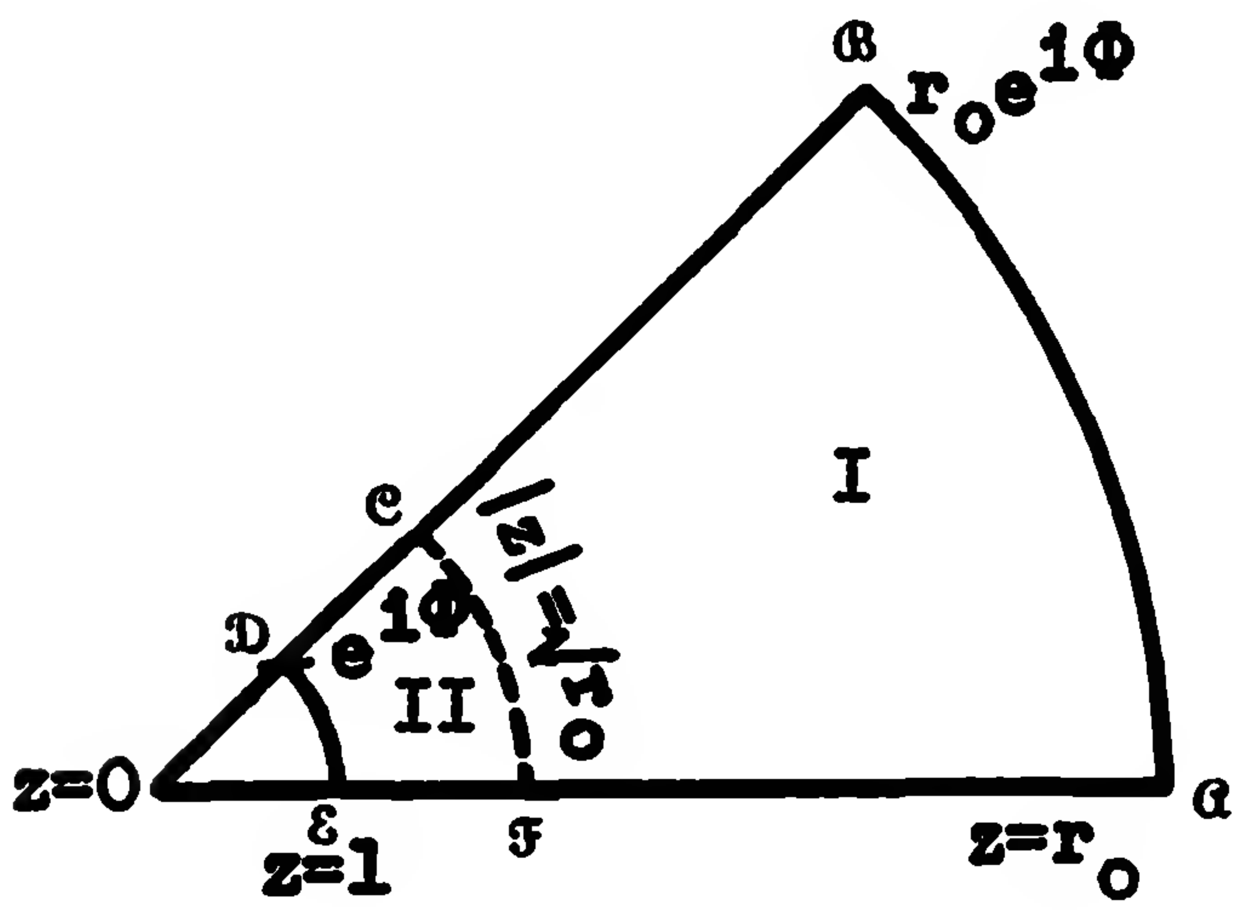
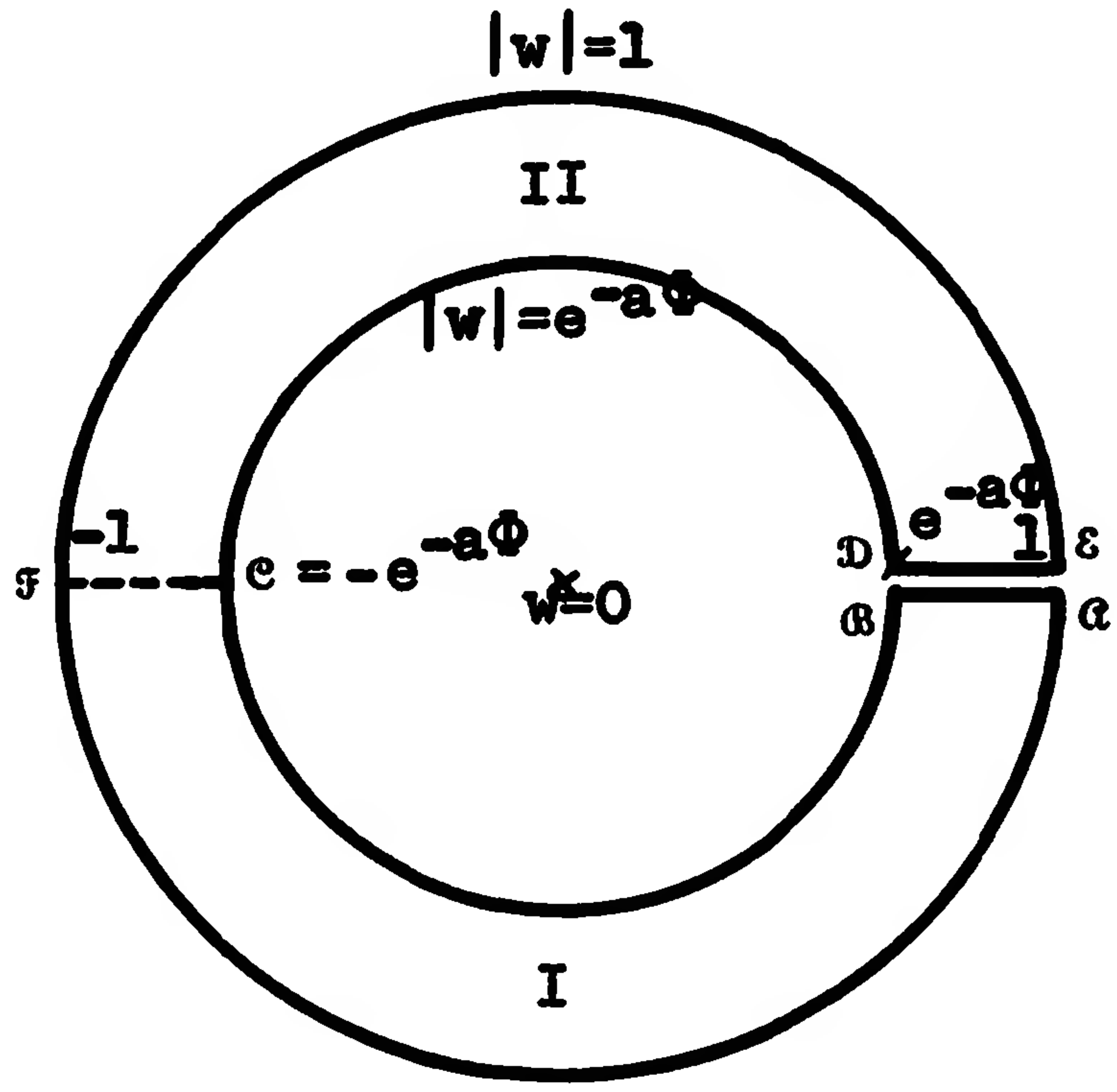
11.10 $\boxed{w = z^{ai}}$, $a > 0$; $z = w^{-bi}$ $(b = 1/a)$.

Critical points: $z = 0, \infty$.

$r_0 = e^{2b\pi}$, $R_0 = e^{-2a\pi}$; $z = re^{i\varphi}$, $w = Re^{i\theta}$.

$k = 0, \pm 1, \pm 2, \ldots$; $\ell = 0, \pm 1, \pm 2, \ldots$.

z - plane	w - plane
points $z = z_1 e^{2\pi kb}$	points $w = z_1^{ai} e^{2\pi a \ell}$

z - plane	w - plane
	
circle $\|z\| = c$, $0 \leq \varphi < 2\pi$ $(1 \leq c \leq r_0)$	line-segment $\arg w = a \log c$, $1 \geq \|w\| \geq R_0$
line-segment $\arg z = \Phi$, $1 \leq \|z\| < r_0$ (Φ fixed, $0 \leq \Phi \leq 2\pi$)	circle $\|w\| = e^{-a\Phi}$, $0 \leq \theta < 2\pi$
points $e^{i\pi}$ (i.e. $\mathcal{F}$); r_0 ($\mathcal{A}$); $r_0e^{i\pi}$ ($\mathcal{B}$); $e^{i\pi}\sqrt{r_0}$ ($\mathcal{J}$); $\sqrt{r_0}$ ($\mathcal{H}$); $e^{2i\pi}\sqrt{r_0}$ ($\mathcal{D}$)	points $\sqrt{R_0}$; $e^{2i\pi}$; $e^{2i\pi}\sqrt{R_0}$; $e^{i\pi}\sqrt{R_0}$; $e^{i\pi}$; $R_0e^{i\pi}$
	

z - plane	w - plane
sector $0 < \arg z < \Phi$ of ring $1 < \lvert z \rvert < r_0$ $(\Phi < 2\pi)$	ring $1 > \lvert w \rvert > e^{-a\Phi}$, cut along positive real axis
sector $0 < \arg z < \Phi$ of ring $c_1 < \lvert z \rvert < c_2$ $(\Phi < 2\pi,$ $1 \leq c_1 < c_2 < r_0)$	sector $a \log c_1 < \arg w < a \log c_2$ of ring $1 > \lvert w \rvert > e^{-a\Phi}$.

$$\boxed{w = \left(\frac{1+z}{1-z}\right)^{a/(\pi i)}}\ ; \quad \boxed{z = i \tan\left(\frac{\pi}{2a} \log w\right)}, \quad a > 0.$$

Critical points: $z = -1;\ 1;\ \infty$.

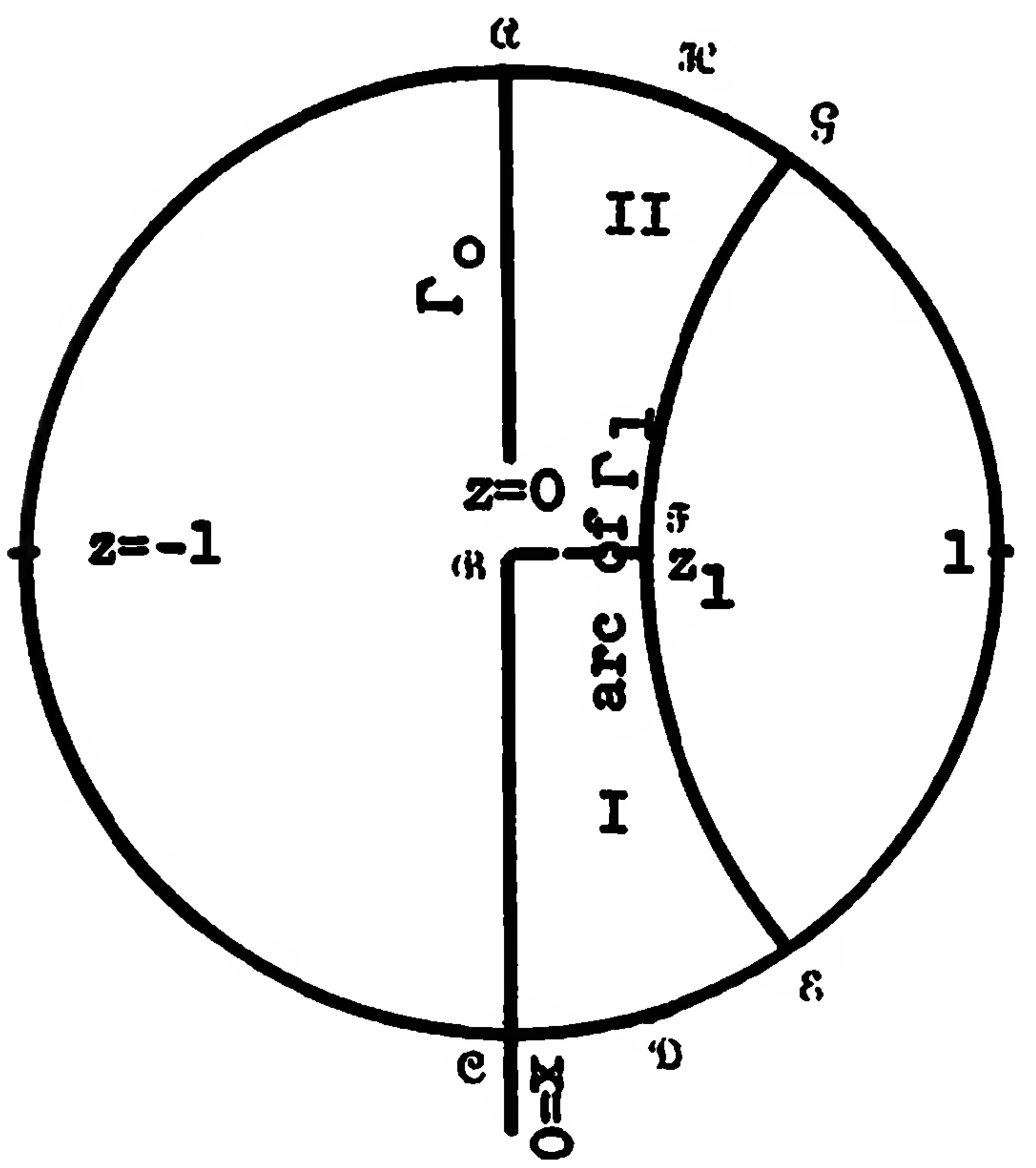
$k = 0, \pm 1, \pm 2, \ldots\ ;\ \ell = 0, \pm 1, \pm 2, \ldots$

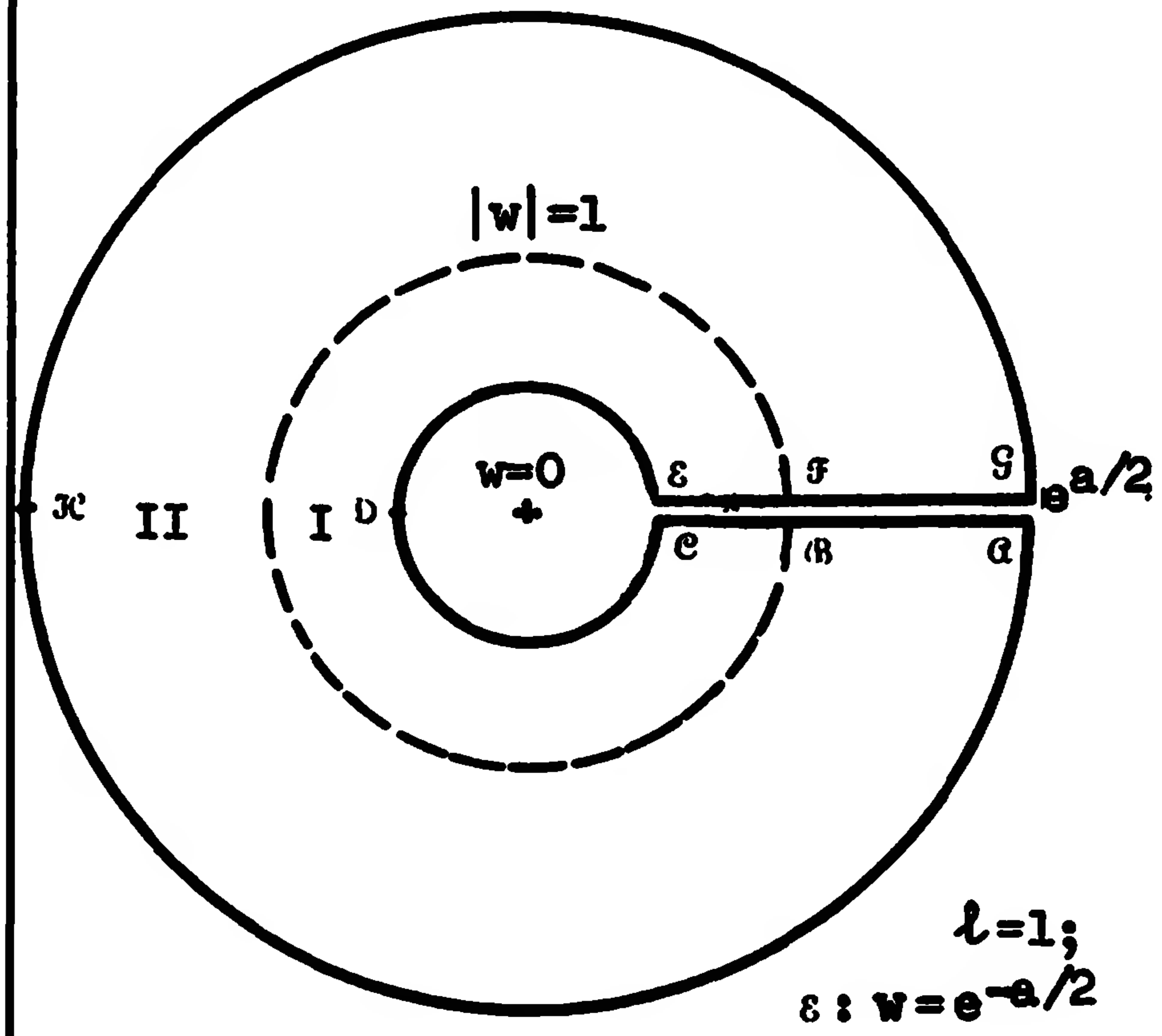
z - plane	w - plane
points $z = \coth \frac{\pi^2 k}{a};\ \tanh \frac{\pi^2 k}{a}$	points $w = e^{(2\ell+1)a};\ e^{2a\ell}$
area of curvilinear quadrilateral, bounded by two arcs of $\lvert z \rvert = 1$ and by any two neighbouring "circles" of the set, belonging to a coaxal set, $\Gamma_k:\ \left\lvert z - \coth \frac{2\pi^2 k}{a} \right\rvert = \frac{1}{\sinh \frac{2\pi^2 \lvert k \rvert}{a}}$ $(k = 0, \pm 1, \ldots,$ for $k = 0$ taking the line $x = 0)$, with limiting points $z = \pm 1$.	ring $e^{-\frac{a}{2} + 2\ell a} < \lvert w \rvert < e^{\frac{a}{2} + 2\ell a}$, cut along positive real axis.

See figures on next page

line-segment $-1 \leq y \leq 0,\ x = 0$	line-segment $e^{-a/2} \leq u \leq 1,\ v = 0$.
line-segment $0 \leq y \leq 1,\ x = 0$	line-segment $1 \leq u \leq e^{a/2},\ v = 0$.
semi-circle $\lvert z \rvert = 1,\ 0 \leq \arg z < \pi$	circle $\lvert w \rvert = e^{a/2}$

z - plane	w - plane





semi-circle $\lvert z \rvert = 1$, $\pi \leq \arg z < 2\pi$	circle $\lvert w \rvert = e^{-a/2}$
line-segment $-1 < x < 1$, $y = 0$	circle $\lvert w \rvert = 1$ (Each of these circles is covered an infinity of times.)
line-segment $0 \leq x < z_1 = \tanh \frac{\pi^2}{a}$, $y = 0$	circle $\lvert w \rvert = 1$, $2\pi \geq \arg w > 0$.

11.11 $\boxed{w = \cos (a \log z)}$, $a > 0$; $z = e^{(\cos^{-1} w)/a}$.

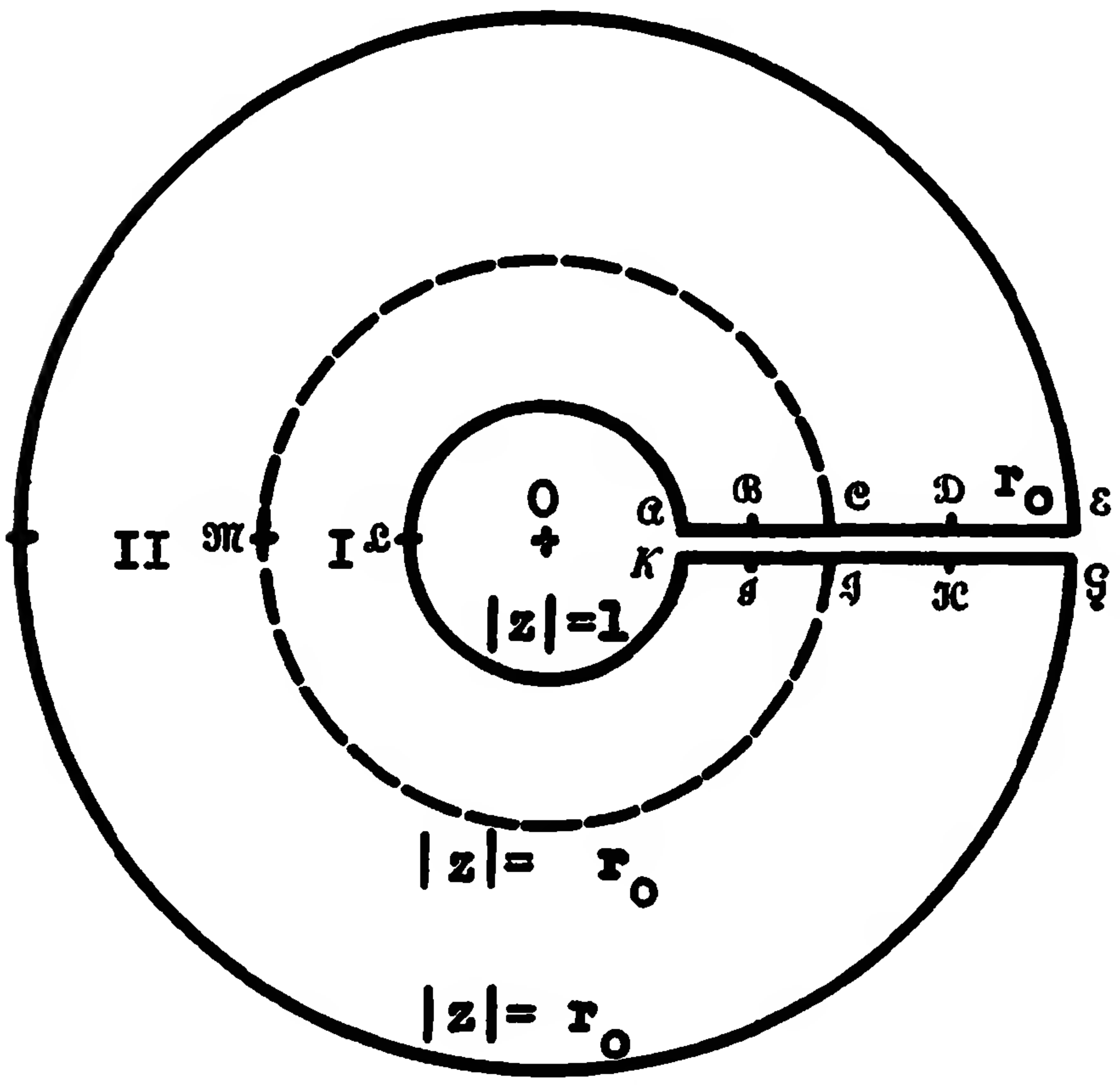
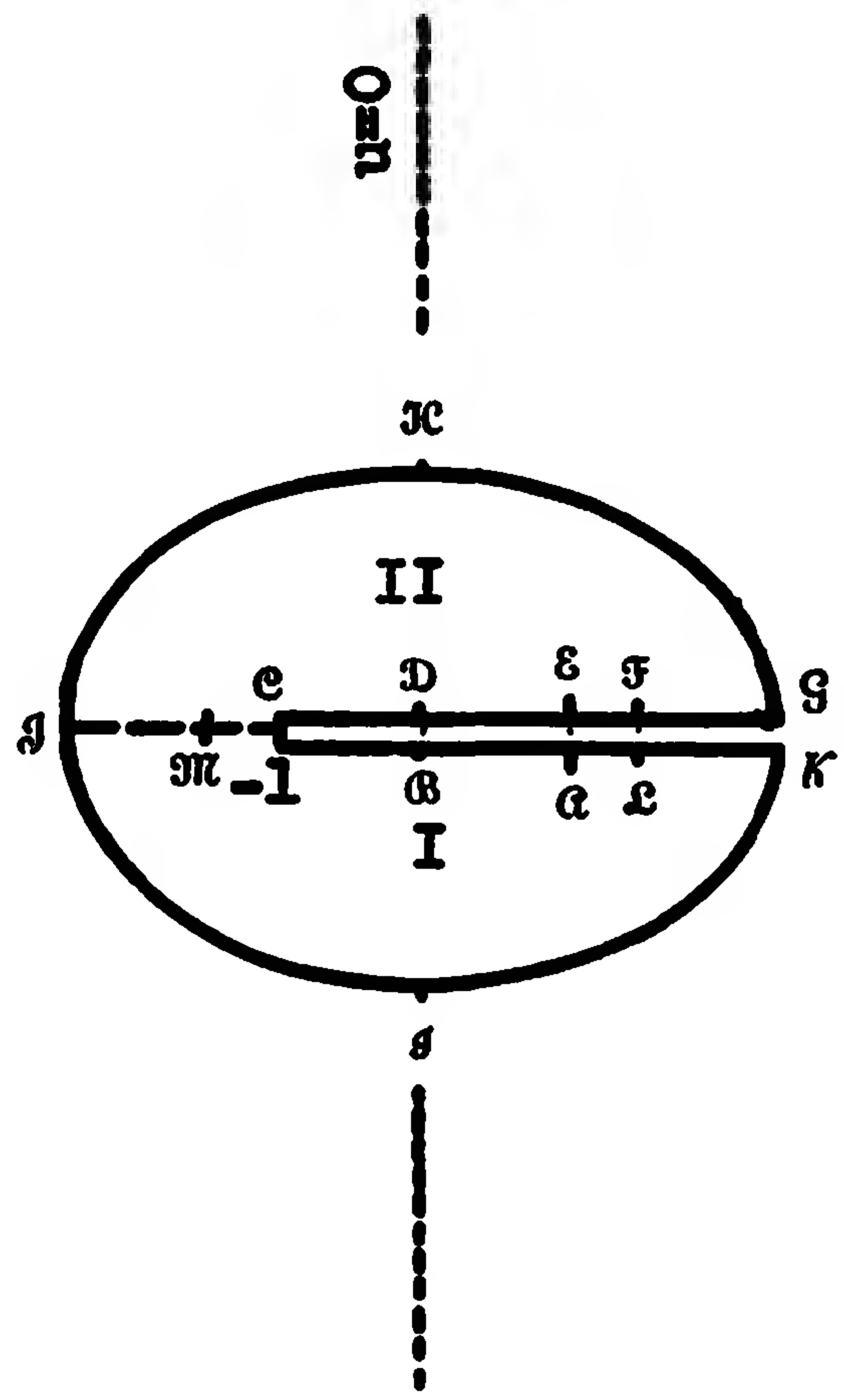
h, k any integers; $r_o = e^{2\pi/a}$.

Critical points: $z = 0$; ∞; $e^{h\pi/a} = r_o^{h/2}$.

z - plane	w - plane

See figures on next page

points $z = \beta r_o^h$ ($\beta \neq 0$) or $\beta^{-1} r_o^h$	points $w = \cos (a \log \beta + 2ak\pi i)$.
points $z = 1$ (i.e α); $r_o^{\frac{1}{4}}$ (β); $\sqrt{r_o}$ (γ)	points $w = 1$; 0; -1

z - plane	w - plane
	
points $z = r_o^{3/4}$ ($\mathfrak{D}$); r_o ($\mathfrak{E}$); $r_o e^{i\pi}$ ($\mathfrak{F}$)	points $w = 0;\ 1;\ \cosh a\pi$
$z = r_o e^{2i\pi}$ ($\mathfrak{G}$); $r_o^{3/4} e^{2\pi i}$ ($\mathfrak{H}$); $r_o^{1/2}\, e^{2\pi i}$ ($\mathfrak{J}$)	$w = \cosh 2a\pi;\ i \sinh 2a\pi;\ -\cosh 2a\pi$
$z = r_o^{1/4}\, e^{2i\pi}$ ($\mathfrak{I}$); $e^{2\pi i}$ (K); $e^{\pi i}$ ($\mathfrak{L}$)	$w = -i \sinh 2a\pi;\ \cosh 2a\pi;\ \cosh a\pi$
line-segment $\mathfrak{G}\,\mathfrak{H}\,\mathfrak{J}$	part $v > 0$ of ellipse $\vert w-1\vert+\vert w+1\vert = 2 \cosh 2a\pi$
line-segment $\mathfrak{J}\,\mathfrak{I}\,K$	part $v < 0$ of same ellipse
circle $\vert z\vert = \sqrt{r_o}$, $0 \le \arg z < 2\pi$	line-segment $v = 0$, $-1 \ge u > -\cosh 2a\pi$
circle $\vert z\vert = r_o^{1/4}$, $0 \le \arg z < 2\pi$	line-segment $u = 0$, $0 \ge v > -\sinh 2a\pi$
circle $\vert z\vert = r_o^{3/4}$, $0 \le \arg z < 2\pi$	line-segment $u = 0$, $0 \le v < \sinh 2a\pi$

z - plane	w - plane
circle $\|z\| = r_0^{\alpha}$, $0 \leq \arg z < 2\pi$ $(0 < \alpha < 1,\ \alpha \neq \frac{1}{4}, \frac{1}{2}, \frac{3}{4})$.	part $0 \geq v > -\sin 2\pi\alpha \sinh 2a\pi$ or $0 \leq v < -\sin 2\pi\alpha \sinh 2a\pi$ (for $\sin 2\pi\alpha > 0$ or < 0, respectively) of the hyperbola branch $\|w+1\| - \|w-1\| = 2\cos 2\pi\alpha$.

11.12 ‡ $\boxed{w = \log \frac{z-m}{z+m} + \log \frac{mz-1}{mz+1} = \log \frac{z^2+1-2\ell z}{z^2+1+2\ell z}}$; $0 < m < 1$; $\ell = \frac{1}{2}(m + \frac{1}{m})$

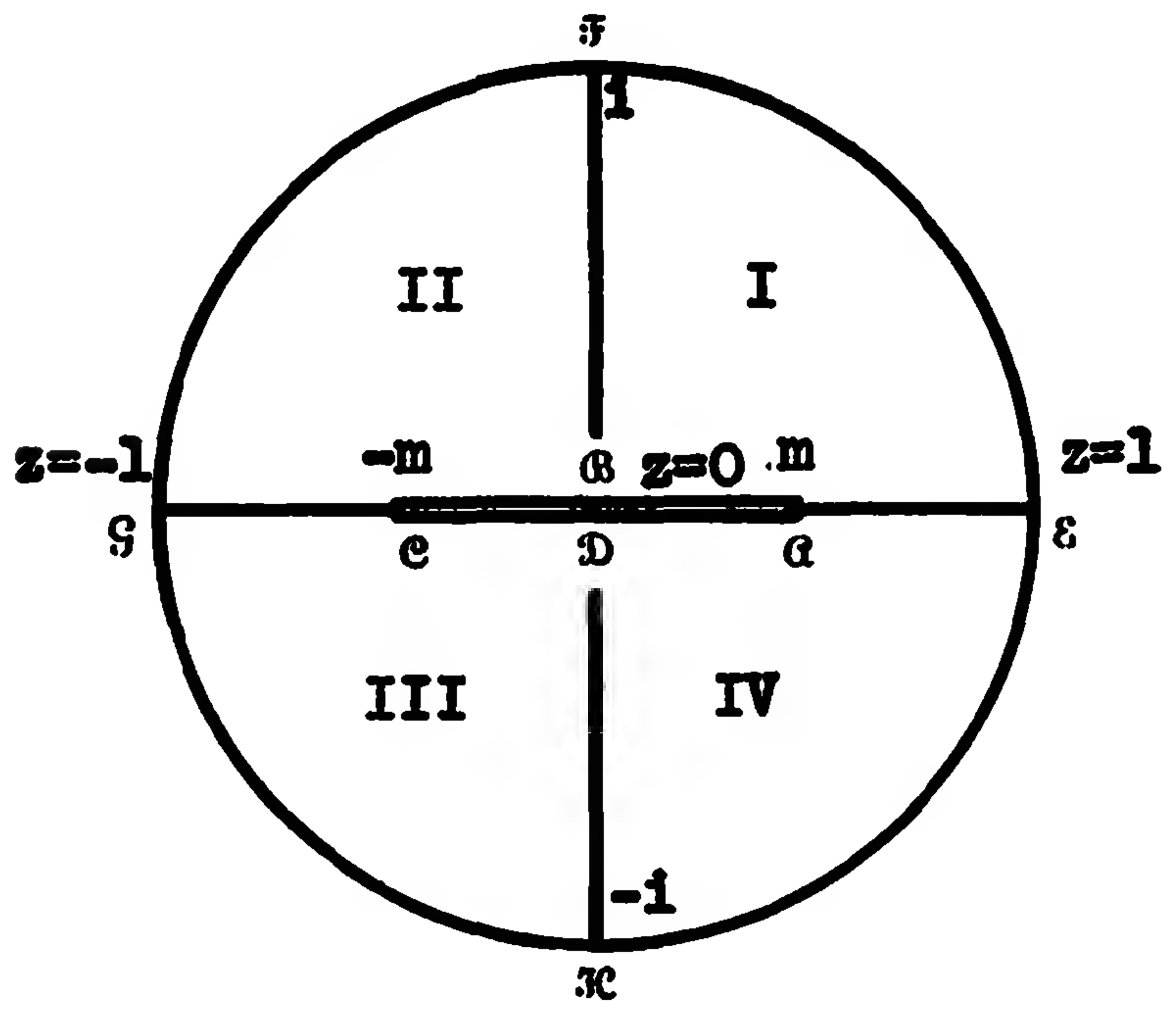
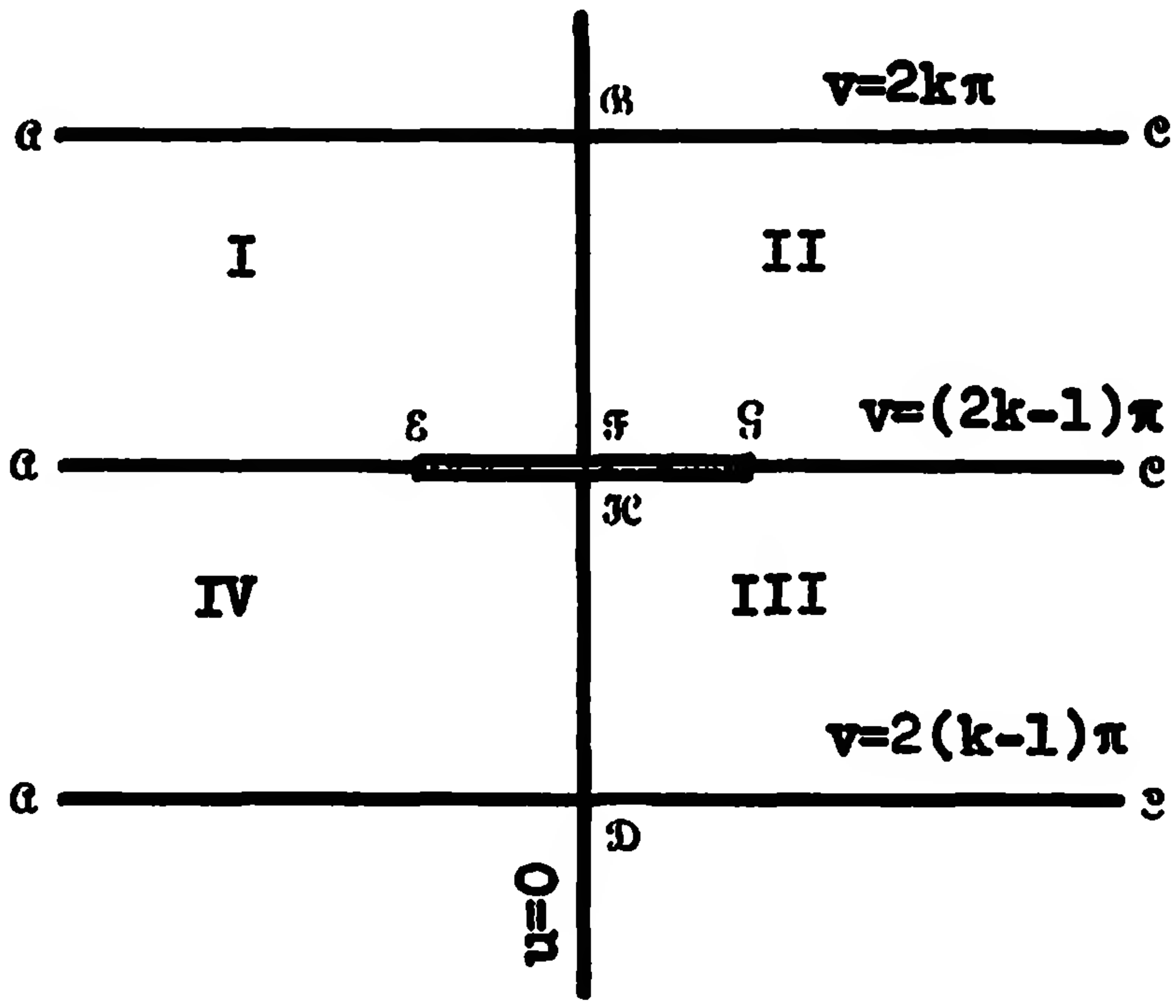
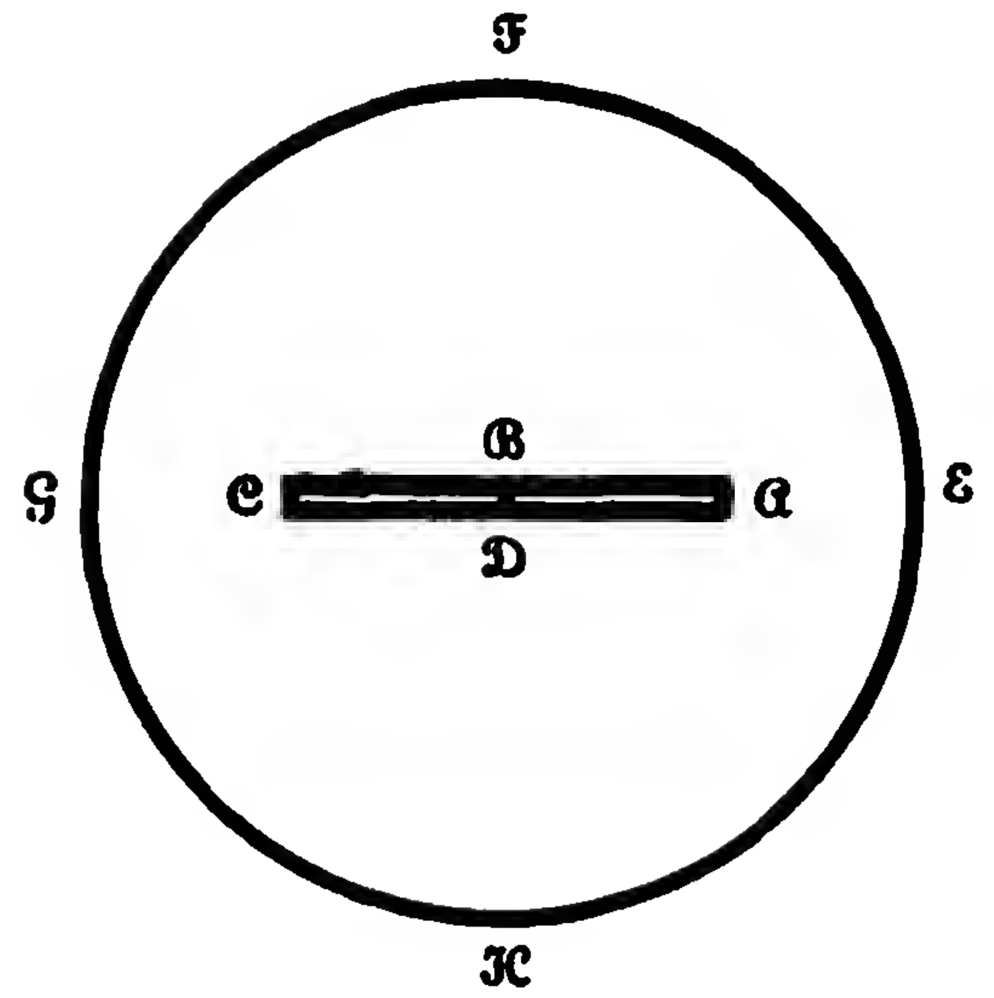
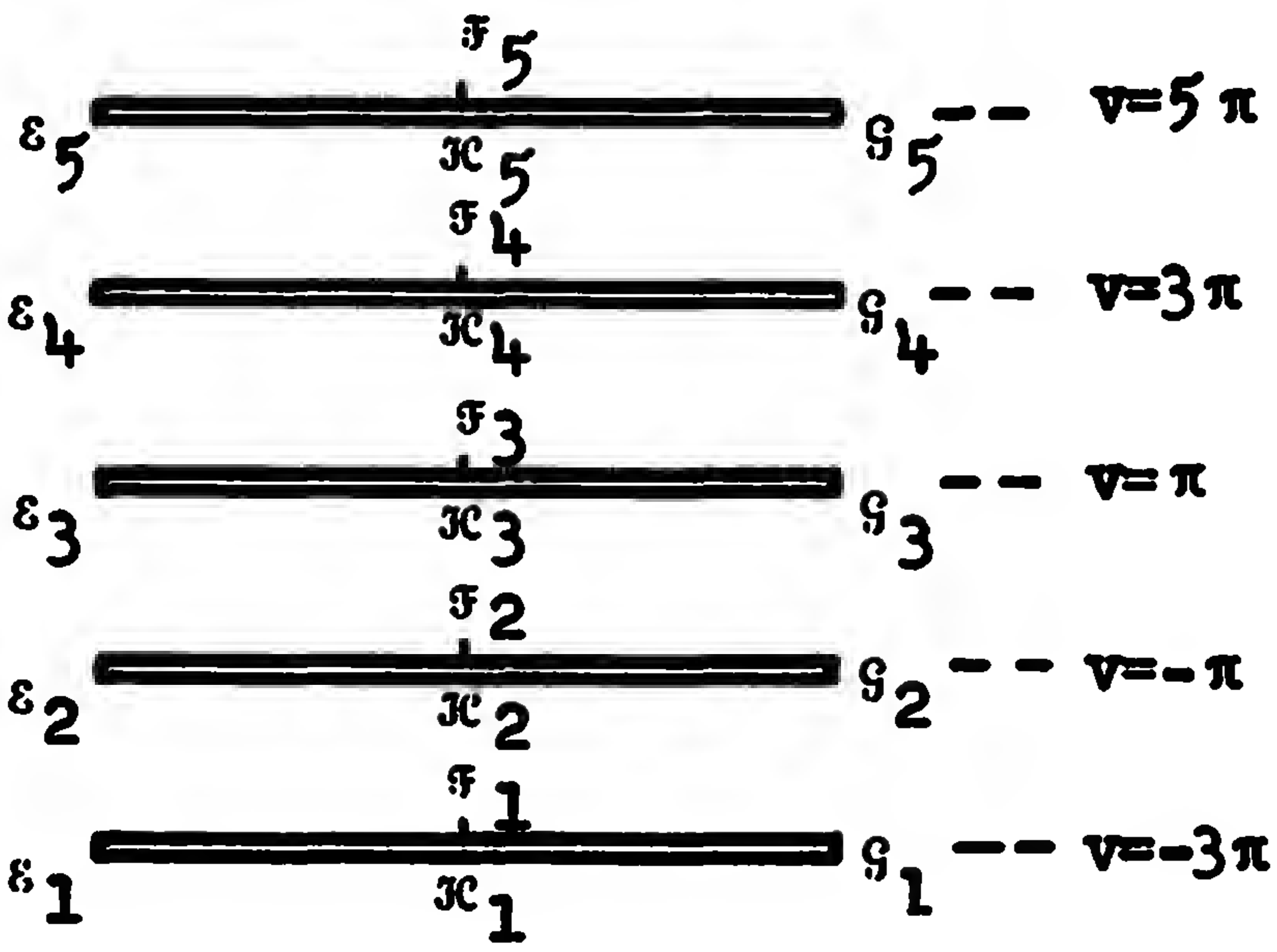
Combination of $\xi = \frac{1}{2}(z + \frac{1}{z})$ and $w = \log \frac{\xi-\ell}{\xi+\ell}$ (Cf. §§ 8, 10.2)

Critical points: $z = m;\ 1/m;\ -m;\ -1/m;\ \infty$.

$c = \log \frac{1+m}{1-m}$; ρ real; $k = 0, \pm 1, \pm 2, \ldots$.

z - plane	w - plane
points z; $1/z$	$w = w(z) + 2k\pi i$
point $z = iy$ $(-1 < y < 1)$	$w = 2k\pi i - 2i \tan^{-1} \frac{2y\ell}{1-y^2}$
line-segment $y = 0$, $-m < x < m$	line $v = 2k\pi$, $\infty > u > -\infty$.
line-segment $y = 0$, $m < x \leq 1$	half-line $v = (2k-1)\pi$, $-\infty < u \leq -2c$.
line-segment $y = 0$, $-m > x \geq -1$	half-line $v = (2k-1)\pi$, $\infty > u \geq 2c$.
line-segment $x = 0$, $0 \leq y \leq 1$	line-segment $u = 0$, $2k\pi \geq v \geq (2k-1)\pi$.
semi-circle $z = e^{i\varphi}$, $\left.\begin{matrix} 0 \leq \varphi \leq \pi \\ 2\pi \geq \varphi \geq \pi \end{matrix}\right\}$	line-segment $v = (2k-1)\pi$, $-2c \leq u \leq 2c$.

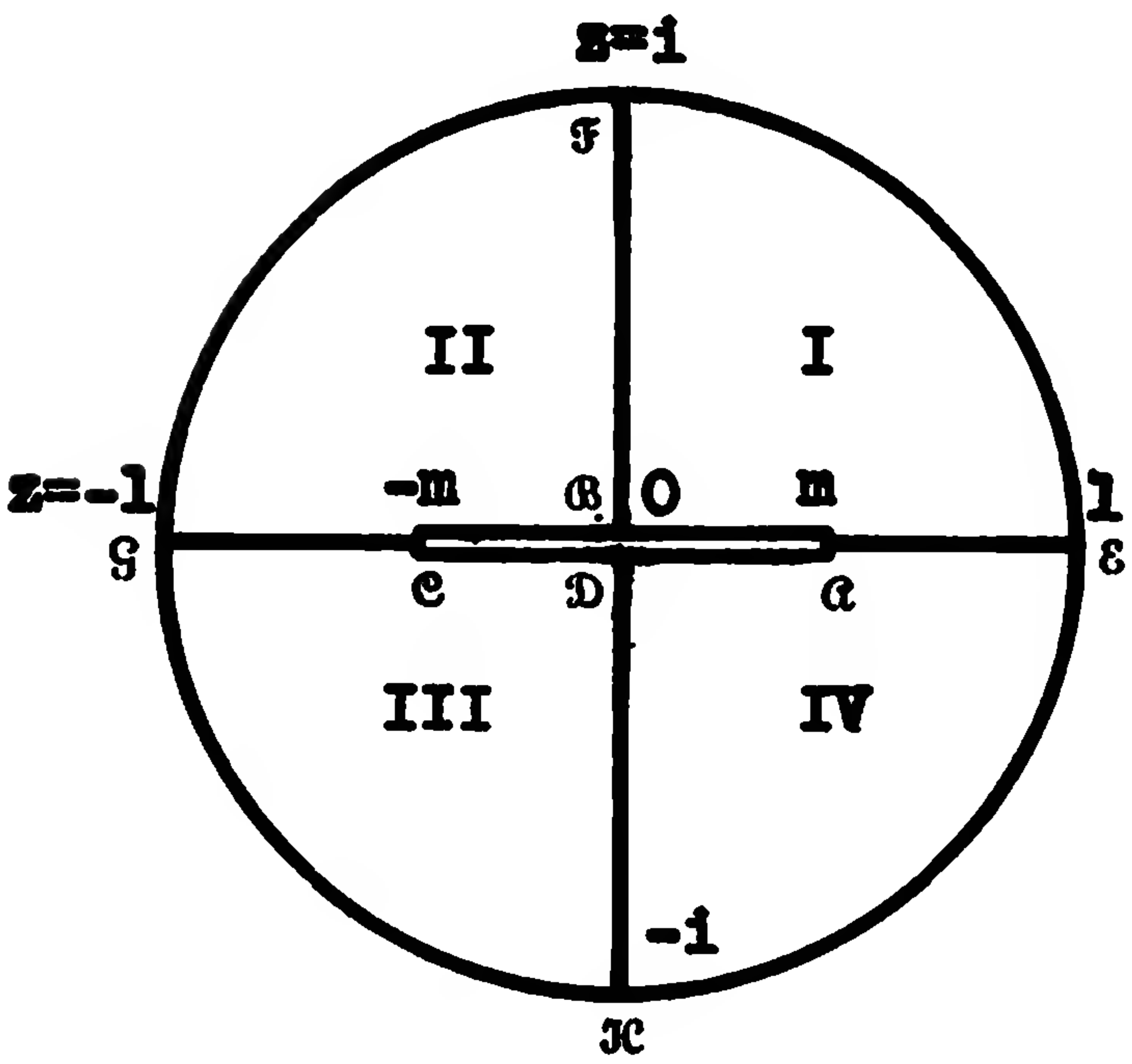
‡ See E. König, S. Kawada, listed in §8.5.

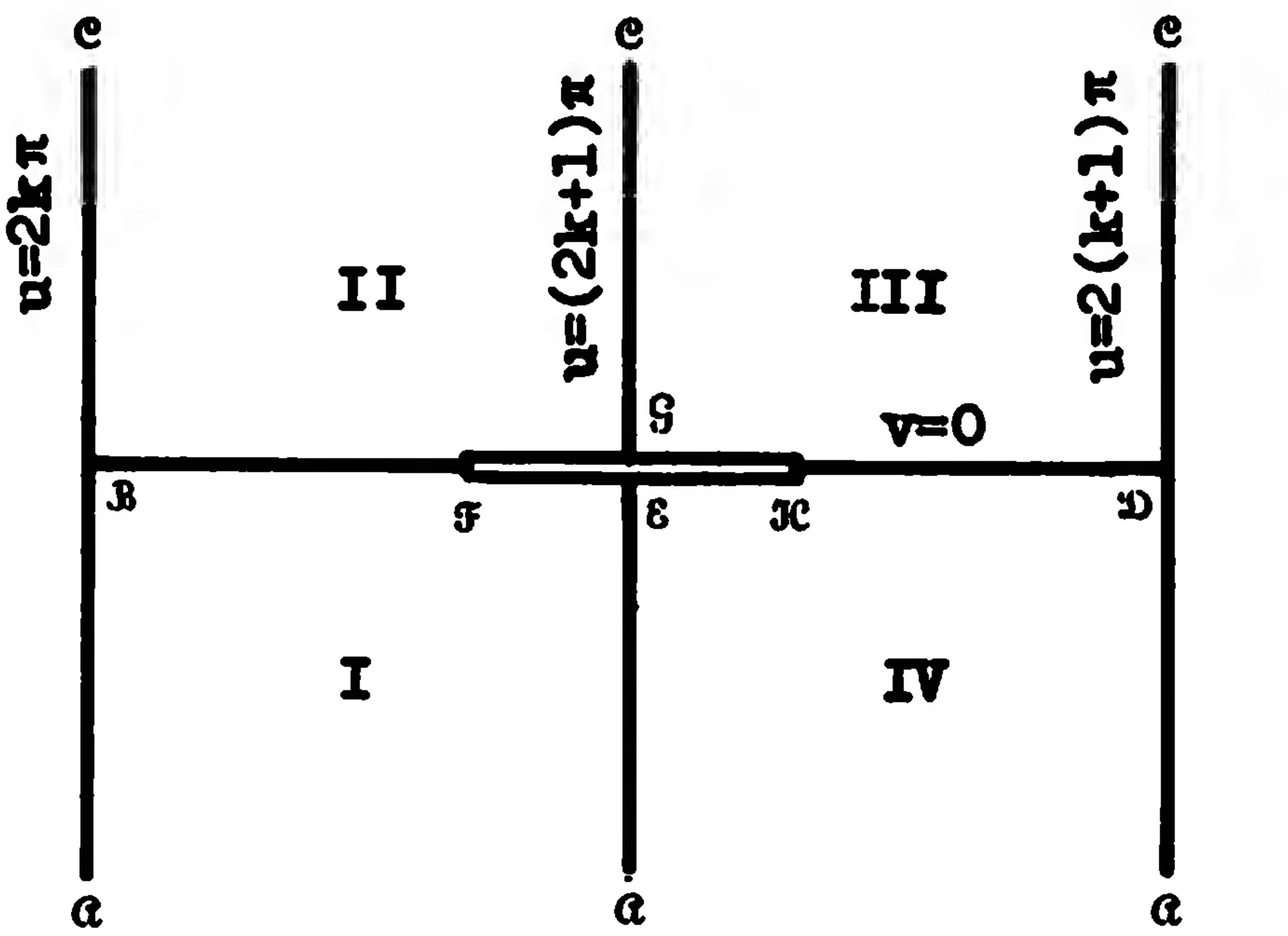
z - plane	w - plane
	 ε : $-2c + (2k-1)\pi i$; $\mathfrak{G}$: $2c + (2k-1)\pi i$
	
circle $\lvert z \rvert = 1$	each of the slits $\varepsilon\mathfrak{G}$.
point $z = e^{i\varphi}$	$w = -2 \tanh^{-1} \frac{2m \cos \varphi}{1 + m^2} + (2k-1)i\pi$
circle $\lvert z-\rho \rvert = 1-\rho$, $0 < \rho < \frac{1-m}{2}$, touching $\lvert z \rvert = 1$ at ε, counted an infinity of times	set of aerofoils, surrounding the slits and exterior to one another, touching $\varepsilon\mathfrak{G}$ at ε on both sides

z - plane	w - plane
region formed by the interiors of this circle, which is counted an infinity of times, the interiors being connected through slit $\mathcal{A}\,\mathcal{B}\,\mathcal{C}\,\mathcal{D}$	region exterior to all these aerofoils
region bounded by $\lvert z \rvert = 1$ and $\lvert z-\rho \rvert = 1-\rho$	interior, cut from $\mathcal{E}$ to $\mathcal{G}$, of an aerofoil

$$\boxed{w = i \log \frac{z-m}{z+m} - i \log \frac{mz-1}{mz+1} = i \log \frac{z^2-1+2\ell z}{z^2-1-2\ell z}} \; ; \quad 0 < m < 1, \quad \ell = \frac{1}{2}\left(\frac{1}{m} - m\right).$$

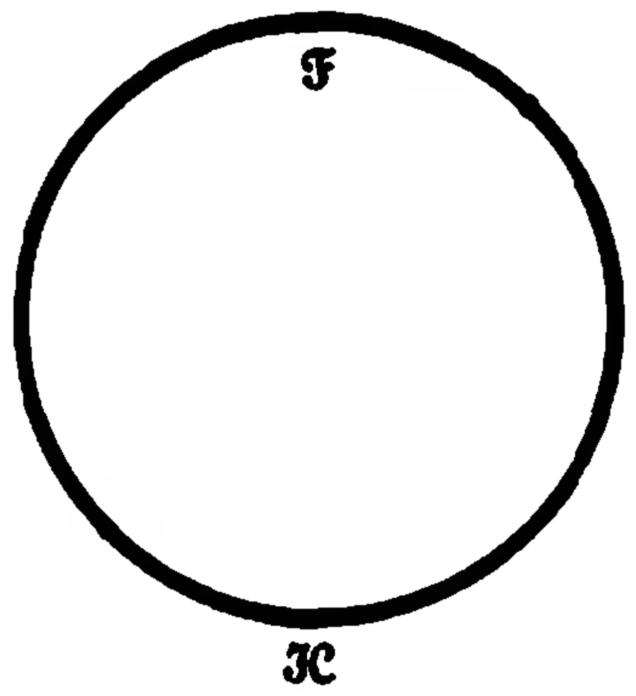
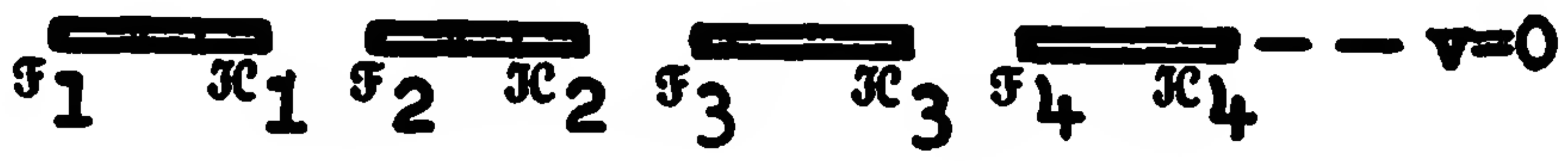
z - plane	w - plane
points z; $1/z$; $iy(-\infty < y < \infty)$; $e^{i\varphi}$; i; $-i$; ± 1	points w; $-w$; $2 \tan^{-1} \frac{2\ell y}{1+y^2}$; $\pi - 2 \tan^{-1} \frac{2m \sin\varphi}{1-m^2}$; $\pi - 4 \tan^{-1} m$; $\pi + 4 \tan^{-1} m$; π [add $2k\pi$ to each value.]





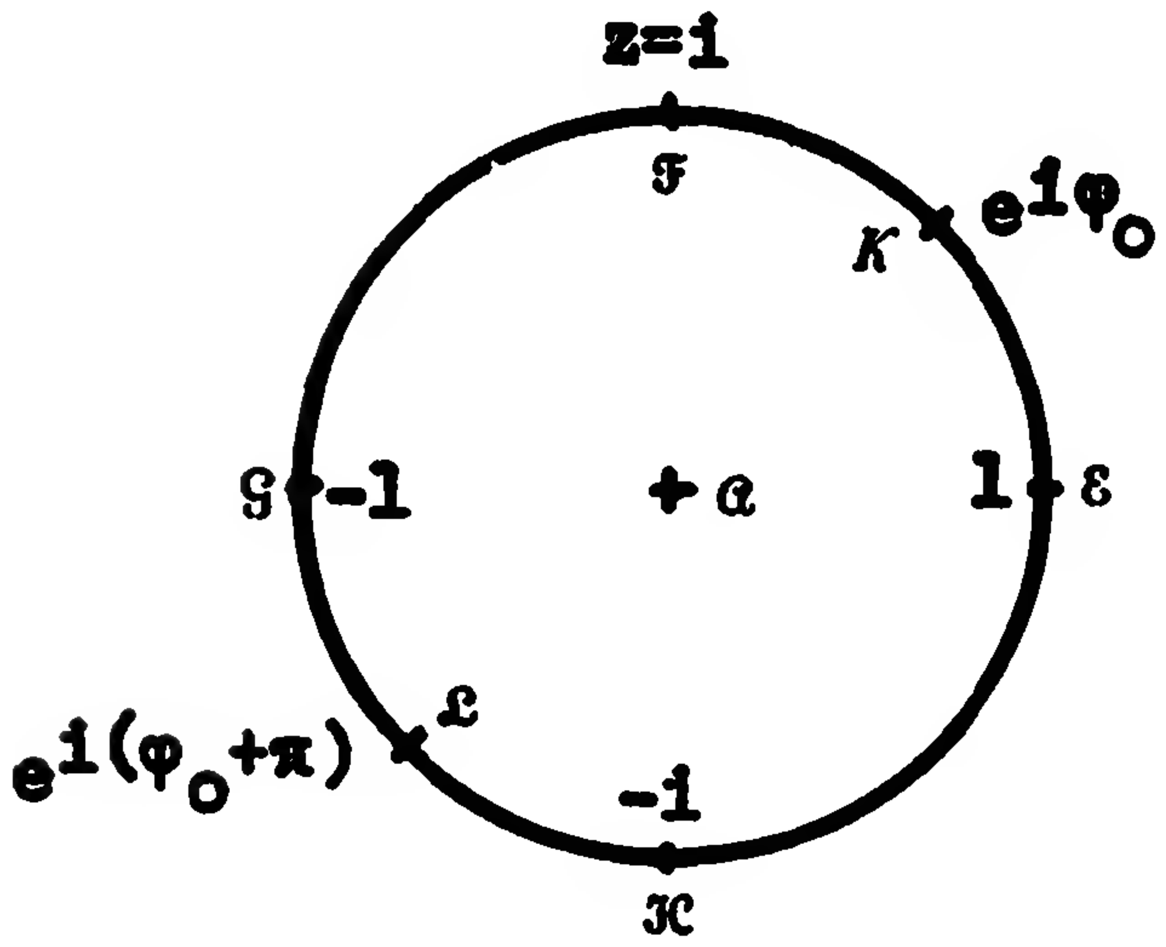
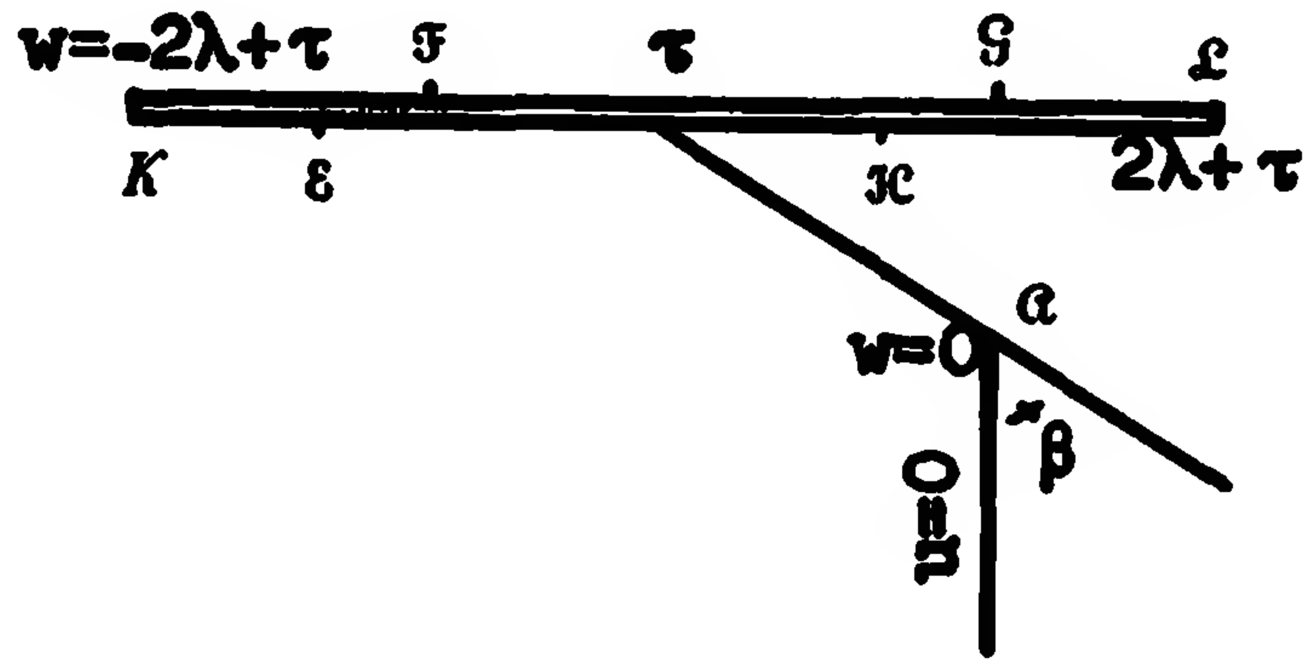
$\mathcal{F}$: $(2k+1)\pi - 4 \tan^{-1} m$;

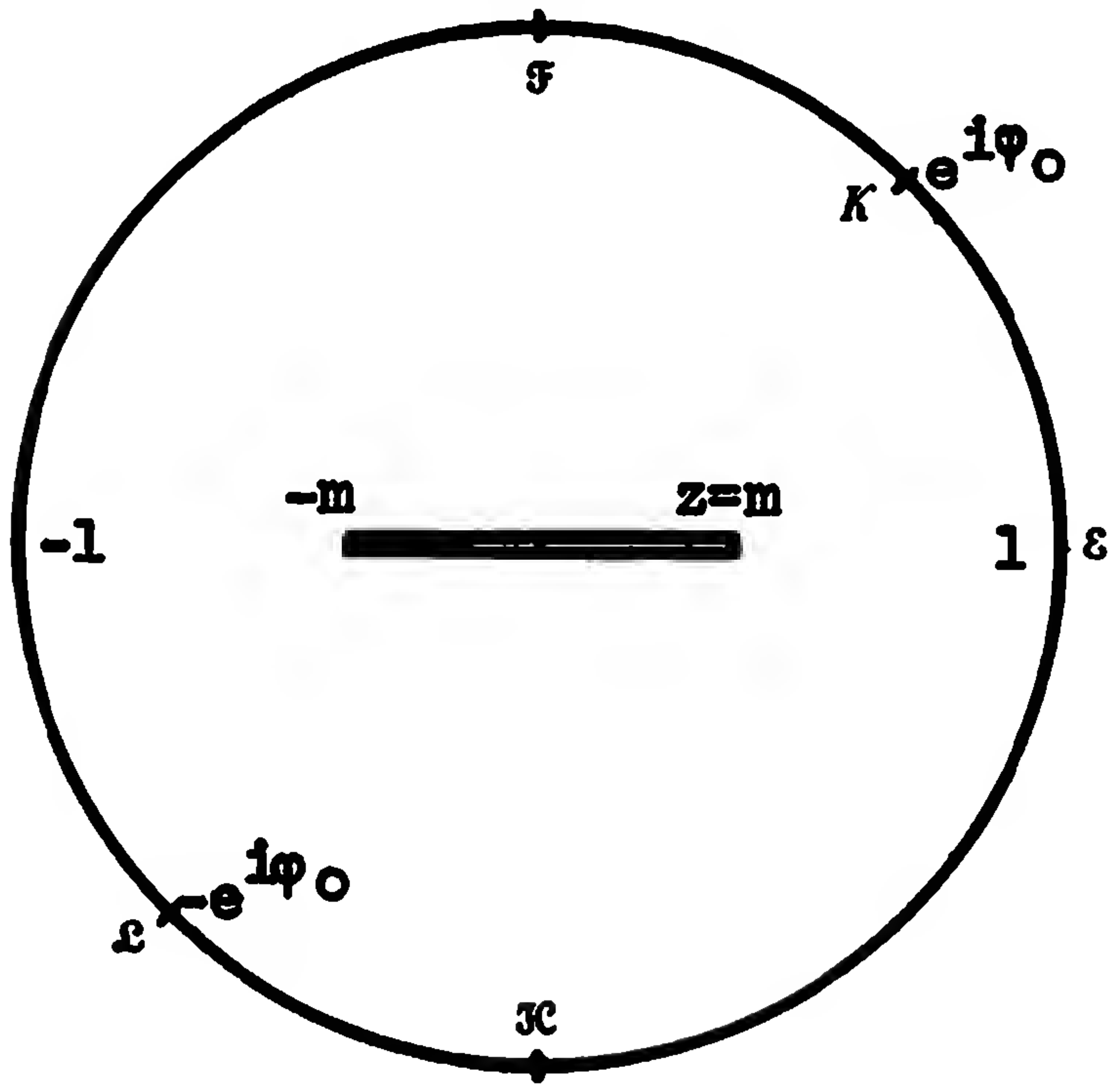
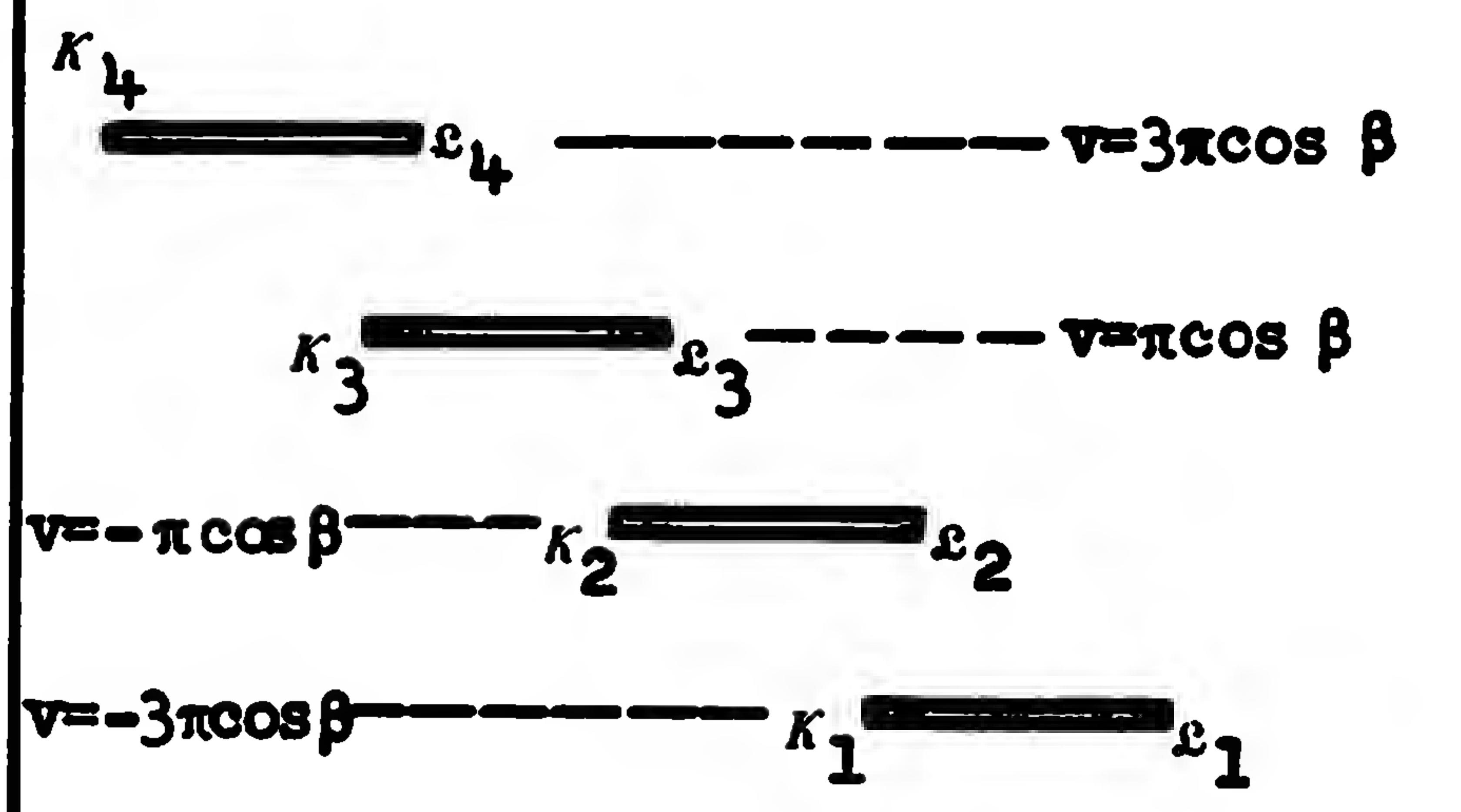
$\mathcal{K}$: $(2k+1)\pi + 4 \tan^{-1} m$

z - plane	w - plane $(k=0,\pm 1,\pm 2,\dots)$
	
circle $\lvert z\rvert = 1$, counted an infinity of times	set of slits
circle $\lvert z-i\rho\rvert = 1-\rho$, $0 < \rho < \frac{1-m^2}{2}$, counted an infinity of times	set of aerofoils, surrounding the slits, with angle = 0 at $\mathfrak{F}$

$$\boxed{w = e^{i\beta}\log\frac{m-z}{m+z} + e^{-i\beta}\log\frac{1-mz}{1+mz}}\,;\quad 0 < m < 1,\quad 0 < \beta < \frac{\pi}{2}.$$

$$c = \log\frac{1+m}{1-m};\quad \tan\varphi_0 = \frac{1-m^2}{1+m^2}\tan\beta\quad \left(0 < \varphi_0 < \frac{\pi}{2}\right);\quad k = 0, \pm 1, \dots .$$

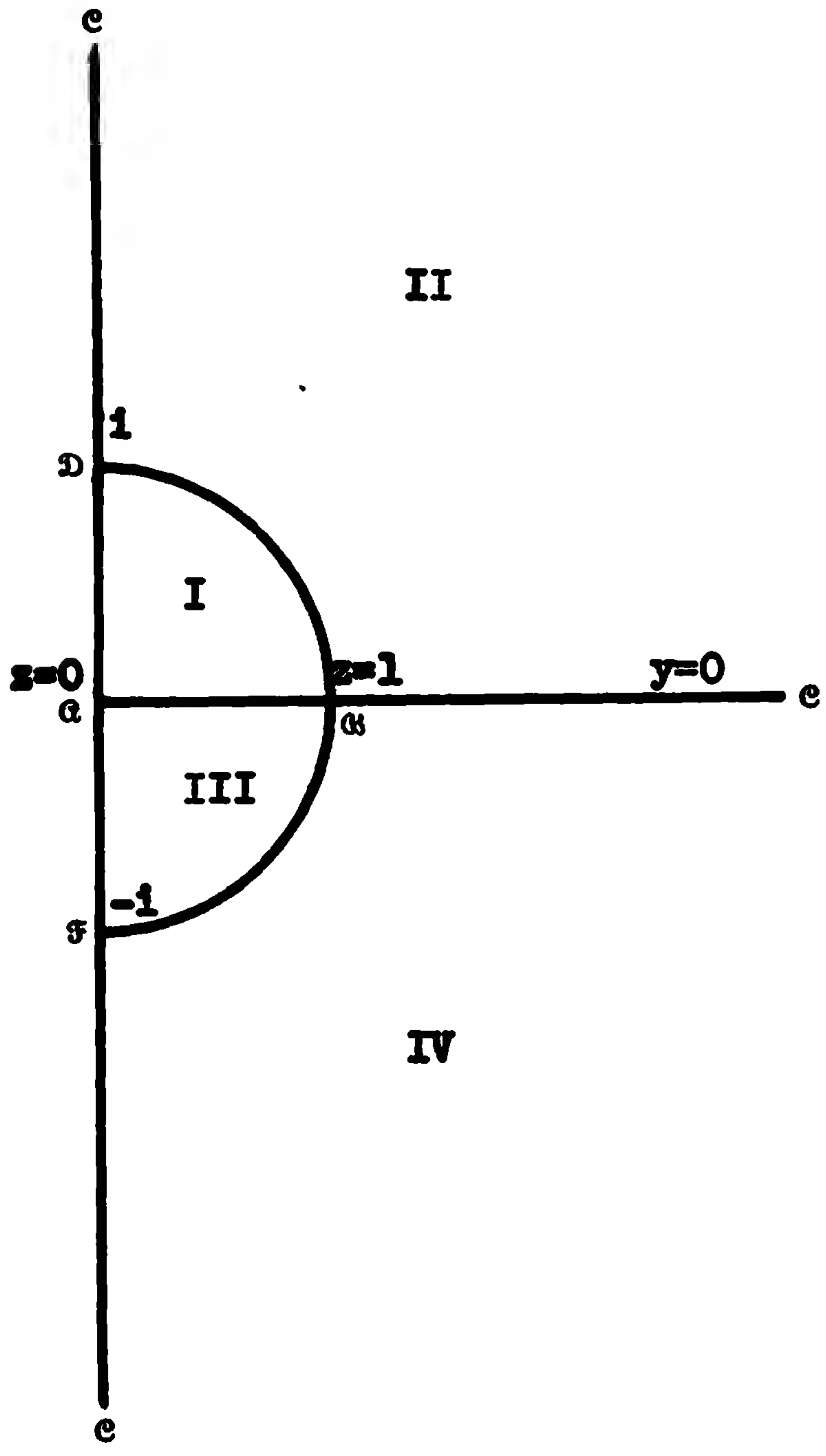
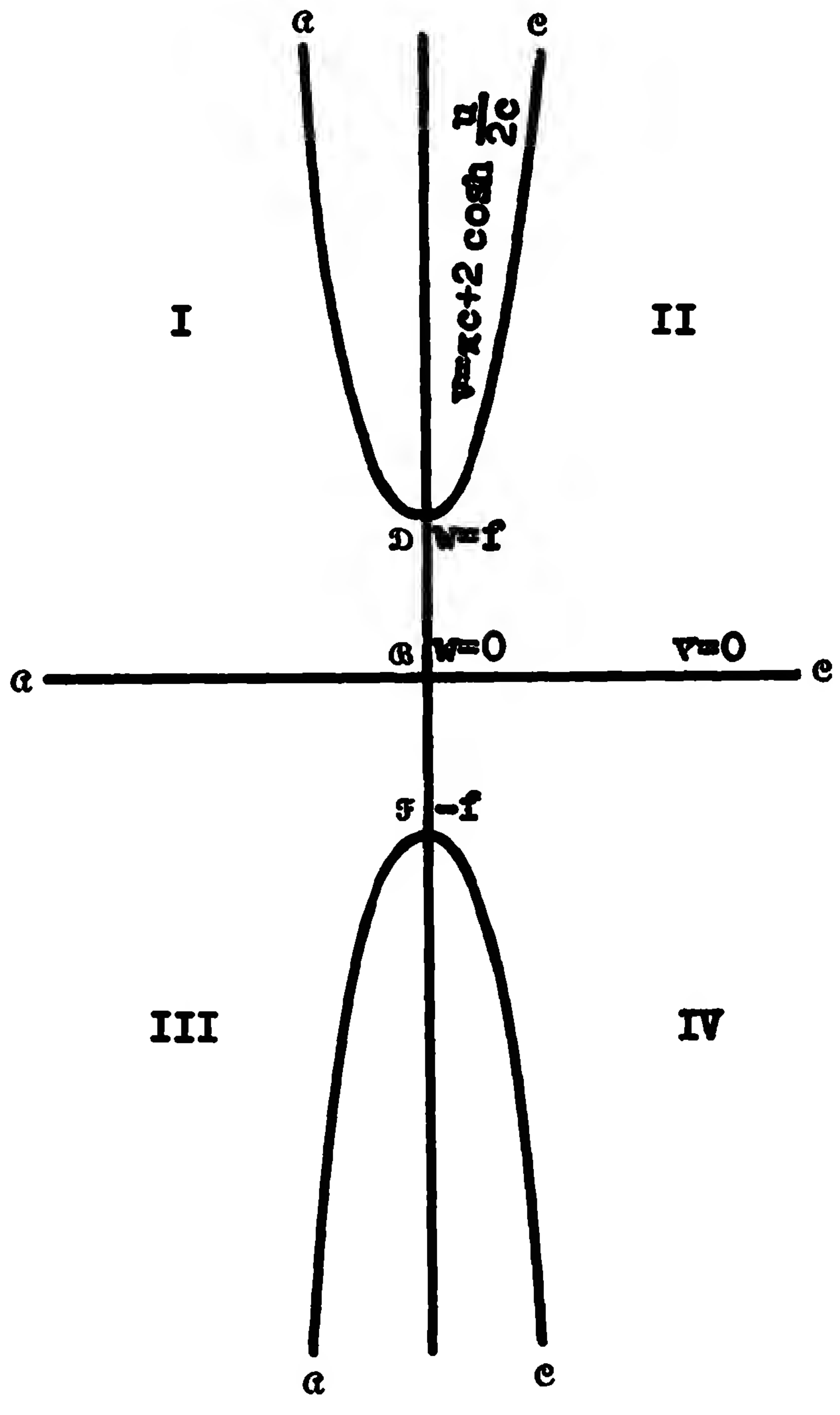
z - plane; $\lvert z\rvert < \frac{1}{m}$	w - plane
points $z = 0;\ 1;$ $-1;\ z = e^{i\varphi}$	points $w = 0;\ \pi i e^{i\beta} - 2c\cos\beta;$ $\pi i e^{i\beta} + 2c\cos\beta;$ $w = \pi i e^{i\beta} - 2\cos\beta\tanh^{-1}\frac{2m\cos\varphi}{1+m^2} - 2\sin\beta\tan^{-1}\frac{2m\sin\varphi}{1-m^2}$ [add $2k\pi i e^{i\beta}$ to each value]
	

z - plane; $\lvert z\rvert < \frac{1}{m}$	w - plane
	$\tau = \pi i e^{i\beta}$; $\lambda = \cos\beta \sinh^{-1} \frac{2m \cos\beta}{1-m^2} + \sin\beta \sin^{-1} \frac{2m \sin\beta}{1+m^2}$ $\mathcal{S}$: $-4 \sin\beta \tan^{-1} m + \tau$; $\varkappa$: $4 \sin\beta \tan^{-1} m + \tau$. ε : $-2c \cos\beta + \tau$; $\mathcal{G}$: $2c \cos\beta + \tau$.
	 $\angle \mathcal{L}_2\mathcal{L}_1\kappa_1 = \mathcal{L}_3\mathcal{L}_2\kappa_2 = \kappa_1\kappa_2\mathcal{L}_2 = \dots = \pi/2-\beta$
circle $\lvert z\rvert = 1$	each of the slits $\kappa\,\mathcal{L}$
circle touching $\lvert z\rvert = 1$ internally at κ, and such that $z = m$ and $z = -m$ lie in its interior; counted an infinity of times	set of aerofoils, surrounding slits and exterior to one another; angle at $\kappa = 0$ (cf. case $\beta = 0$).
its interior, counted an infinity of times, cut from $z = -m$ to $z = m$	region exterior to all the aerofoils (cf. case $\beta = 0$).
interior of $\lvert z\rvert = 1$, counted an infinity of times, cut from $z = -m$ to $z = m$	region exterior to all the slits.

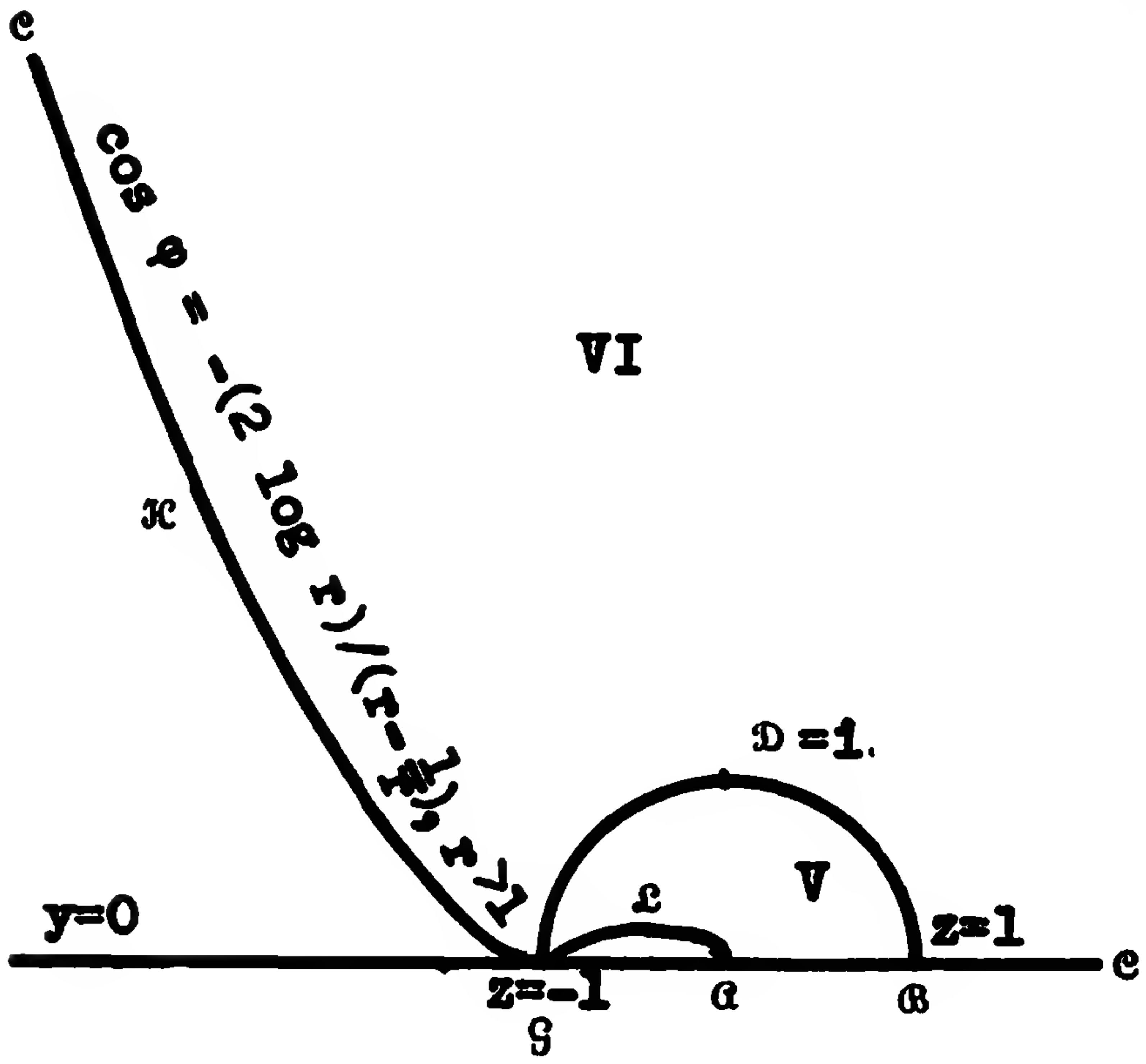
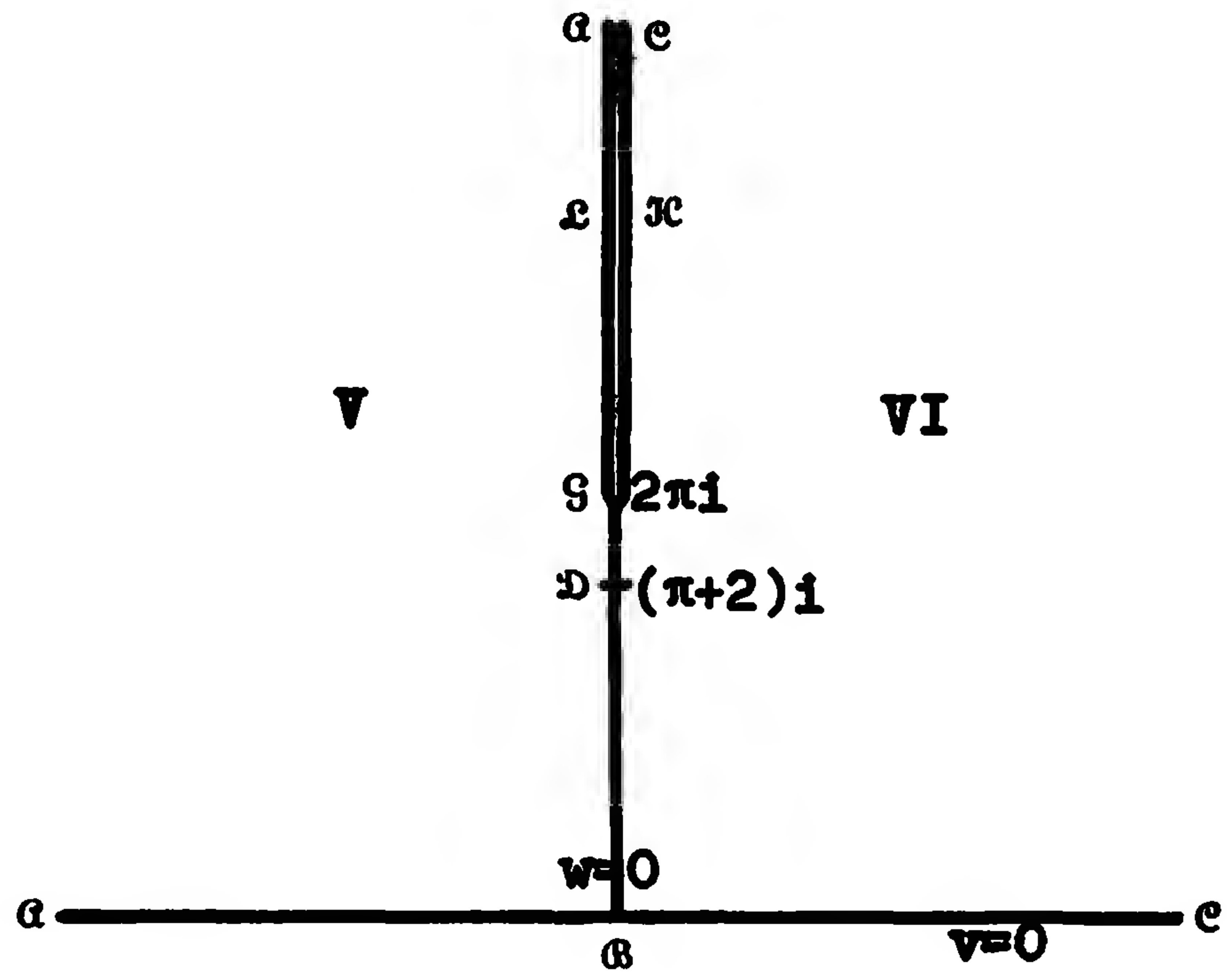
11.13 $\boxed{w = z - \frac{1}{z} + 2c \log z}$; $c > 0$. For $w = z + \frac{1}{z} + 2c \log z$, see part IV, §12.3.

Critical points: $z = 0;\ \infty;\ -c + \sqrt{(c^2-1)};\ -c - \sqrt{(c^2-1)}$.

Set $f = (2 + \pi c)i$.

z - plane; $z = re^{i\varphi}$	w - plane
	
points $z = i;\ -i;\ 1$	points $w = f;\ -f;\ 0$
half-line $y = 0,\ 0 < x < \infty$	line $v = 0,\ -\infty < u < \infty$
half-line $x = 0,\ 0 < y < \infty$	curve $v = \pi c + 2 \cosh \frac{u}{2c},\ -\infty < u < \infty$
half-line $x = 0,\ 0 > y > -\infty$	curve $v = -\pi c - 2 \cosh \frac{u}{2c},\ -\infty < u < \infty$.

Case $c = 1$.

z - plane	w - plane
circle $z = e^{i\varphi}$, $-\pi < \varphi < \pi$	line-segment $-2\pi < v < 2\pi$, $u = 0$
part $1 \leq r < \infty$ of curve $\mathfrak{e}\,\mathfrak{K}\,\mathfrak{g}\,\mathfrak{L}\,\mathfrak{a}$ $\cos(\pi-\varphi) = \dfrac{2\log r}{r-1/r}$, $\pi \geq \varphi > \dfrac{\pi}{2}$ part $1 \geq r > 0$ of the same curve	half-line $2\pi \leq v < \infty$, $u = 0$
	

Case $0 < c < 1$

$c = \cos\varphi_0$; $0 < \varphi_0 < \pi/2$; $\lambda = \pi - \varphi_0 + \tan\varphi_0$; notice that $2c\lambda > 2+\pi c$ =-if

arc $\varphi_0 - \pi \leq \varphi \leq \pi - \varphi_0$ of circle $z = e^{i\varphi}$	line-segment $-2c\lambda \leq v \leq 2c\lambda$, $u = 0$.
part $1 < r < \infty$ of curve $\cos(\pi-\varphi) = \dfrac{2c\log r}{r-1/r}$, $\pi - \varphi_0 > \varphi > \pi/2$ part $1 > r > 0$ of the same curve	half-line $2c\lambda < v < \infty$, $u = 0$.

Case $c > 1$

$c = \cosh \alpha$, $\alpha > 0$; $r_0 = e^{\mu} > 1$, where $\sinh \mu = c\mu$.

z - plane	w - plane
circle $z = re^{i\varphi}$, $-\pi < \varphi < \pi$	line-segment $-2\pi c < v < 2\pi c$, $u = 0$
part $r_0 < r < \infty$ of curve $\cos(\pi-\varphi) = \dfrac{2c \log r}{r-1/r}$, $\pi > \varphi > \pi/2$; part $1/r_0 > r > 0$ of the same curve	half-line $2\pi c < v < \infty$, $u = 0$
segment $-1 < x < -e^{-\alpha}$ of $y = 0$; segment $-1/r_0 > x > -e^{-\alpha}$ of $y = 0$	segment $0 > u > -2(\alpha \cosh \alpha - \sinh \alpha)$ of $v = 2\pi c$
segment $-1 > x > -e^{\alpha}$ of $y = 0$; segment $-r_0 < x < -e^{\alpha}$ of $y = 0$	segment $0 < u < 2(\alpha \cosh \alpha - \sinh \alpha)$ of $v = 2\pi c$.
points $z = -e^{\alpha}$; $-e^{-\alpha}$	points $w = 2(\alpha \cosh \alpha - \sinh \alpha) \pm 2i\pi c$; $2(\sinh \alpha - \alpha \cosh \alpha) \pm 2i\pi c$

See figures on next page.

$$\boxed{w = P\left(z + \frac{a^2}{z}\right) - iQ\left(z - \frac{a^2}{z}\right) + \frac{iK}{2\pi} \log \frac{z}{a}} \;;$$

$a > 0$, $K > 0$, P and Q real, $P^2 + Q^2 > 0$.

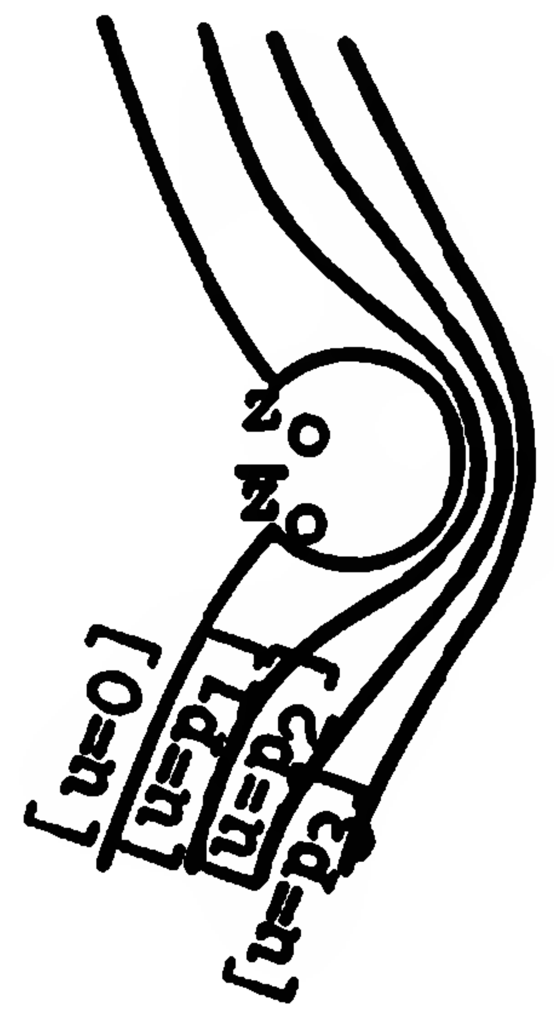
This is a combination of

$$w = ai\sqrt{(P^2 + Q^2)}\,\xi - \left(\frac{K}{4} + \frac{K}{2\pi} \tan^{-1} \frac{Q}{P}\right), \quad z = ai\sqrt{\left(\frac{P + iQ}{P - iQ}\right)}\,\zeta,$$

$$\text{and } \xi = \zeta - \frac{1}{\zeta} + 2c \log \zeta, \quad \text{where } c = \frac{K}{4a\pi}(P^2 + Q^2)^{-1/2}$$

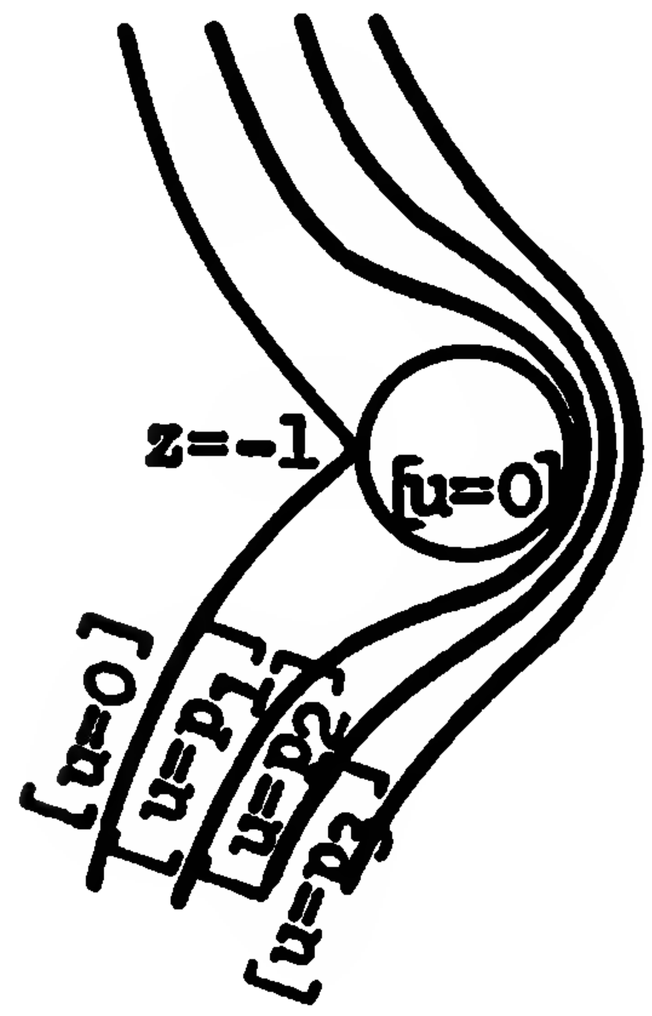
Curves in the z - plane corresponding to the lines $u = p$, $p \geq 0$

z - plane; $0 < p_1 < p_2 < \dots$. $z - 1/z + 2c \log z = w$.

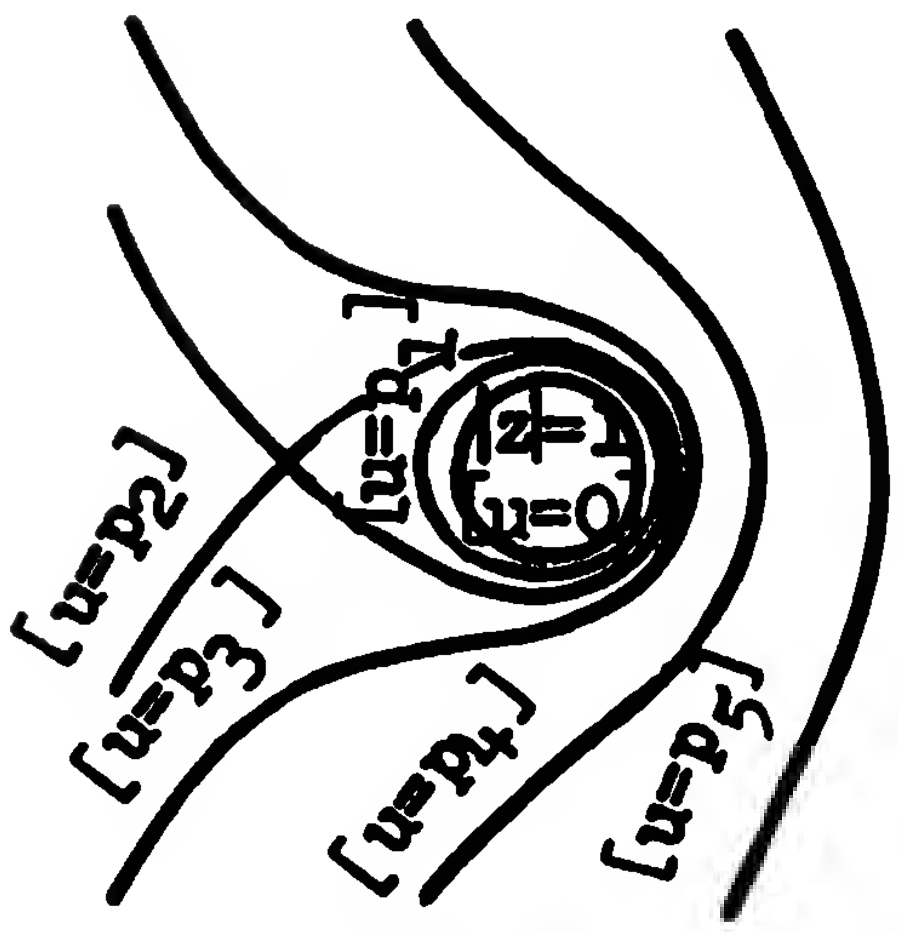


$0 < c < 1$

$z_o = -e^{-i\varphi_o} = -c + i(1-c^2)^{1/2}$;
$\bar{z}_o = -e^{i\varphi_o}$.



$c = 1$



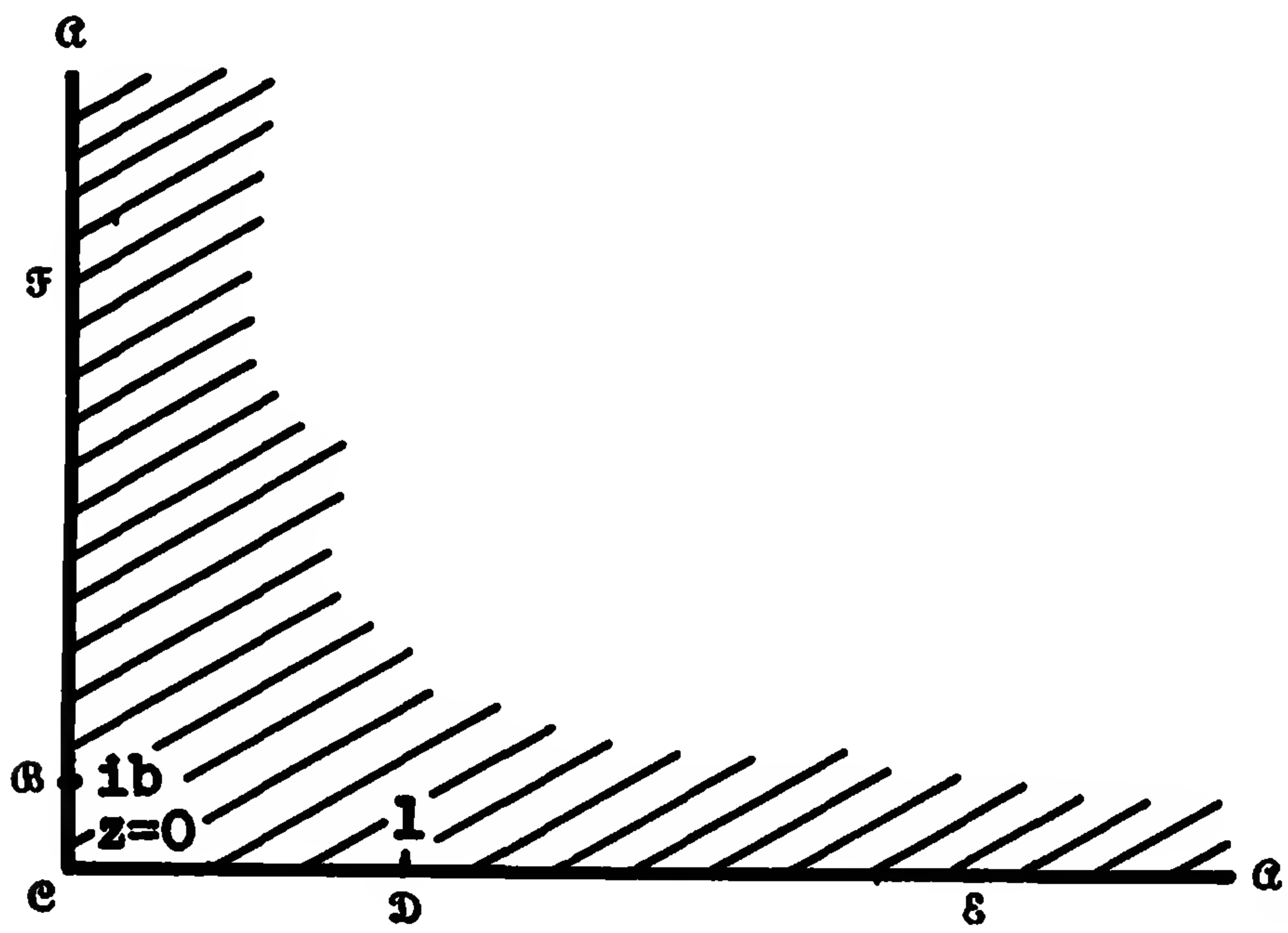
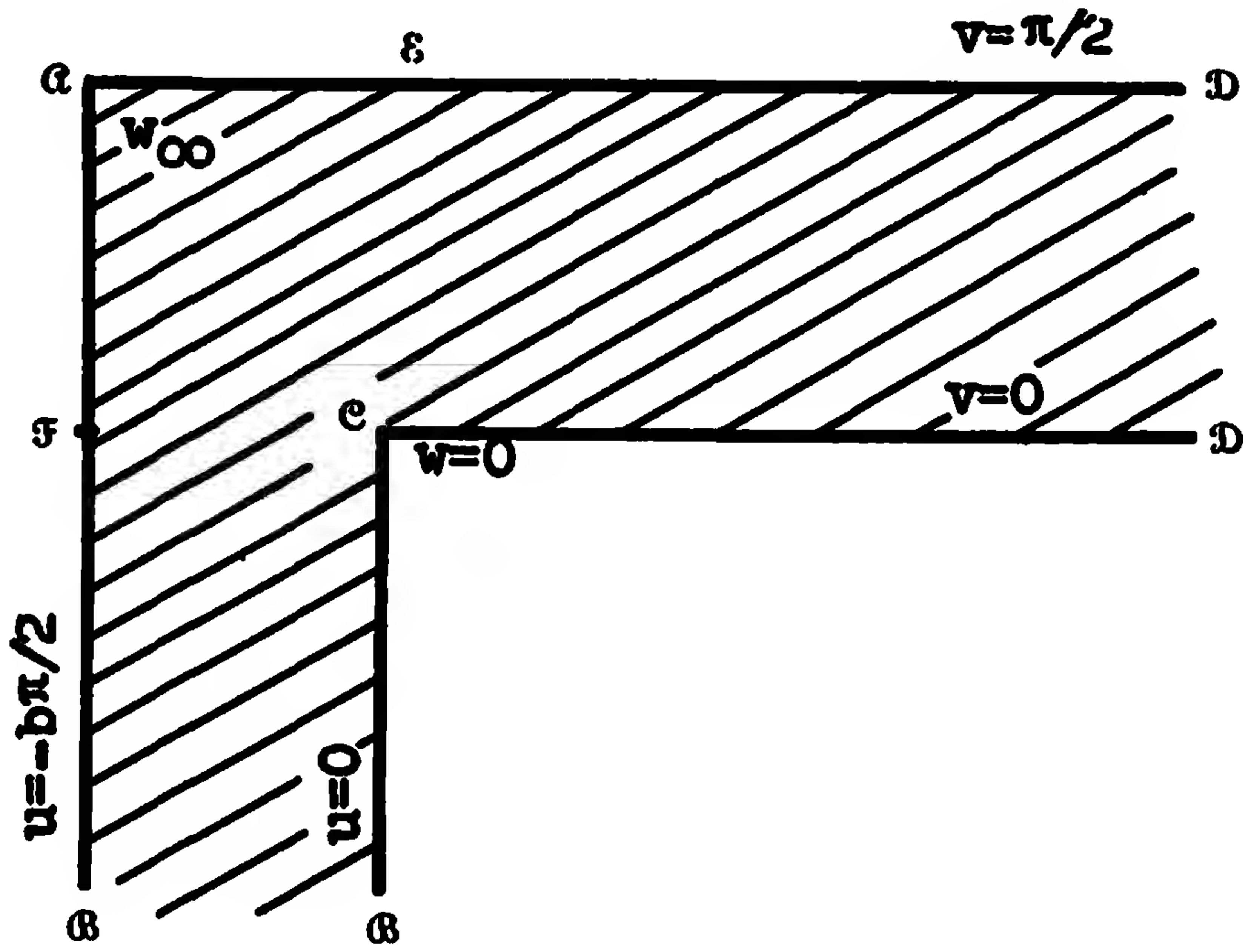
$c > 1$

The circle $|z| = 1$, i.e. $[u=0]$, is counted an infinity of times; so is $[u=p_1]$.

The essential curves of figs. 56a, b, c are copied in these diagrams.

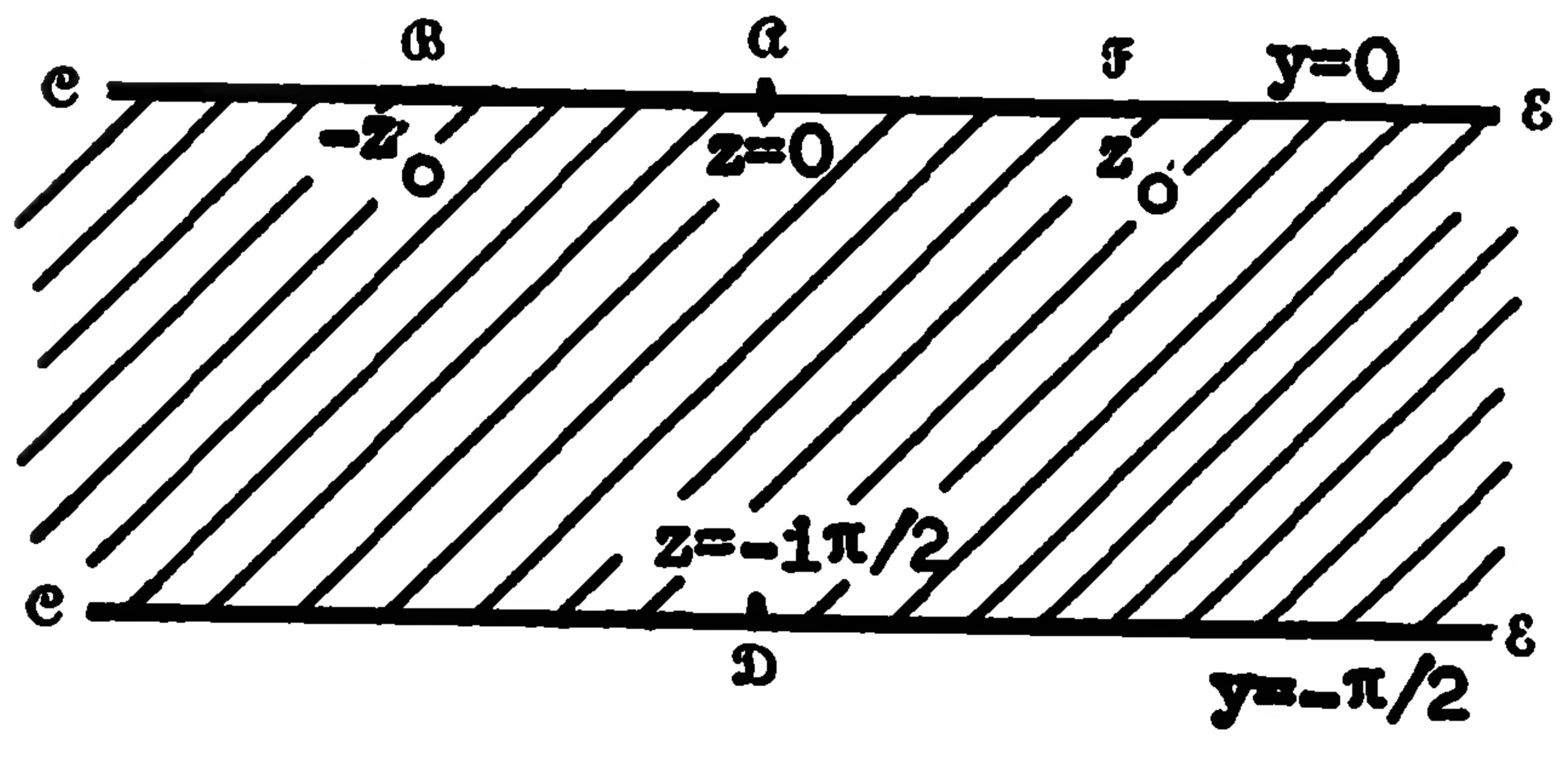
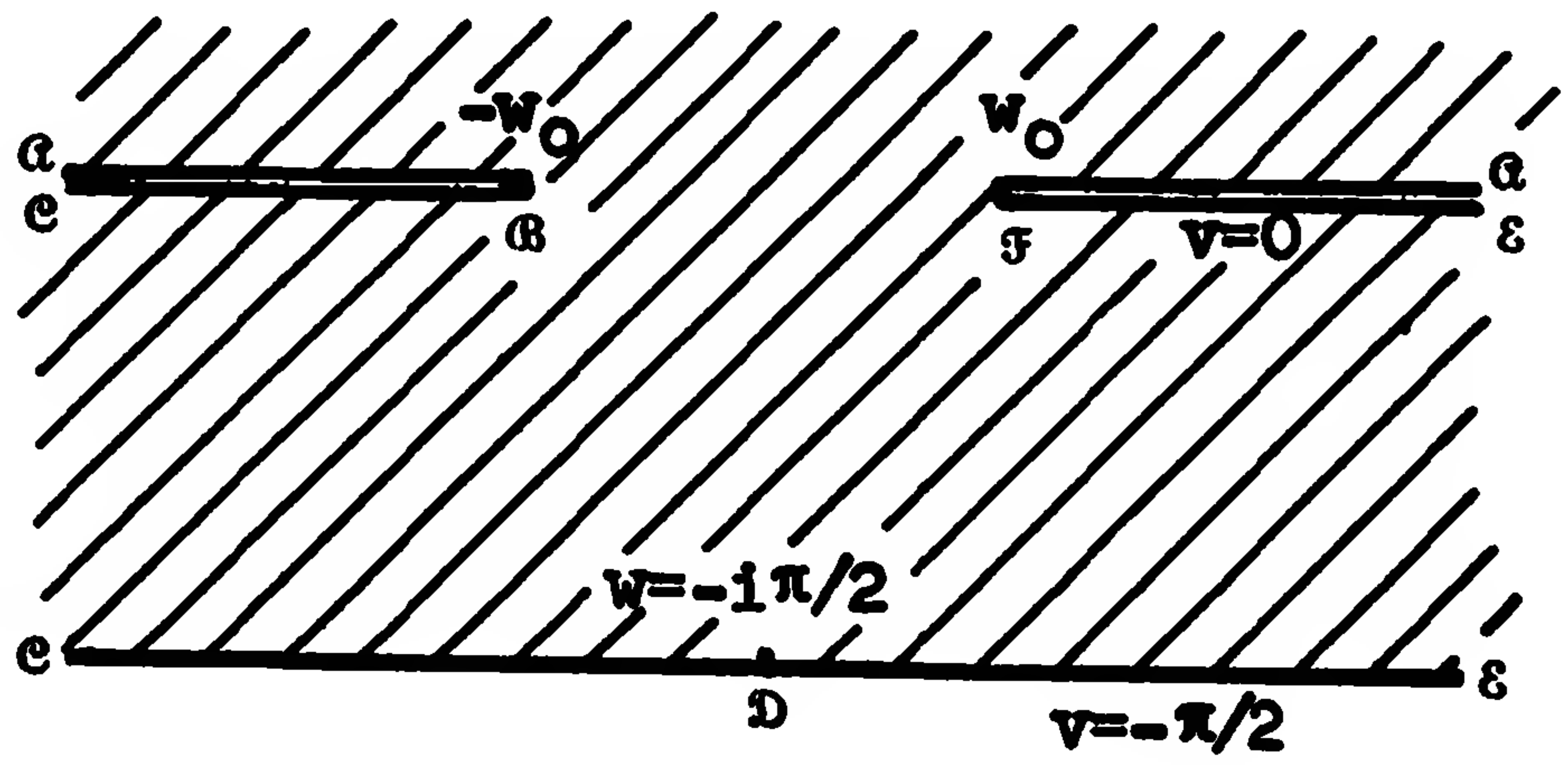
11.14 $\boxed{w = \tanh^{-1} z - b \tan^{-1} \frac{z}{b}}$, $b > 0$; cf. part IV, §12.6.

Critical points: $z = 0; +1; -1; ib; -ib; \infty$.

z - plane	w - plane
point $z = \infty$	point $w = -b\pi/2 + i\pi/2 = w_\infty$
	

$\boxed{w = z + c \coth z}$, $c > 0$; cf. part IV, example (Ai).

Critical points: $z = \infty$; $k\pi i$ $(k = 0, \pm 1, \ldots)$; $k\pi i \pm \sinh^{-1}\sqrt{c}$.

z - plane	w - plane
points $z = \sinh^{-1}\sqrt{c} = z_0$; $-z_0$; $-\frac{1}{2}i\pi$	points $w = \sinh^{-1}\sqrt{c} + \sqrt{c(c+1)} = w_0$; $-w_0$; $-\frac{1}{2}i\pi$
	

PART FOUR

SCHWARZ-CHRISTOFFEL TRANSFORMATIONS REPRESENTED IN TERMS OF ELEMENTARY TRANSFORMATIONS

INTRODUCTION.

Remarks on the method of the Schwarz-Christoffel transformation.

The Schwarz-Christoffel transformation.

$$\frac{dz}{dw} = (w-a_1)^{-\alpha_1/\pi}(w-a_2)^{-\alpha_2/\pi}\ldots(w-a_n)^{-\alpha_n/\pi}, \text{ or}$$

$z = f(w)$, where

$$-\infty < a_1 < a_2 < \ldots < a_n < \infty,$$

$$-\pi \leq \alpha_j \leq 3\pi \ (j = 1,2, \ldots,n); \quad -\pi \leq \sum_{j=1}^{n} \alpha_j \leq 3\pi; \quad \alpha_j \neq 0;$$

$$\alpha_{n+1} = 2\pi - \sum_{1}^{n} \alpha_j;$$

$$A_j = f(a_j) \quad (j = 1,2, \ldots,n); \quad A_{n+1} = f(\infty);$$

and where the $A_1, A_2, \ldots, A_{n+1}$ are the vertices of a "polygon" in the z-plane.

(1) The sides of this polygon must not intersect one another, but self-contacts may occur, and vertices may lie at infinity; the sides must form the boundary of a simply connected region, the "interior" of the polygon. If this region lies to the left when we transverse the sides in order, then it is mapped on $v > 0$; angle α_j is the change of direction when we pass through A_j.

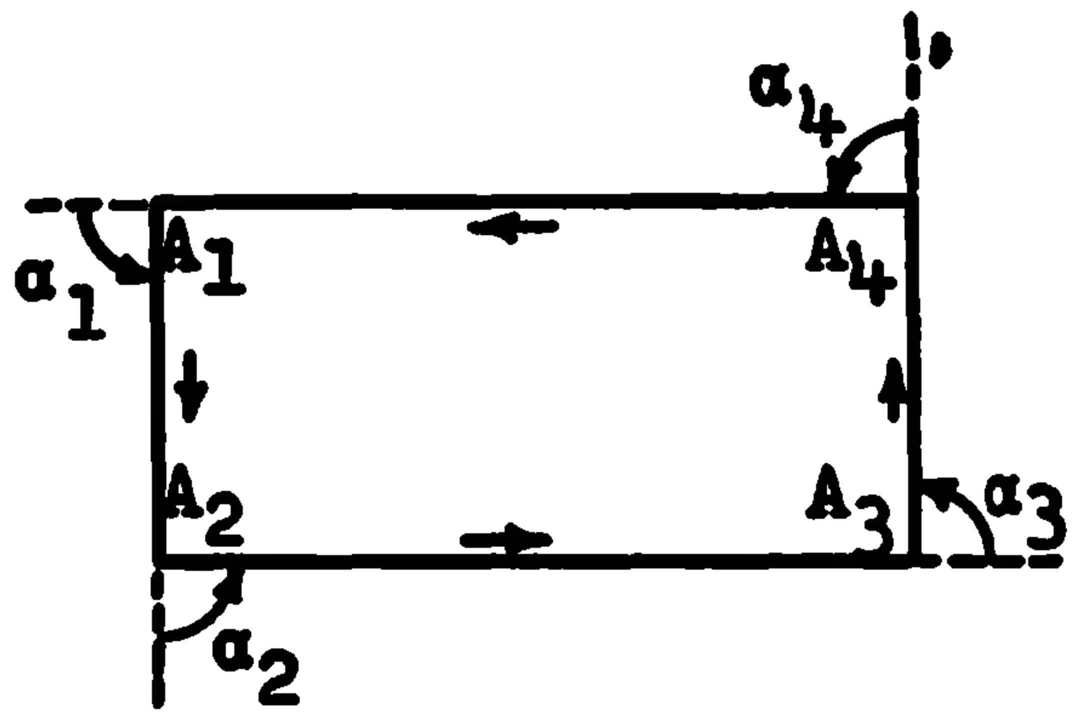
If $\alpha_{n+1} = 0$, then the polygon has n vertices only, $A_1, A_2, \ldots, A_n$. A vertex A_j lies at infinity if, and only if,

$$\pi \leq \alpha_j \leq 3\pi.$$

All the vertices of the polygon are finite if

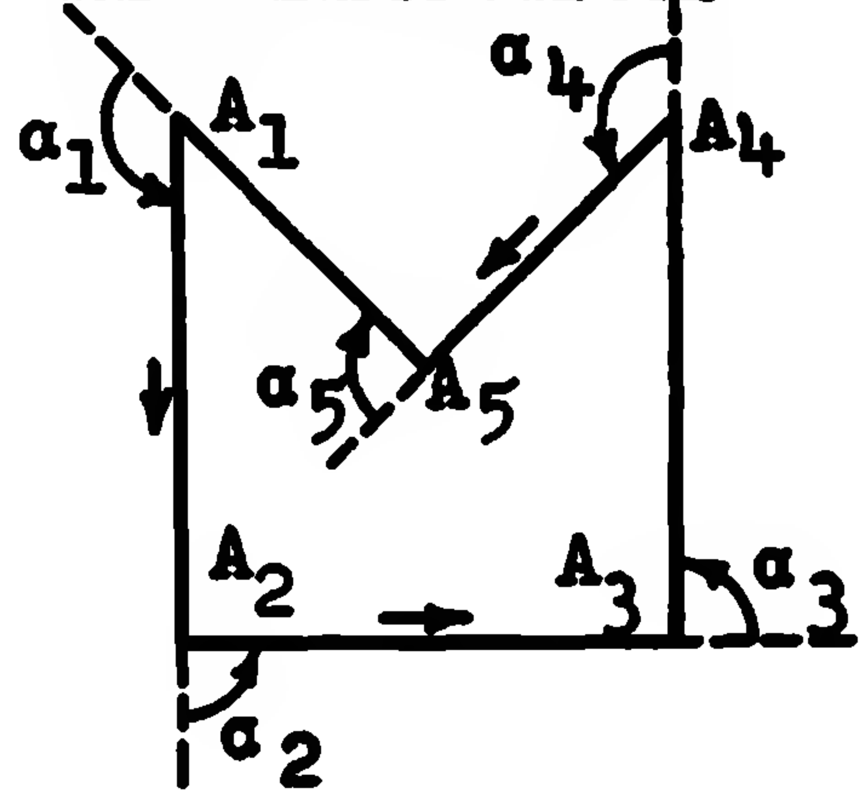
$$-\pi \leq \alpha_j < \pi, \qquad j = 1,2, \ldots, n+1.$$

Finite polygons



$\angle\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}\pi$;

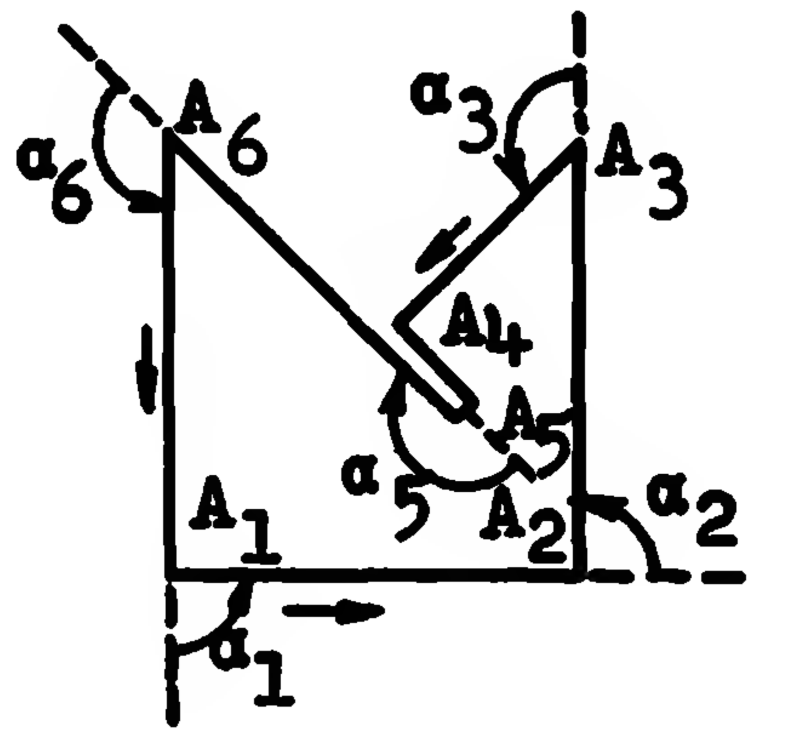
$\Sigma\alpha_j = 2\pi.$



$\angle\alpha_2 = \alpha_3 = \frac{1}{2}\pi$; $\alpha_1 = \alpha_4$,

$\alpha_1 = \frac{3}{4}\pi$; $\angle\alpha_5 = -\frac{1}{2}\pi$;

$\Sigma\alpha_j = 2\pi.$



$\angle\alpha_1 = \alpha_2 = \alpha_4 = \frac{1}{2}\pi$;

$\alpha_3 = \alpha_6 = \frac{3}{4}\pi$;

$\alpha_5 = -\pi$;

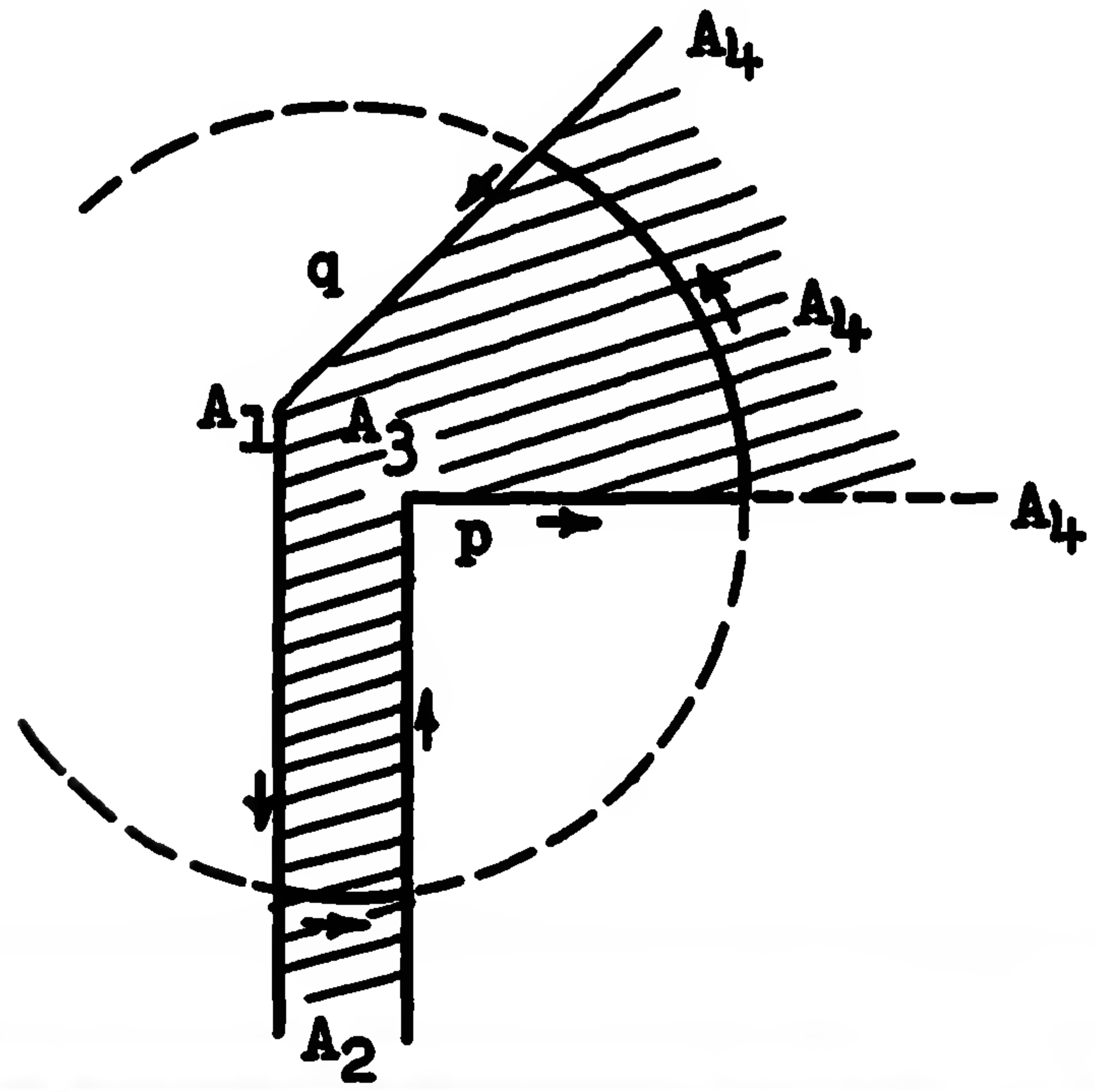
$\Sigma\alpha_j = 2\pi$;

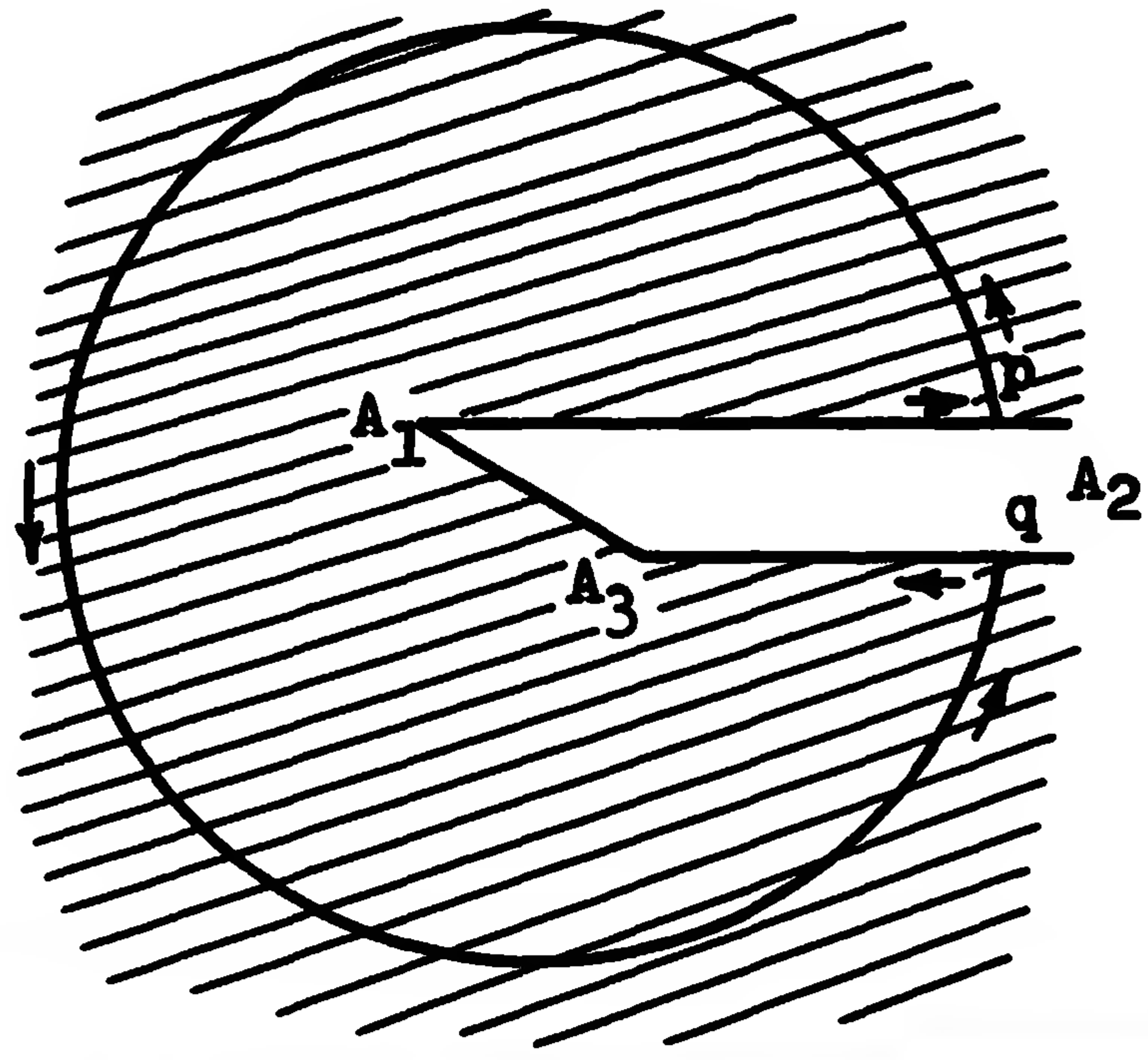
Self-contact along A_4A_5

(2) Change of direction, i.e. angle α, when passing through a vertex A at infinity.‡

A sufficiently large circle is drawn which contains all the finite vertices in its interior. Instead of passing through A, we pass from the "side" p to the "side" q along that arc which, joining p to q, lies in the interior of the polygon, and find the total change of direction. The arc can be replaced by a suitable polygonal chain.

Vertices at infinity.





‡ Cf. W. Mangler.

$\angle\,\alpha_1 = \frac{\pi}{4}$, $\alpha_3 = -\frac{\pi}{2}$; $\alpha_2 = \pi$;

$\alpha_4 = \frac{5\pi}{4}$.

$\Sigma\alpha_j = 2\pi$.

"interior of polygon" is exterior of semi-infinite strip.

$\angle\,\alpha_1 = -\frac{5\pi}{6}$; $\alpha_2 = 3\pi$; $\alpha_3 = -\frac{\pi}{6}$.

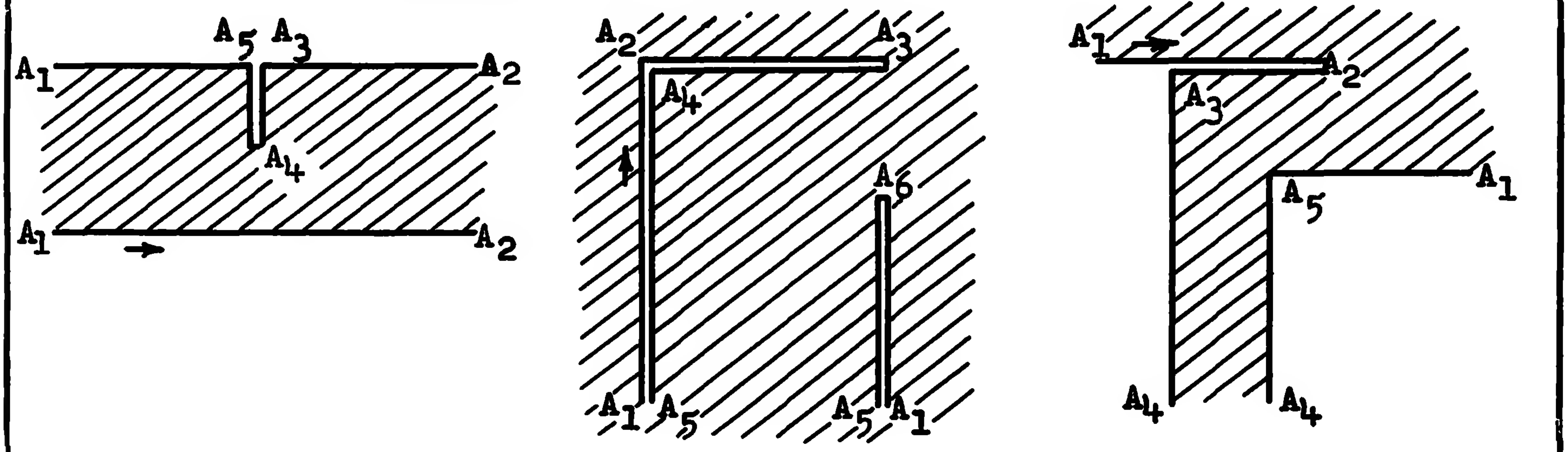
$\Sigma\alpha_j = 2\pi$.

(3) Not every differential equation $dz/dw = (w-a_1)^{-\alpha_1/\pi}\ldots$, where the a_j, α_j satisfy the conditions stated above, is a Schwarz-Christoffel transformation; for instance,

$$\frac{dz}{dw} = w^{-1}(w-2)^{-\alpha_2/\pi}(w-3)^{-\alpha_3/\pi}$$

is not whenever $-2\pi \leq \alpha_2 + \alpha_3 < -\pi$. But for every "polygon" defined as above a Schwarz-Christoffel transformation can be constructed. The latter can be represented in terms of functions treated in SS1-11 certainly if not more than two angles α_j are $\pm\frac{1}{2}\pi$ or odd multiples of $\pm\frac{1}{2}\pi$, while the other angles are $\pm\pi$, 2π, or 3π.

__Examples__ (not mentioned any further)



$\alpha_1 = \alpha_2 = -\alpha_4 = \pi$;
$\alpha_3 = \alpha_5 = \frac{1}{2}\pi$.

$\alpha_1 = 3\pi$; $\alpha_5 = \pi$;
$\alpha_3 = \alpha_6 = -\pi$;
$\alpha_4 = -\alpha_2 = \frac{1}{2}\pi$.

$\alpha_1 = 2\pi$; $\alpha_4 = -\alpha_2 = \pi$;
$\alpha_3 = -\alpha_5 = \frac{1}{2}\pi$.

All the transformations of part four concern polygons of which at least one vertex lies at infinity.

(4) The <u>exterior of a finite polygon</u> is transformed into the upper half of the w-plane by

$$\frac{dz}{dw} = (w-a_1)^{-\alpha_1/\pi} \dots (w-a_n)^{-\alpha_n/\pi} (w^2+1)^{-2}.$$

The angles α_j, as the changes of direction, are measured by transversing the sides in order so that the interior of the polygon lies to the right. The point $z = \infty$ is mapped on $w = i$.

(5) <u>Transformations which are equivalent</u> from the topological point of view.

When $z = f(w)$ maps the half-plane $v > 0$ conformally on a region in the z-plane, then $\tilde{z} = g(w)$ has the same property if, and only if, it can be represented in the form

$$\tilde{z} = f\left(\frac{aw+b}{cw+d}\right); \quad a, b, c, d \text{ real}; \quad ad - bc > 0.$$

<u>Example</u>:

The transformation $w = e^z$ maps the strip $0 < y < \pi$ (i.e. polygon with two vertices, both at infinity; $\alpha_1 = \alpha_2 = \pi$) on $v > 0$. The following transformations have the same property:

$$w = \tanh \tfrac{1}{2} z = \frac{e^z-1}{e^z+1}; \quad w = -\coth \tfrac{1}{2} z; \quad w = e^{z/2} \operatorname{sech} \tfrac{1}{2} z;$$

$$w = -e^{z/2} \operatorname{cosech} \tfrac{1}{2} z; \quad w = -e^{-z/2} \operatorname{sech} \tfrac{1}{2} z; \quad w = -e^{-z/2} \operatorname{cosech} \tfrac{1}{2} z; \text{ etc.}$$

(6) <u>Combinations of Schwarz-Christoffel transformations.</u>

(i) <u>Simple combination</u>. Interior of polygon on strip. To find the "equipotentials" and "stream lines", it is necessary to map the interior of the polygon concerned on that of a strip so that the boundary lines of the strip correspond to the sides of the polygon, taken in a certain order according to the condition of the physical problem.

An example will show how the construction of the transformation required is reduced to that of two Schwarz-Christoffel transformations.

Actually a considerable number of the transformations of Part Three can be represented as Schwarz-Christoffel transformations or as combinations of two of them.

(ii) Combination, using a differential equation. Let

$$w = w(\tau) \quad \text{and} \quad \lambda = \lambda(\tau)$$

be Schwarz-Christoffel transformations. The transformation required is

$w = f(z)$, where

$$\frac{dz}{d\tau} = -\frac{dw}{d\tau}\, e^{\lambda(\tau)}.$$

This method plays a considerable part in the theory of "free stream-lines" in Hydrodynamics; the velocity-vector is $\bar{\zeta}^{-1}$ where

$$\zeta = -\frac{dz}{dw}, \quad \log \zeta = \lambda.$$

References:

Kirchhoff, Zur Theorie Freier Flüssigkeitsstrahlen, (1869) Gesammelte Abhandlungen, p. 416.

Helmholtz, Berliner Monatsber. April 23, 1868.

Rayleigh, (i) On the resistance of fluids, Phil. Mag. Dec. 1876 (Papers, 1. 287)

(ii) Notes on Hydrodynamics, Phil. Mag. Dec. 1876 (Papers, 1. 297)

Love, A.E.H., On the theory of discontinuous fluid motions in two dimensions, Proc. Camb. Phil Soc. vol. 7 (1892), p. 175.

Greenhill, G., Theory of a stream-line past a plane barrier, R. &. M. 19 (1910).

For further references concerning the subject see

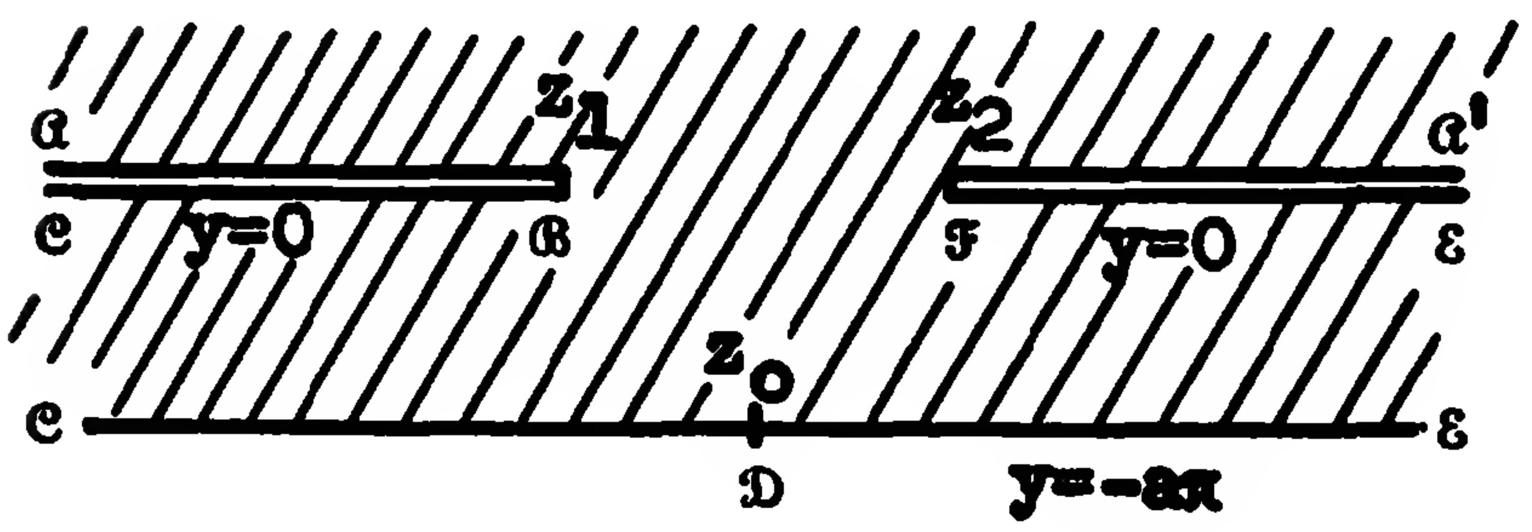
Lamb, H., Hydrodynamics, Cambridge 1932, Chapter IV.

Hanson, J.H., The practical application of conformal representation, Thesis October 1934, University of Liverpool.

(7) Principle of reflection. Suppose a region R in the z-plane to be symmetrical with respect to a line ℓ. Then the transformation which maps R on $v > 0$ can be reduced to a transformation mapping one of the two regions into which R, obviously excluding the points of R lying on ℓ, is divided by ℓ on $v > 0$. This will be shown by an example. In the third problem of §12.9 the principle is applied twice.

(8) Examples.

(A) Example for the simple combination.



To map the half-plane $y > -a\pi$, with the two slits $-\infty < x < z_1$; $z_2 < x < \infty$,

[$a > 0$; z_1, z_2 real; $z_2 > z_1$;

$$z_0 = \frac{z_1+z_2}{2} - ai\pi \;]$$

on the strip $0 < \Im(\zeta) < \pi$ of the ζ-plane so that

(i) $\mathfrak{C}\mathfrak{D}\mathfrak{E}$ $(y = -a\pi)$ is transformed into $\Im(\zeta) = 0$, the two slits together into $\Im(\zeta) = \pi$, or

(ii) $\mathfrak{C}\mathfrak{D}\mathfrak{E}$ and the slit $\mathfrak{A}\mathfrak{B}\mathfrak{C}$ together into $\Im(\zeta) = 0$, slit $\mathfrak{A}'\mathfrak{F}\mathfrak{E}$ into $\Im(\zeta) = \pi$.

We take $z_1+z_2 = 0$, i.e., $z_0 = -ai\pi$, and ρ as the real root of

$$\rho + \log[\rho + (\rho^2+1)^{1/2}] = \frac{z_2}{a} \qquad [\rho > 0,\ (\rho^2+1)^{1/2} > 0];$$

$w_1 = -a\rho$, $w_2 = a\rho$, $k = a \log[\rho + (\rho^2+1)^{1/2}]$, $b = a(\rho^2+1)^{1/2} - a$.

Then

$$z_2 = w_2+k, \quad z_1 = -z_2 = w_1-k, \quad k = a \log \frac{w_2+b}{w_2-b}.$$

Using a result of §12.2, with $\beta = 1$ and $c = -b$, we see that

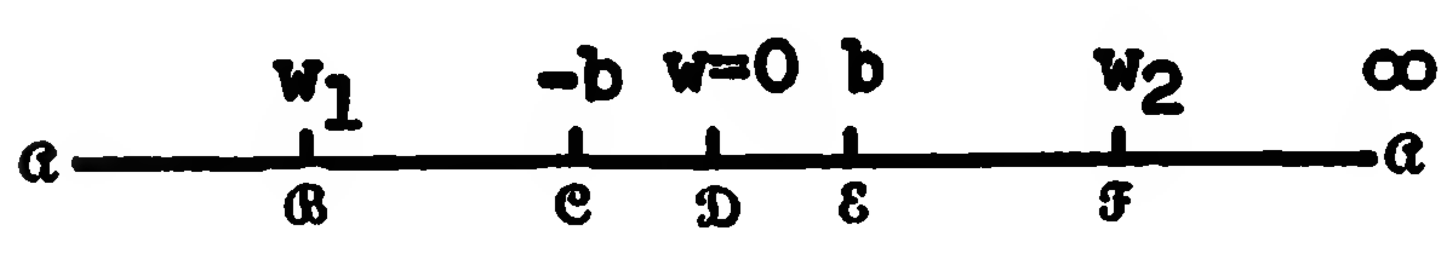
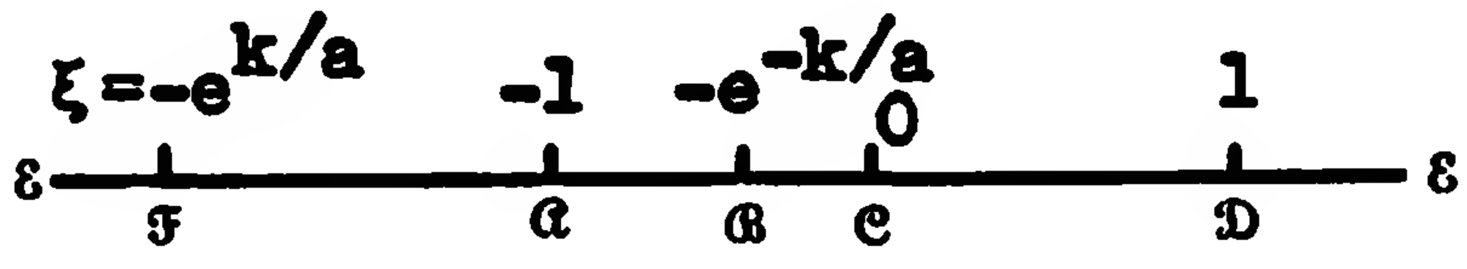
(1) $\quad z = w + a \log[(w+b)(w-b)^{-1}]$

is the transformation which maps the above region on $\Im(w) > 0$, as in §12.2 (second figure), p. 152.

Case (i) Now

(2) $$\xi = -\frac{w+b}{w-b}; \qquad w = b\,\frac{\xi-1}{\xi+1}$$

maps the half-plane $\Im(w) > 0$ on $\Im(\xi) > 0$.

w - plane	ξ - plane
points w_1, $-b$; 0; b; w_2; ∞	points $-e^{-k/a}$; 0; 1; ∞; $-e^{k/a}$; -1
	
line-segment $-b < w < b$	half-line $\xi > 0$
half-lines $b < w \leq \infty$ and $-\infty \leq w < -b$ together	half-line $\xi < 0$

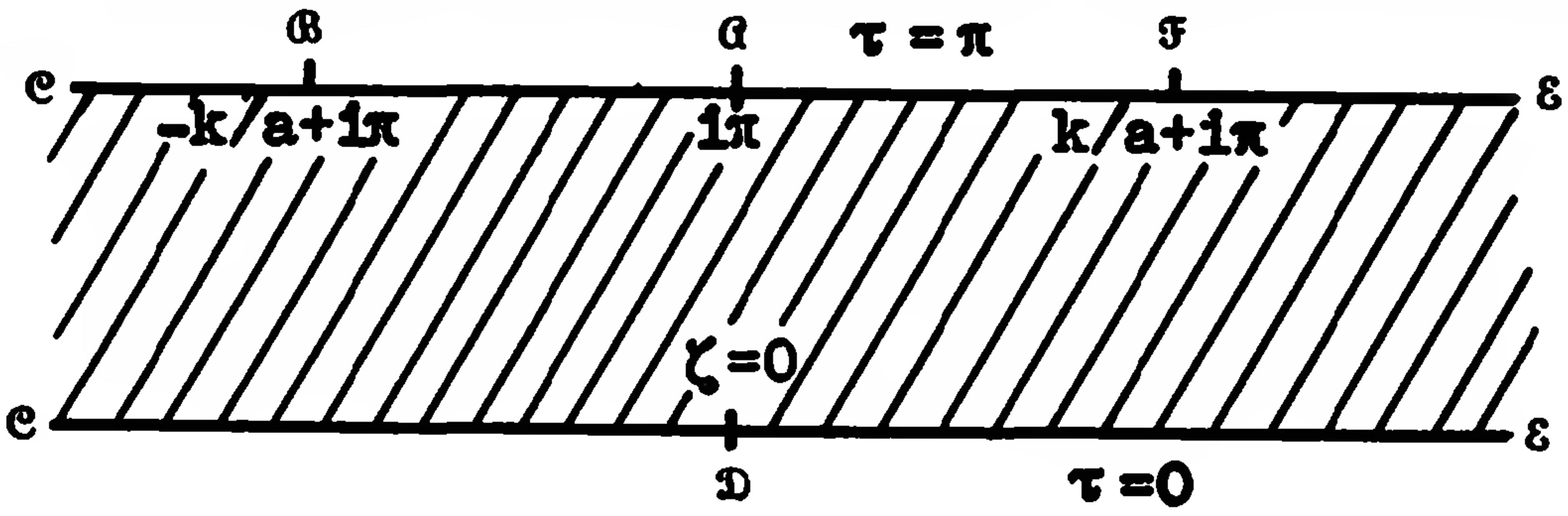
The transformation

$$(3) \quad \zeta = \log \xi$$

maps the half-plane $\Im(\xi) > 0$ on the strip $0 < \Im(\zeta) < \pi$, transforming the half-line $\xi > 0$ into the line $\Im(\zeta) = 0$, and the half-line $\xi < 0$ into the line $\Im(\zeta) = \pi$ (cf. §10.1). Combining (1), (2) and (3), we arrive at the transformation required (cf. §11.14):

$$\boxed{z = b \tanh \frac{\zeta}{2} + a\zeta - ai\pi.}$$

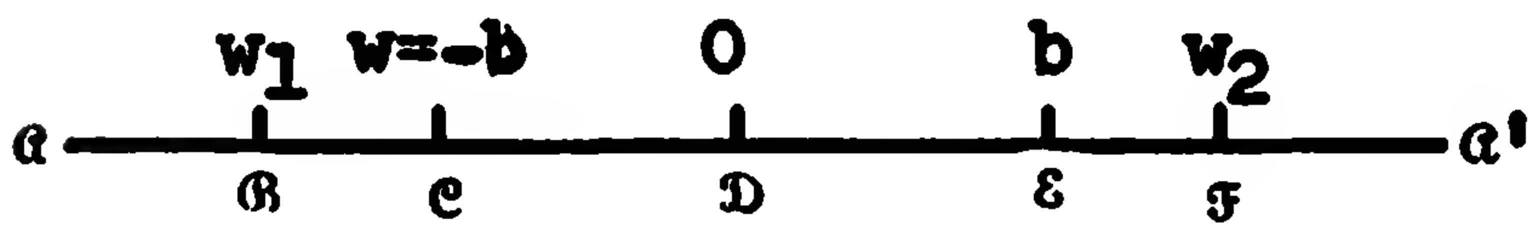
ζ - plane; $\zeta = \alpha + i\tau$



Case (ii) The linear transformation

$$(2') \qquad w = \xi + b$$

maps the half-plane $\Im(w) > 0$ on $\Im(\xi) > 0$, the half-line $\mathfrak{A}\mathfrak{B}\mathfrak{C}$ and the segment $\mathfrak{C}\mathfrak{D}\mathfrak{E}$ together on the half-line $\xi < 0$, and the half-line $\mathfrak{E}\mathfrak{F}\mathfrak{A}'$ on $\xi > 0$.



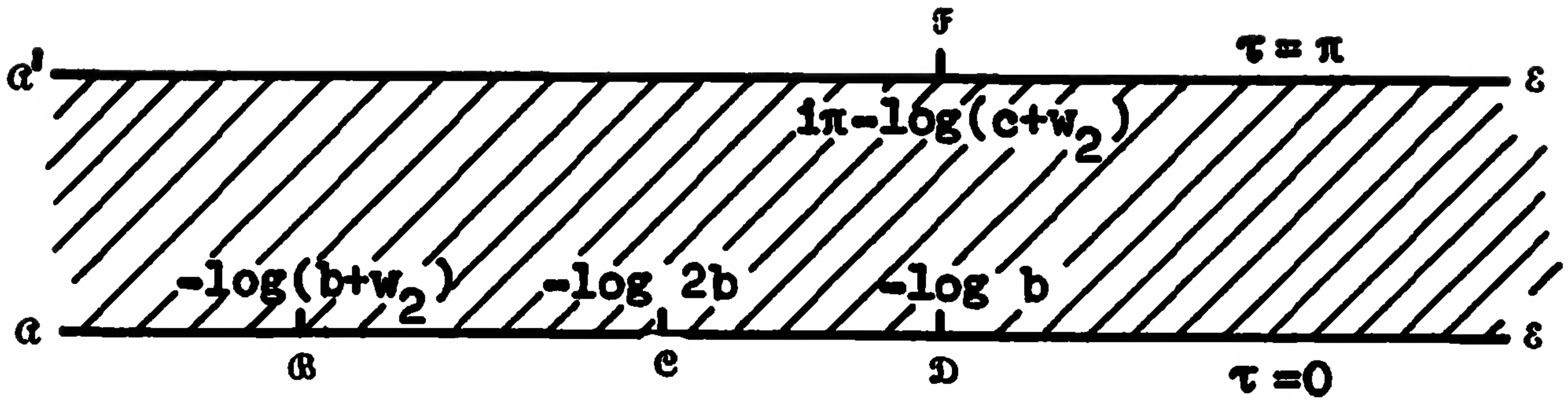
$$(3') \qquad \zeta = i\pi - \log \xi; \quad \xi = -e^{-\zeta}$$

maps $\Im(\xi) > 0$ on the strip $0 < \Im(\zeta) < \pi$, the half-lines $\xi > 0$ or $\xi < 0$, respectively, on the lines $\Im(\zeta) = \pi$ or $\Im(\zeta) = 0$ (cf. §10.1).

Combining (1), (2') and (3'), we arrive at the transformation required

$$\boxed{z = b - e^{-\zeta} + a \log\{1 - 2be^{\zeta}\}}$$

ζ - plane; $\zeta = \sigma + i\tau$

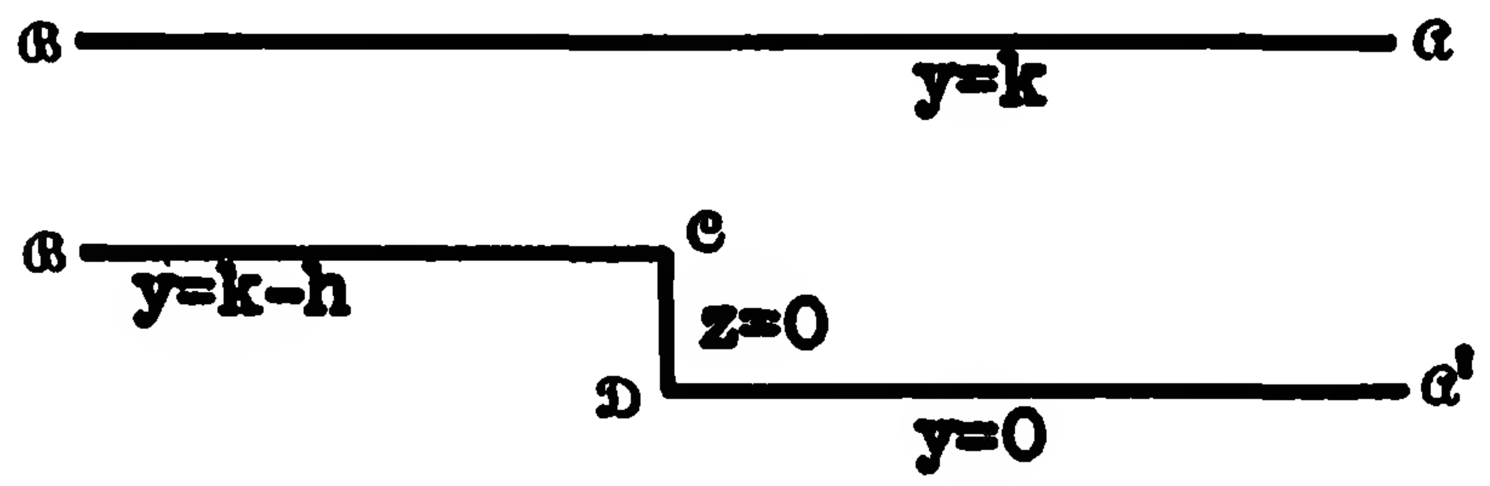


(B) <u>Example for reflection</u>: The transformation (see §12.8, p. 161)

$$\frac{dz}{dw} = \frac{k(w-1)^{1/2}}{\pi w(w-c)^{1/2}}; \quad c > 1, \quad k > 0; \quad z = F(w)$$

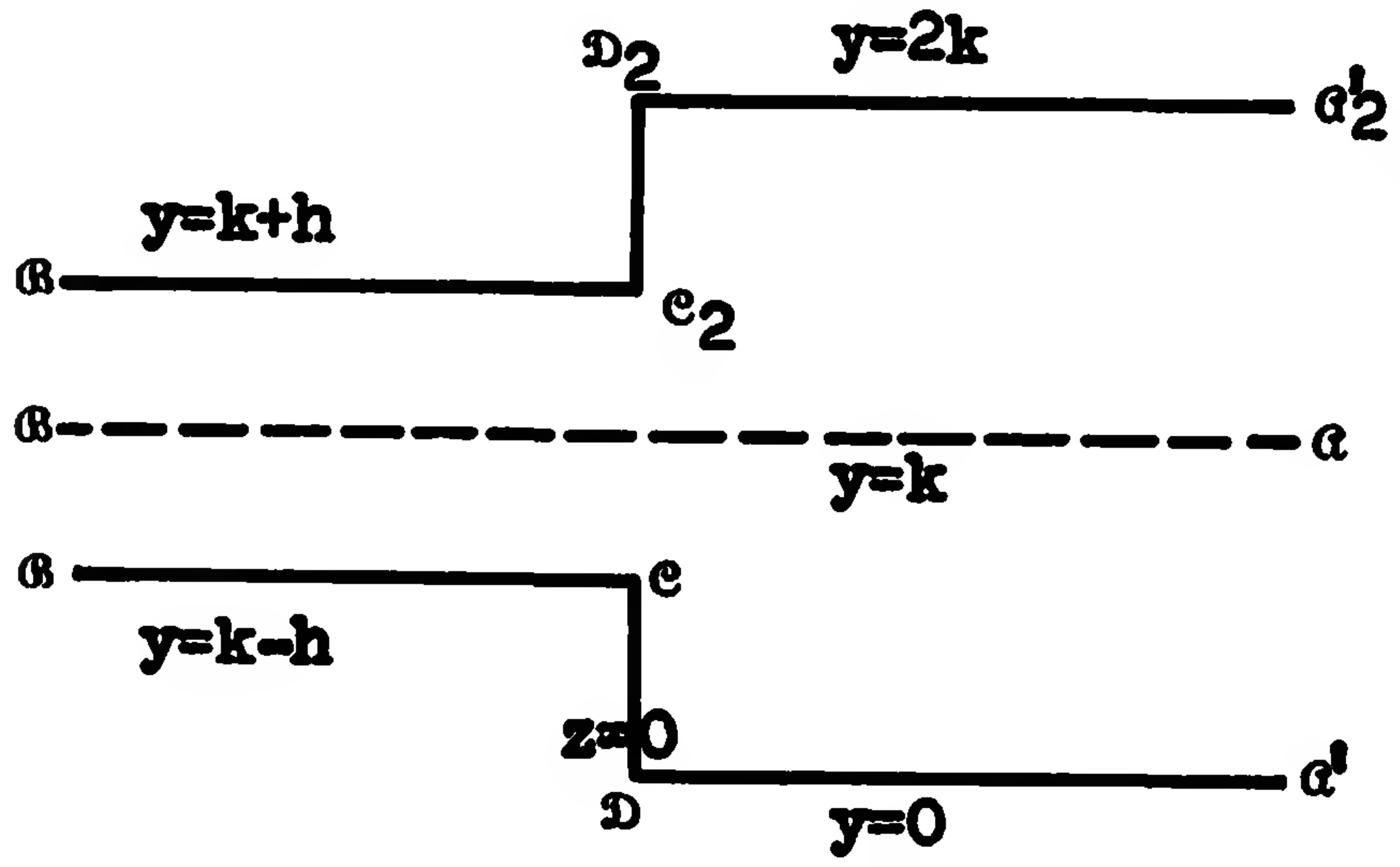
maps the domain $\mathcal{A}\,\mathcal{B}\,\mathcal{C}\,\mathcal{D}\,\mathcal{A}'$ on $v \geq 0$.

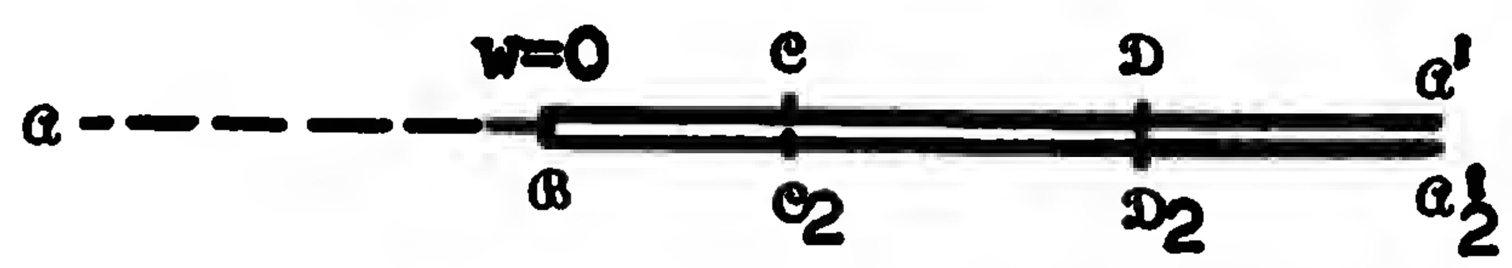
z - plane; $h = kc^{-1/2}$	w - plane



The same transformation, therefore, maps the region

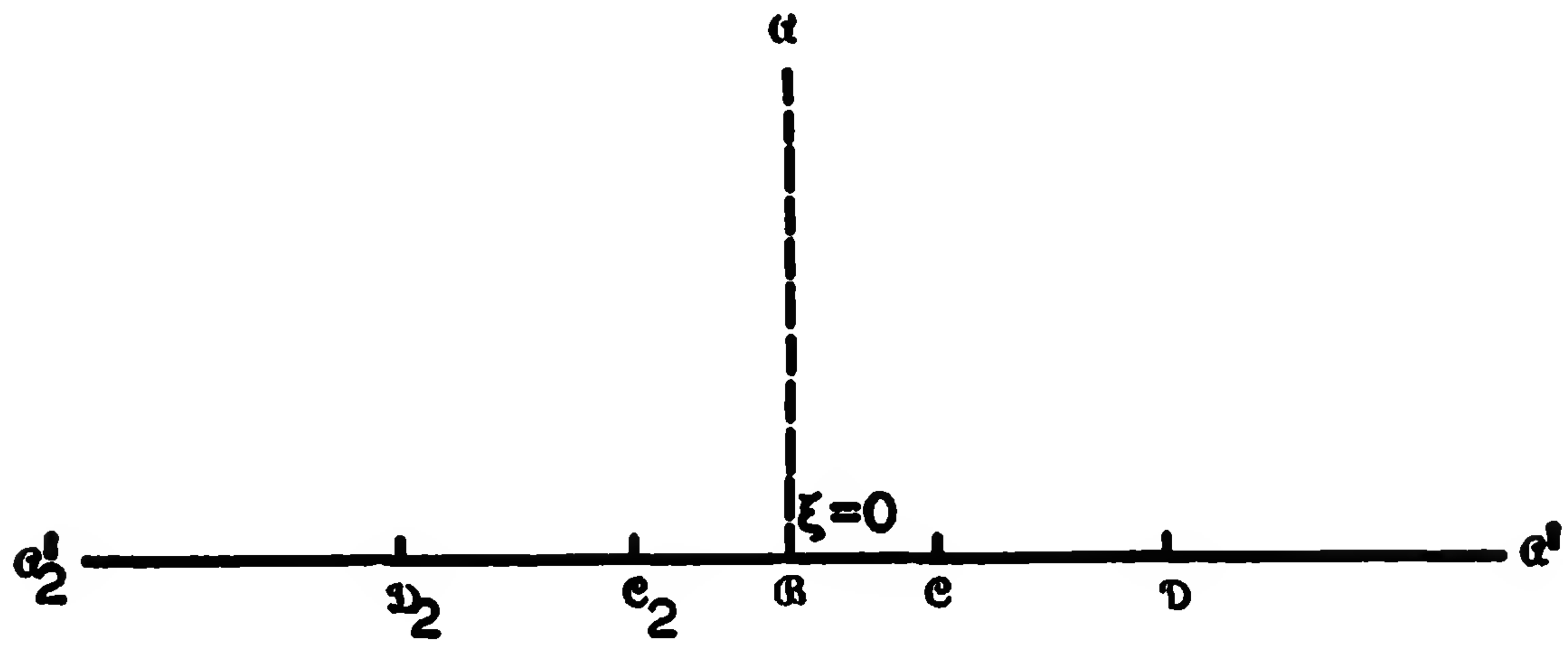
z - plane	w - plane





interior to $\mathfrak{a}_2'\mathfrak{d}_2\mathfrak{c}_2\mathfrak{B}\,\mathfrak{c}\,\mathfrak{d}\,\mathfrak{a}$ on the whole w - plane, cut from $\mathfrak{B}$ to $\mathfrak{a}'$. Now $w = \xi^2$ maps this cut w - plane on $\Im(\xi) > 0$. Combining $z = F(w)$ with $w = \xi^2$, we arrive at $z = F(\xi^2)$ which maps the domain $\mathfrak{a}_2'\mathfrak{d}_2\mathfrak{c}_2\mathfrak{B}\,\mathfrak{c}\,\mathfrak{d}\,\mathfrak{a}'$ of the z - plane on $\Im(\xi) \geq 0$.

ξ - plane



(9) <u>A converse of the method</u>. Given a function $w = f(z)$, it may be possible to represent it (i) in the Schwarz-Christoffel form, or (ii) as a combination of two Schwarz-Christoffel transformations. Thus information is gained about fundamental regions which are mapped on each other.

<u>Example</u> (i): $w = (1-z^2)^{1/2}$ (cf. §9.2) $dz/dw = -w(1-w)^{-1/2}(1+w)^{-1/2}$.

Take $a_1 = -1$, $a_2 = 0$, $a_3 = 1$, $a_4 = \infty$. $\alpha_1 = \alpha_3 = \pi/2$, $\alpha_2 = -\pi$, $\alpha_4 = 2\pi$.
Hence the interior of the polygon in the z-plane is a cut half-plane, as in the first figure on p. 76; except for its orientation. It is mapped on $v > 0$.

<u>Example</u> (ii): $w = z + c \coth z$.

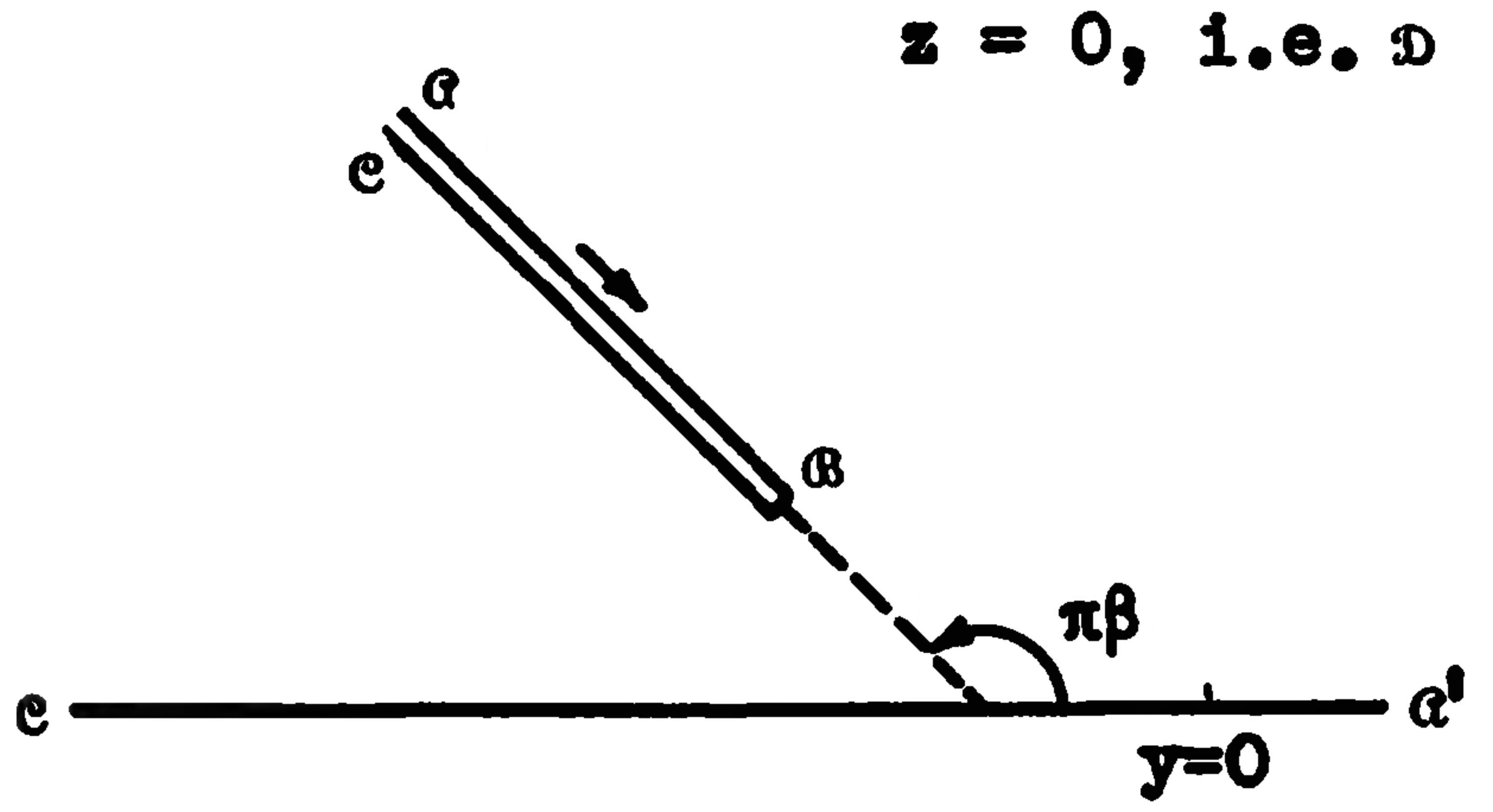
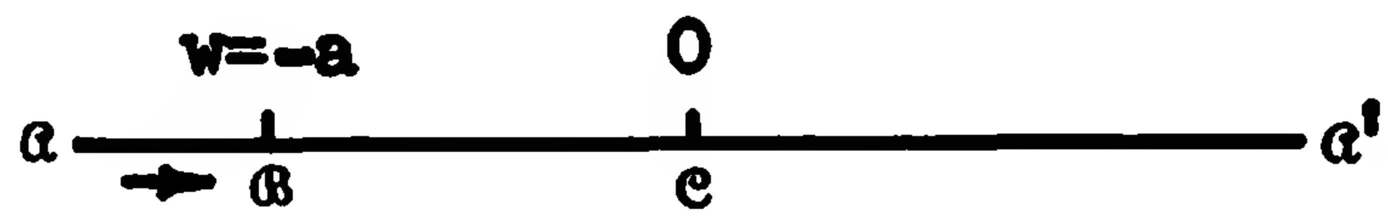
(a) $c > 0$, cf. p.140. This is a combination of the two Schwarz-Christoffel transformations $dz/d\zeta = (1-\zeta)^{-1}(1+\zeta)^{-1}$, i.e. $\zeta = \coth z$, and $dw/d\zeta = c(d-\zeta)(d+\zeta)(1-\zeta)^{-1}(1+\zeta)^{-1}$ where $d = (1+1/c)^{-1/2}$.

(b) $c < -1$. Combination of $dz/d\zeta = 2^{-1}\zeta^{-1/2}(1-\zeta)^{-1}$, i.e. $\zeta = \coth^2 z$, and $dw/d\zeta = 2^{-1}c(\zeta-k)\zeta^{-1/2}(\zeta-1)^{-1}$ where $k = 1+1/c$.
The sides of the polygon in the w-plane are $A_4A_1(u = 0,\ \infty \geq v \geq \pi/2)$, $A_1A_2(v = \pi/2,\ 0 \leq u \leq u_o)$, $A_2A_3(v = \pi/2,\ u_o \geq u \geq -\infty)$, where $u_o = \log(\sqrt{-c} - \sqrt{-c-1}) + (c^2+c)^{1/2}$, and $A_3A_4(v = 0,\ -\infty \leq u \leq \infty)$.
The polygon in the z-plane is a semi-strip, with vertices $w = \pi i/2$, $-\infty$, 0.

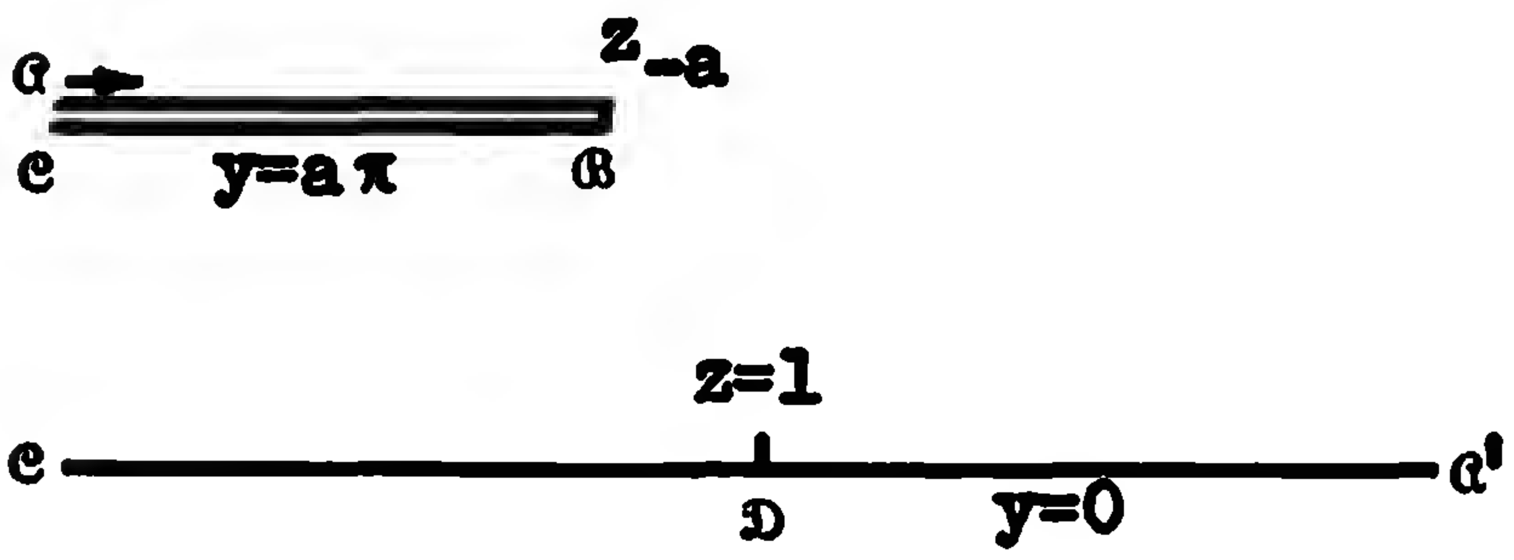
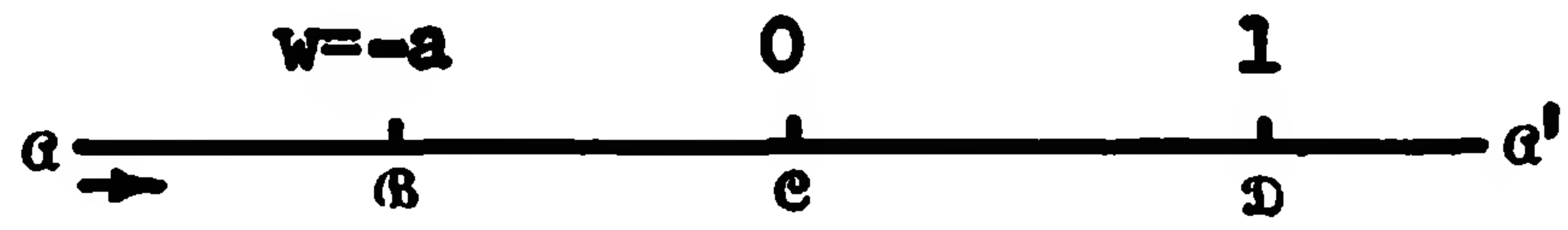
The cases (b') $c = -1$ and (b") $-1 < c < 0$ are treated similarly.

12.1 Half-plane, with one slit (cf. §9.3).†

$$\boxed{\frac{dz}{dw} = \frac{w+a}{w^{2-\beta}}} \;;\quad z = \frac{w^\beta}{\beta} - \frac{aw^{\beta-1}}{1-\beta} \qquad (a > 0,\ 0 < \beta < 1).$$

z - plane	w plane
points $z = a\beta e^{i\pi\beta}/(\beta-\beta^2)$, i.e. $\mathcal{B}$; $z = 0$, i.e. $\mathcal{D}$	point $w = -a$; point $w = a\beta/(1-\beta)$
	
half-plane $y > 0$, with the slit $\mathcal{A}\mathcal{B}\mathcal{C}$	half-plane $v > 0$

$$\boxed{\frac{dz}{dw} = \frac{w+a}{w}} \;,\quad a > 0; \qquad \boxed{z = w + a \log w}$$

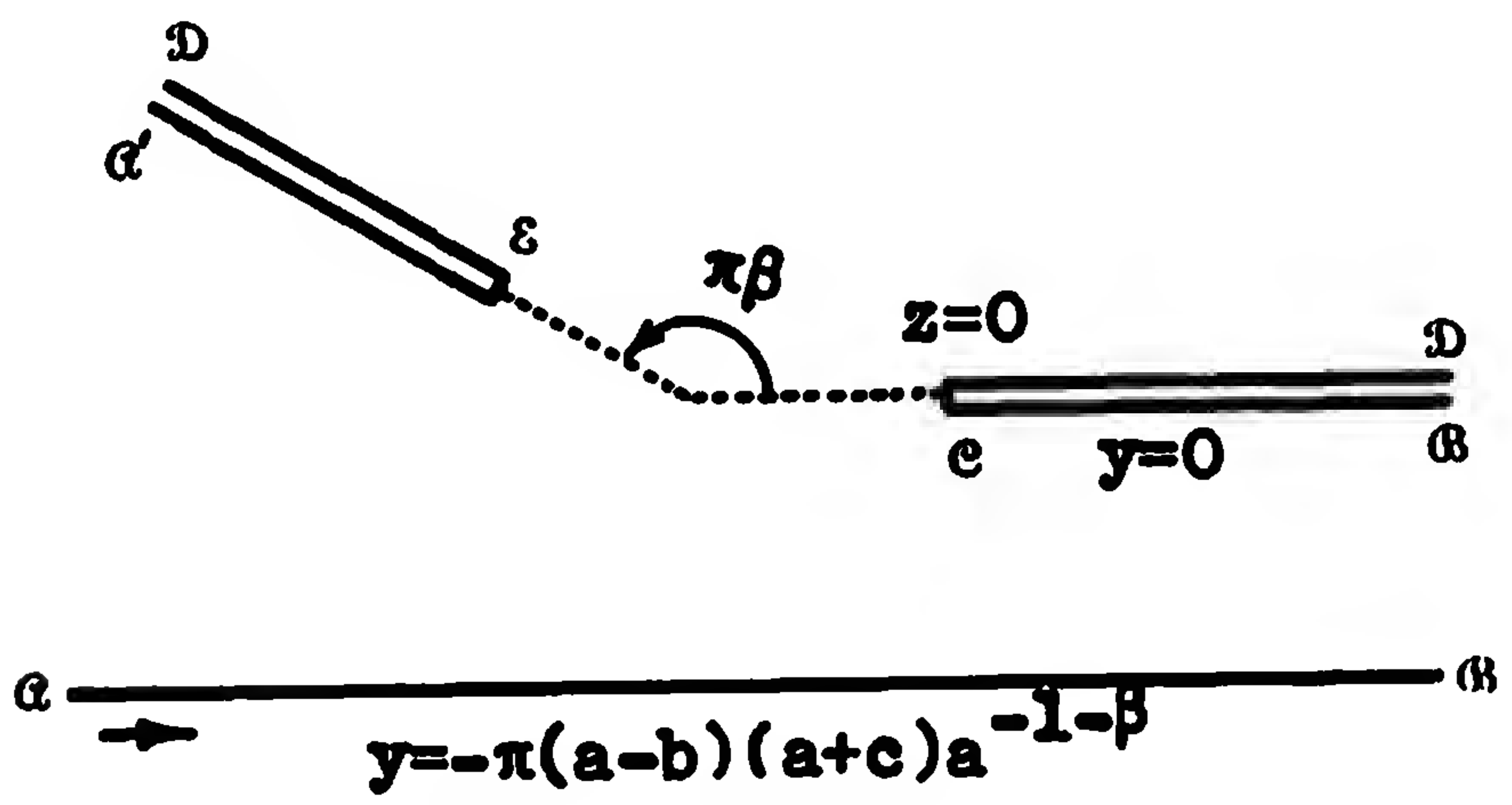
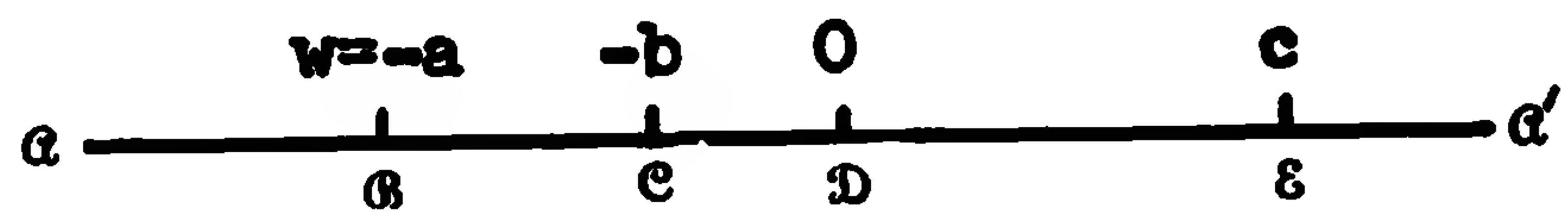
z - plane	w - plane
points $z = 1$; $a(\log a - 1 + i\pi) = z_{-a}$	points $w = 1$; $-a$.
half-plane $y > 0$, with one slit	half-plane $v > 0$.
	

$\boxed{z = w - a \log w + b}$, $a > 0$: combination of $w = -\xi$, $z = -\zeta+(b-a\pi i)$ and $\zeta = \xi + a \log \xi$; $a > 0$.

†Whole plane, with one (i) finite slit: p. 78, (ii) infinite slit: p. 35.

12.2 Half-plane, with two slits, (cf. p. 154, bottom).

$$\boxed{\frac{dz}{dw} = e^{i\pi\beta}\,\frac{(w+b)(w-c)}{w^{\beta+1}(w+a)}}\;;\quad 0 \le \beta \le 1;\quad a > b > 0,\quad c > 0.$$

z - plane	w - plane
 half-plane, with two slits	 half-plane $v > 0$

Representable in terms of elementary functions, if $\beta = p/q$ (p, q positive integers, also $p = 0$, i.e. parallel slits, being admitted, $0 \le p \le q$); then

$$z = q e^{i\pi\beta} \int_{(-b)^{1/q}}^{w^{1/q}} \frac{(s^q+b)(s^q-c)}{s^{p+1}(s^q+a)}\, ds.$$

Example: $\pi\beta = \frac{1}{2}\pi$, i.e. $p = 1$, $q = 2$:

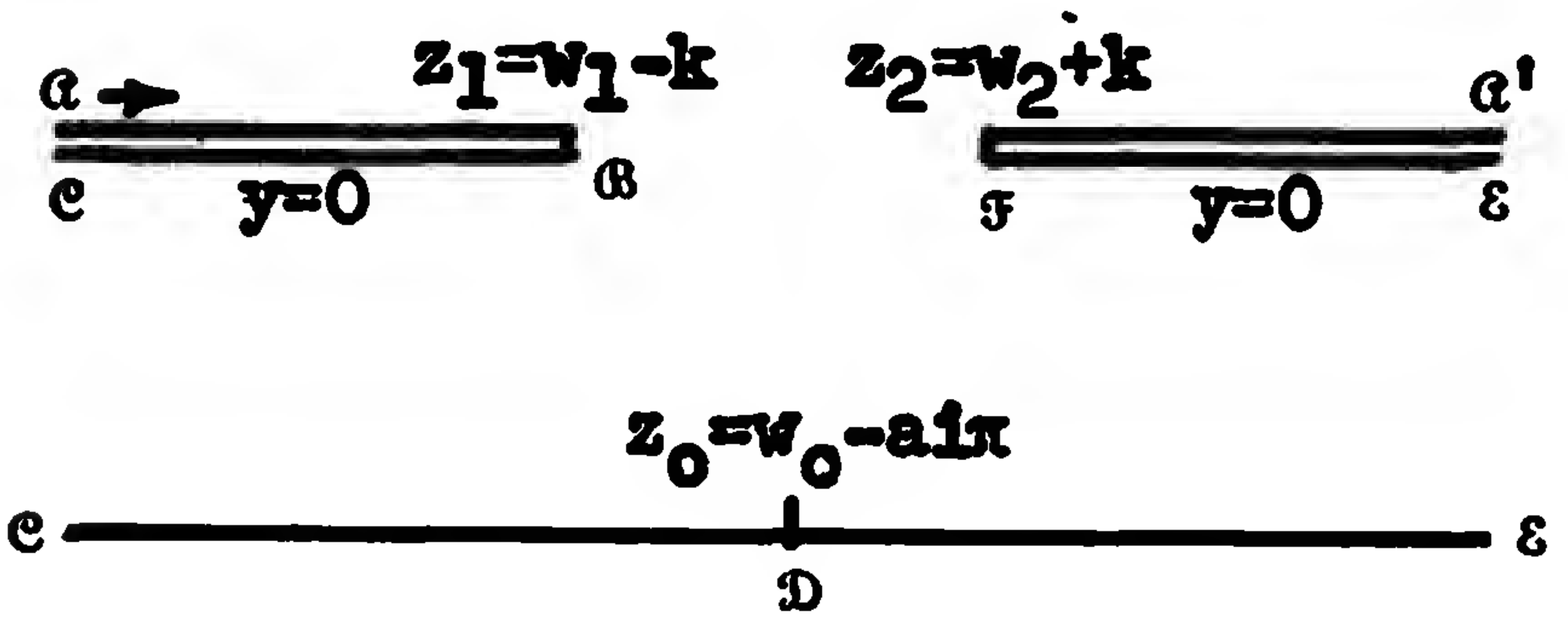
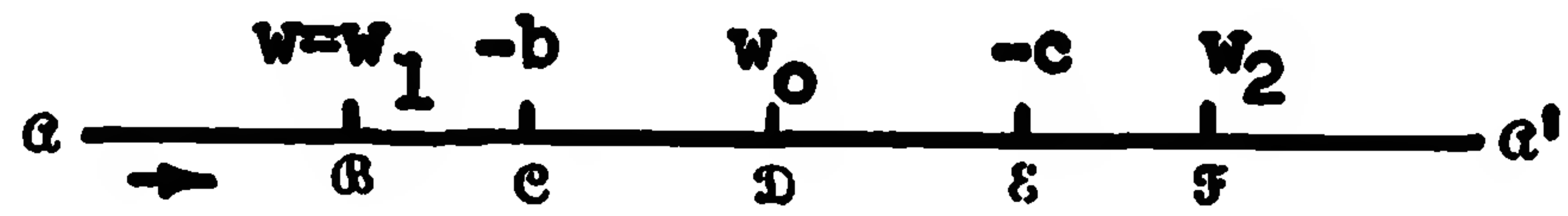
$$z = 2i\left(\sqrt{w} + \frac{bc}{a\sqrt{w}} - \frac{(a-b)(a+c)}{a\sqrt{a}} \tan^{-1}\left(\frac{w}{a}\right)^{1/2}\right) - k,$$

where $$k = \frac{2\sqrt{bc}}{a} - 2\sqrt{b} + \frac{2(a-b)(a+c)}{a\sqrt{a}} \tanh^{-1}\left(\frac{b}{a}\right)^{1/2}.$$

$$\boxed{\frac{dz}{dw} = \frac{(w-w_1)(w-w_2)}{(w+b)(w+c)}}\;;\qquad \boxed{z = w + a \log \frac{w+b}{w+c}}\;,$$

where $a > 0$; $b > c$ (a, b, c real); w_1, w_2 roots of $(w+b)(w+c) = a(b-c)$, they are real, $w_1+w_2 = 2w_0$, $w_1 < -b < -\frac{1}{2}(b+c) = w_0 < -c < w_2$;

$$k = a \log \frac{b+w_2}{c+w_2} = -a \log \frac{b+w_1}{c+w_1} > 0.$$

<table>
<tr><th>z - plane</th><th>w - plane</th></tr>
<tr><td>points $z = w_1 - k;\ w_o - a\pi i;\ w_2 + k$.
Half-plane $y > -a\pi$, with two slits

</td><td>points $w = w_1;\ w_o;\ w_2$.
Half-plane $v > 0$.

</td></tr>
</table>

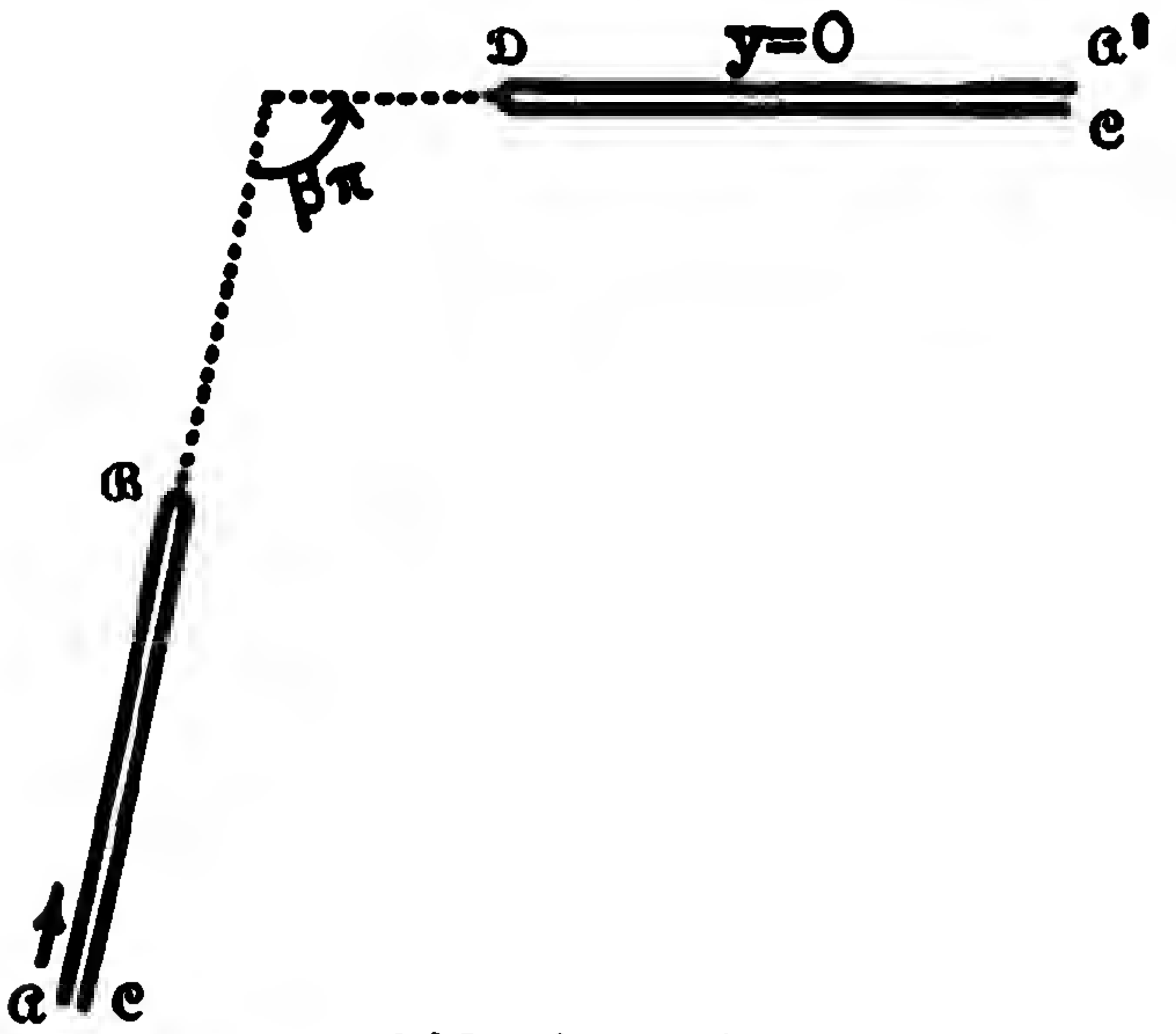
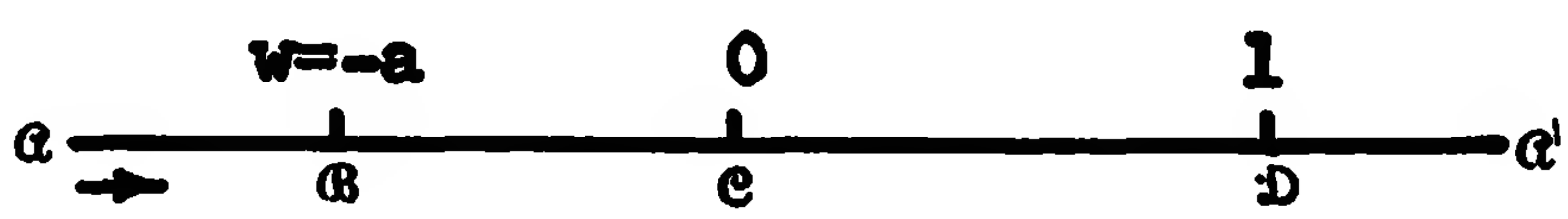
Combination of this transformation and of $\xi^2 = (w+c)/(w+b)$:

Whole plane, with four slits  on $\Im(\xi) > 0$.

12.3 <u>Whole plane, with two slits</u> (cf. §8.1, §9.3, §9.4).

$$\boxed{\frac{dz}{dw} = \frac{(w+a)(w-1)}{w^{\beta+1}}}\ , \quad 0 < \beta < 1;\ a > 0.$$

$$z = \frac{w^{2-\beta}}{2-\beta} + \frac{a-1}{1-\beta}\, w^{1-\beta} + \frac{a}{\beta}\, w^{-\beta} \ \text{(cf. §6.2, end).}$$

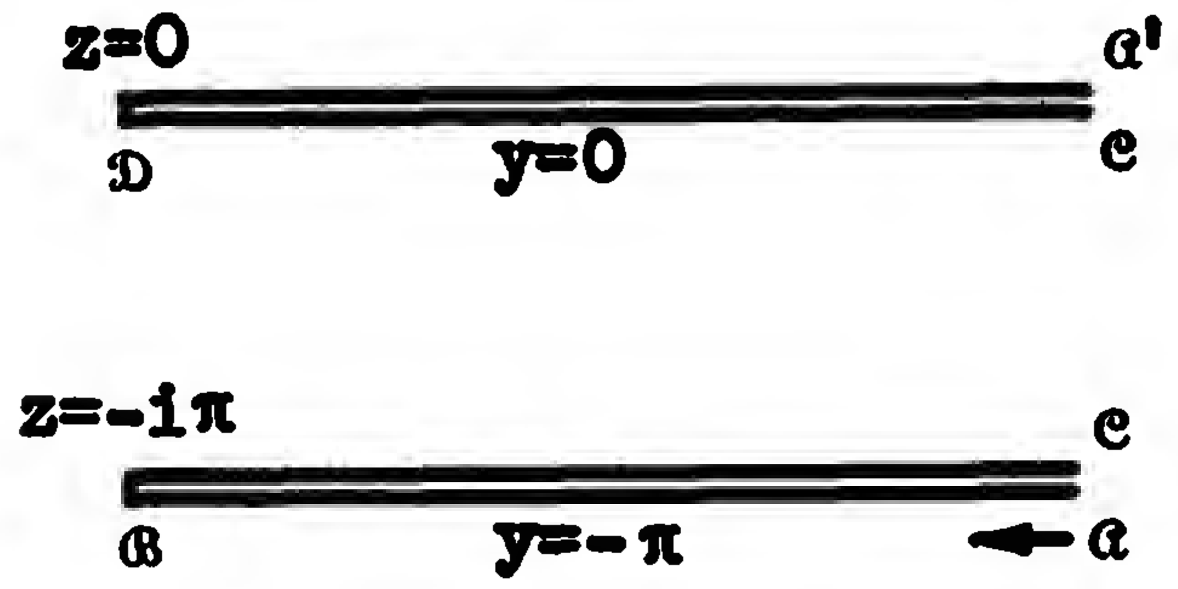
<table>
<tr><th>z - plane</th><th>w - plane</th></tr>
<tr><td>point $z = \left\{\dfrac{a^{1-\beta}}{\beta(1-\beta)} - \dfrac{a^{2-\beta}}{(1-\beta)(2-\beta)}\right\} e^{-i\pi\beta}$;</td><td>point $w = -a$;</td></tr>
<tr><td>point $z = \dfrac{1}{2-\beta} - \dfrac{1}{1-\beta} + \dfrac{a}{\beta(1-\beta)}$

</td><td>point $w = 1$

</td></tr>
<tr><td>whole plane, with the two slits</td><td>half-plane $v > 0$</td></tr>
</table>

$$\boxed{\frac{dz}{dw} = \frac{w^2-1}{w}} \quad ; \quad z = \frac{w^2-1}{2} - \log w.$$

z - plane

points $z = -i\pi$; 0

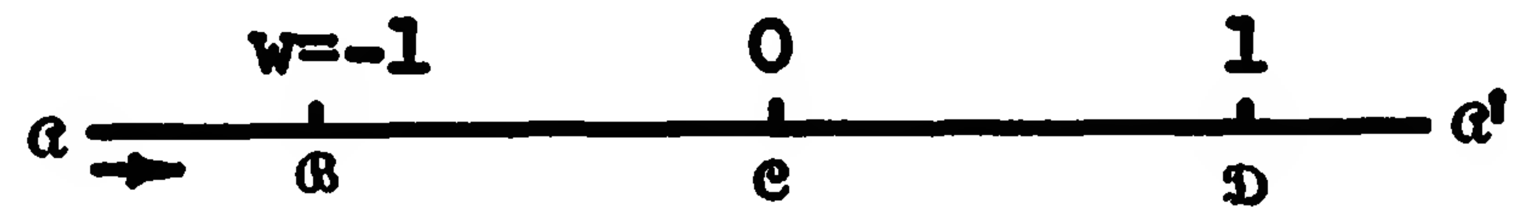
whole plane, with two slits



w - plane

points $w = -1$; 1

half-plane $v > 0$



$$\boxed{z = w + \frac{b}{w} + 2c \log w} \; ; \; b, c, \text{ real}, b > 0.^{‡}$$

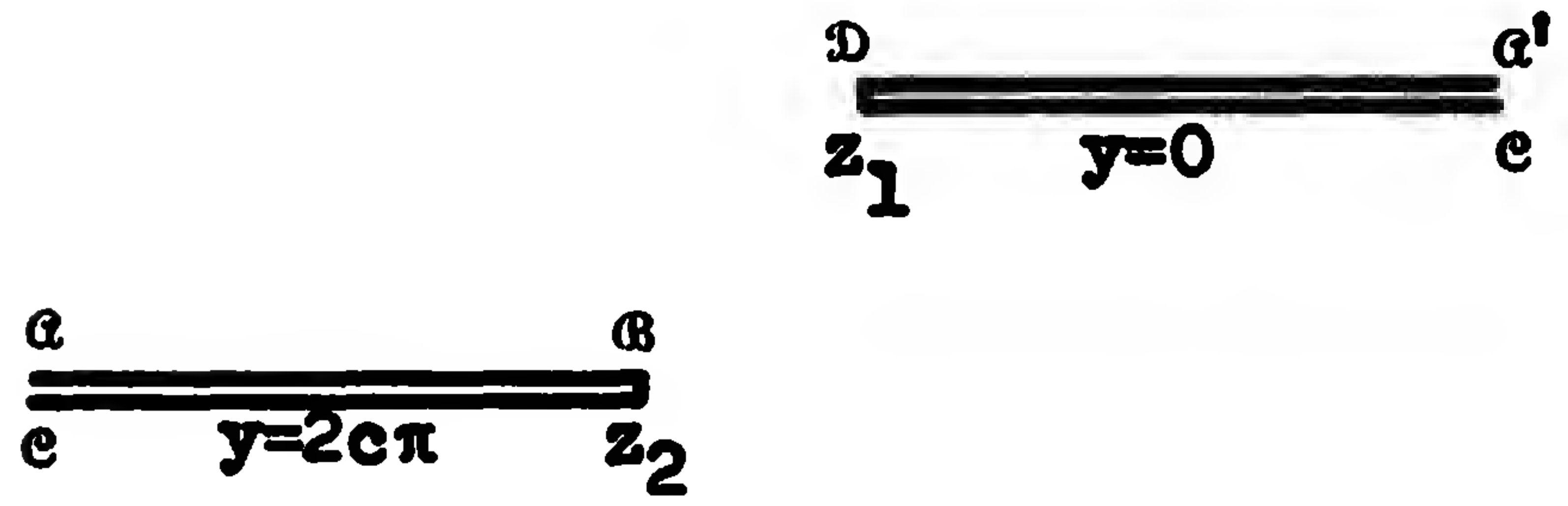
$$\frac{dz}{dw} = \frac{(w-w_1)(w-w_2)}{w^2}, \text{ where } w_{\frac{1}{2}} = -c \pm \sqrt{(b+c^2)}; \; w_2 < 0 < w_1.$$

In the figure, $c < 0$.

z - plane

point $z_1 = w_1 + \frac{b}{w_1} + 2c \log w_1$;

point $z_2 = w_2 + \frac{b}{w_2} + 2c \log|w_2|$
$+ 2ic\pi$

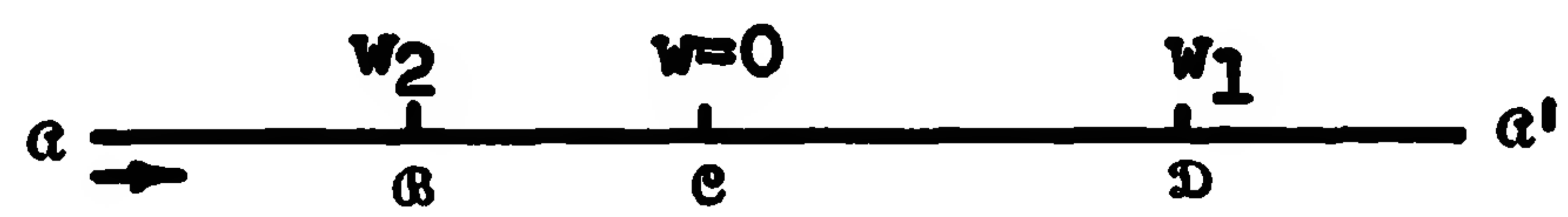


Whole plane, with the two slits

w - plane

point w_1;

point w_2



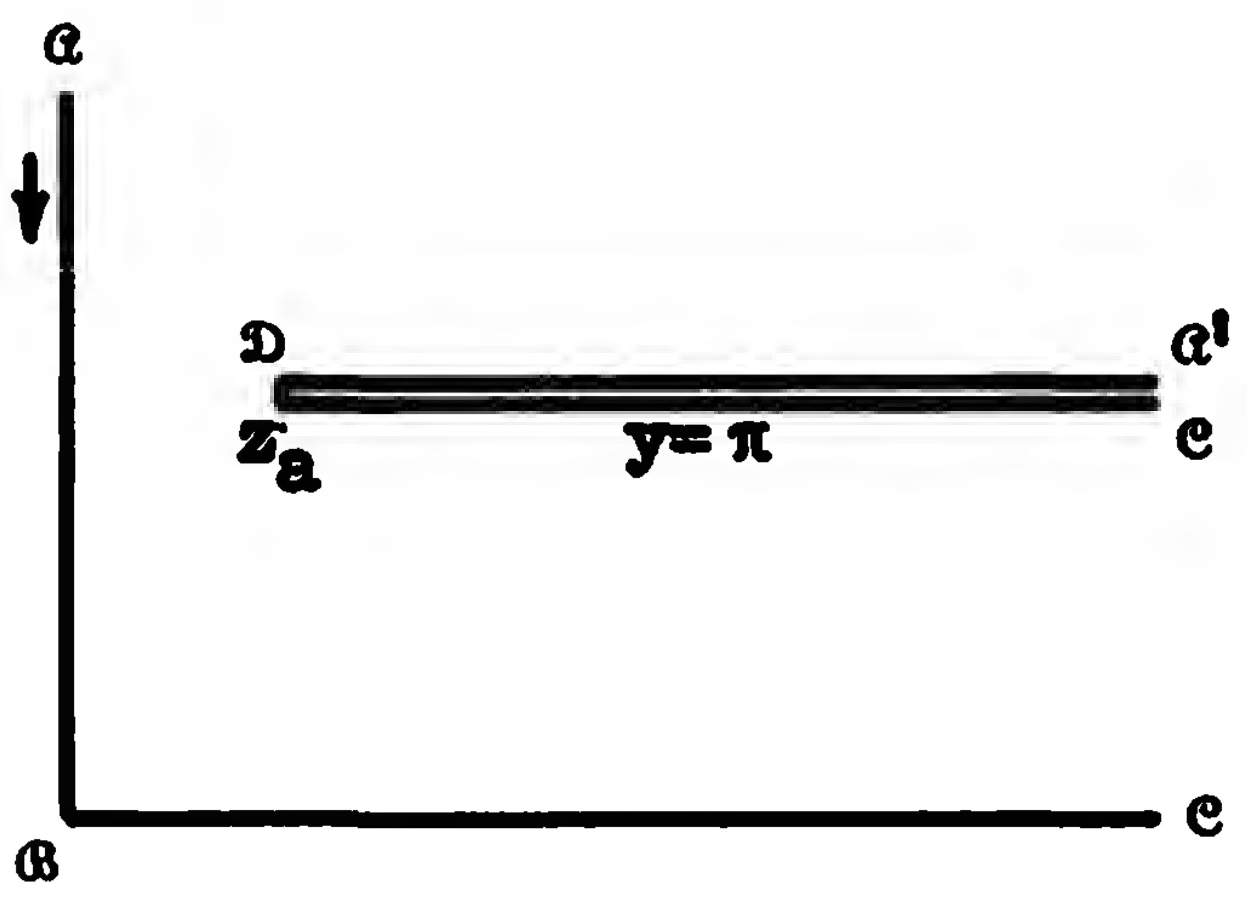
Half-plane $v > 0$

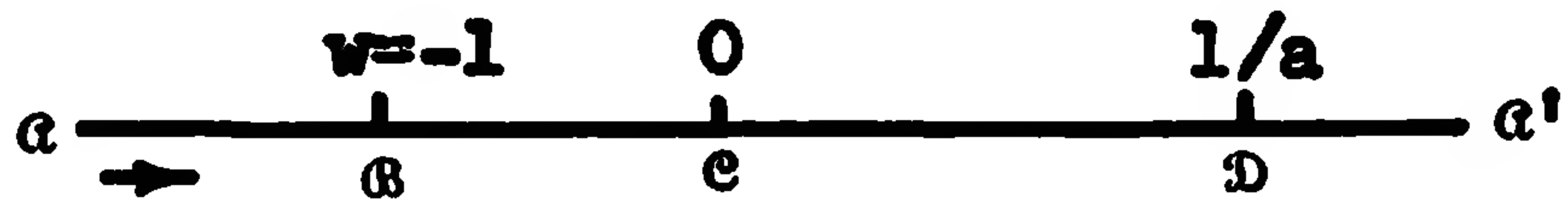
‡ Cf. E. Kehren. Contrast the transformation with that in §11.13.

12.4 Quadrant, with one slit.

$$\boxed{\frac{dz}{dw} = \frac{aw - 1}{w\sqrt{(w+1)}}}, \quad a > 0; \; z = 2a\sqrt{(w+1)} + 2\tanh^{-1}\sqrt{(w+1)}$$
$$= 2a\sqrt{(w+1)} + \cosh^{-1}\left(-\frac{2}{w} - 1\right)$$

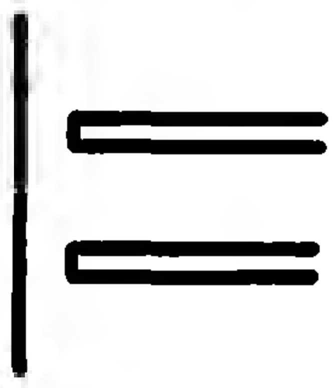
z - plane	w - plane
points $z = 0$; $z_a = 2\sqrt{(a^2+a)} + 2\log\left\{\sqrt{(a+1)} + \sqrt{a}\right\} + i\pi$	points $w = -1$; $1/a$
Quadrant $x > 0$, $y > 0$, with slit	half-plane $v > 0$.





Combining this transformation with $w = \frac{(\xi-1)^2}{4\xi}$;

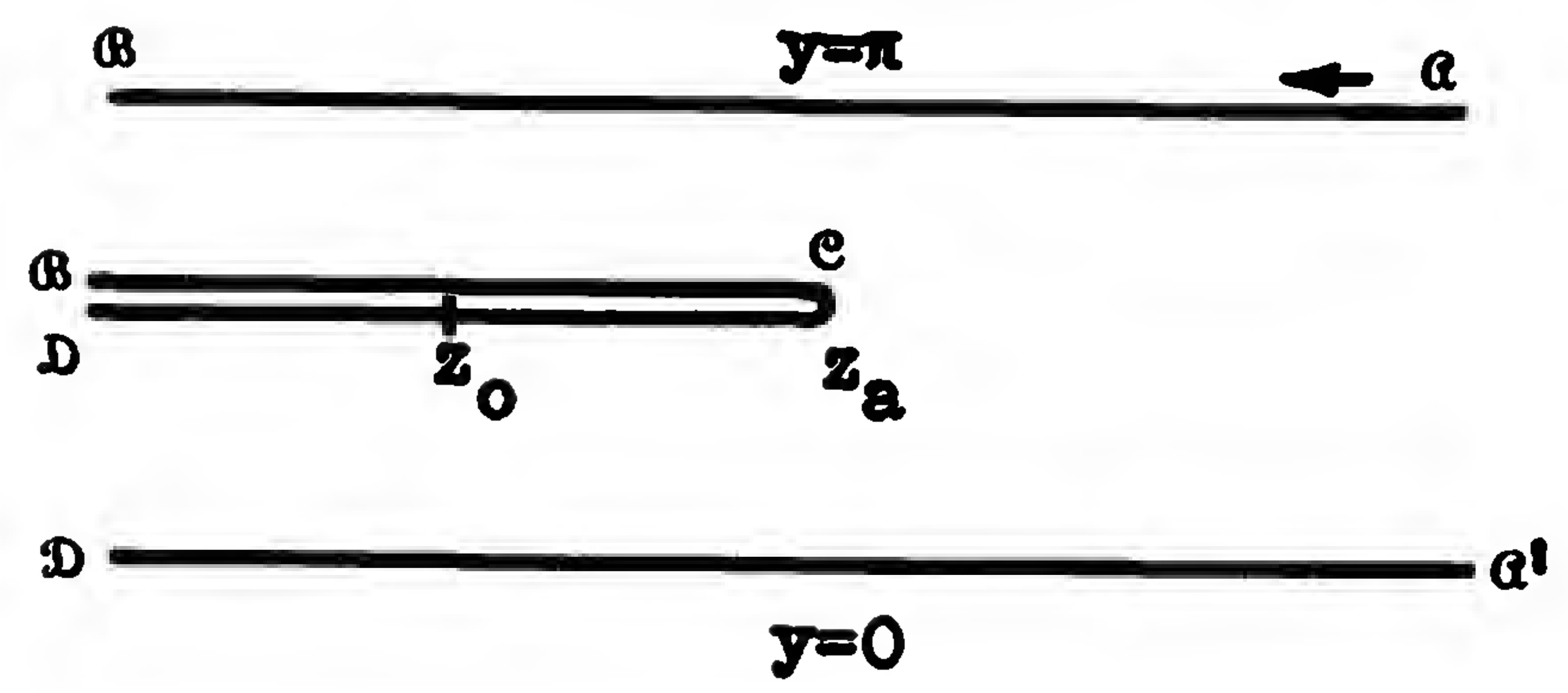
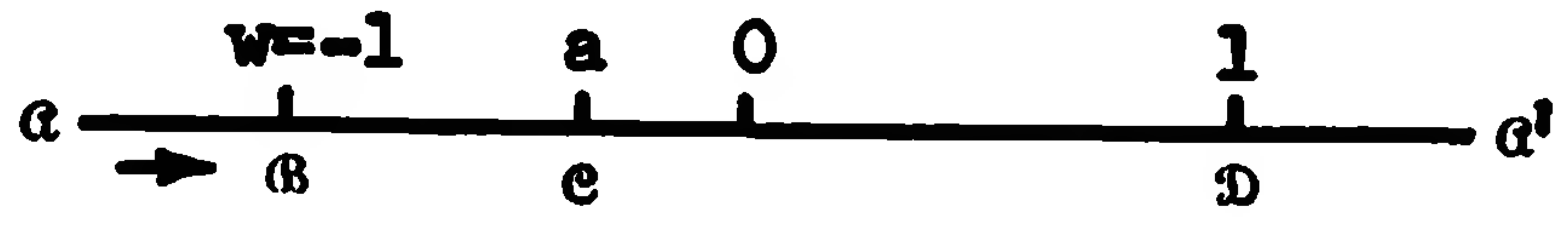
$$\boxed{z = \frac{a}{\sqrt{\xi}}(\xi+1) + 2\tanh^{-1}\frac{\xi+1}{2\sqrt{\xi}}}$$

z - plane	ξ - plane
half-plane $x > 0$, with two horizontal slits 	$\Im(\xi) > 0$

12.5 Cut strip

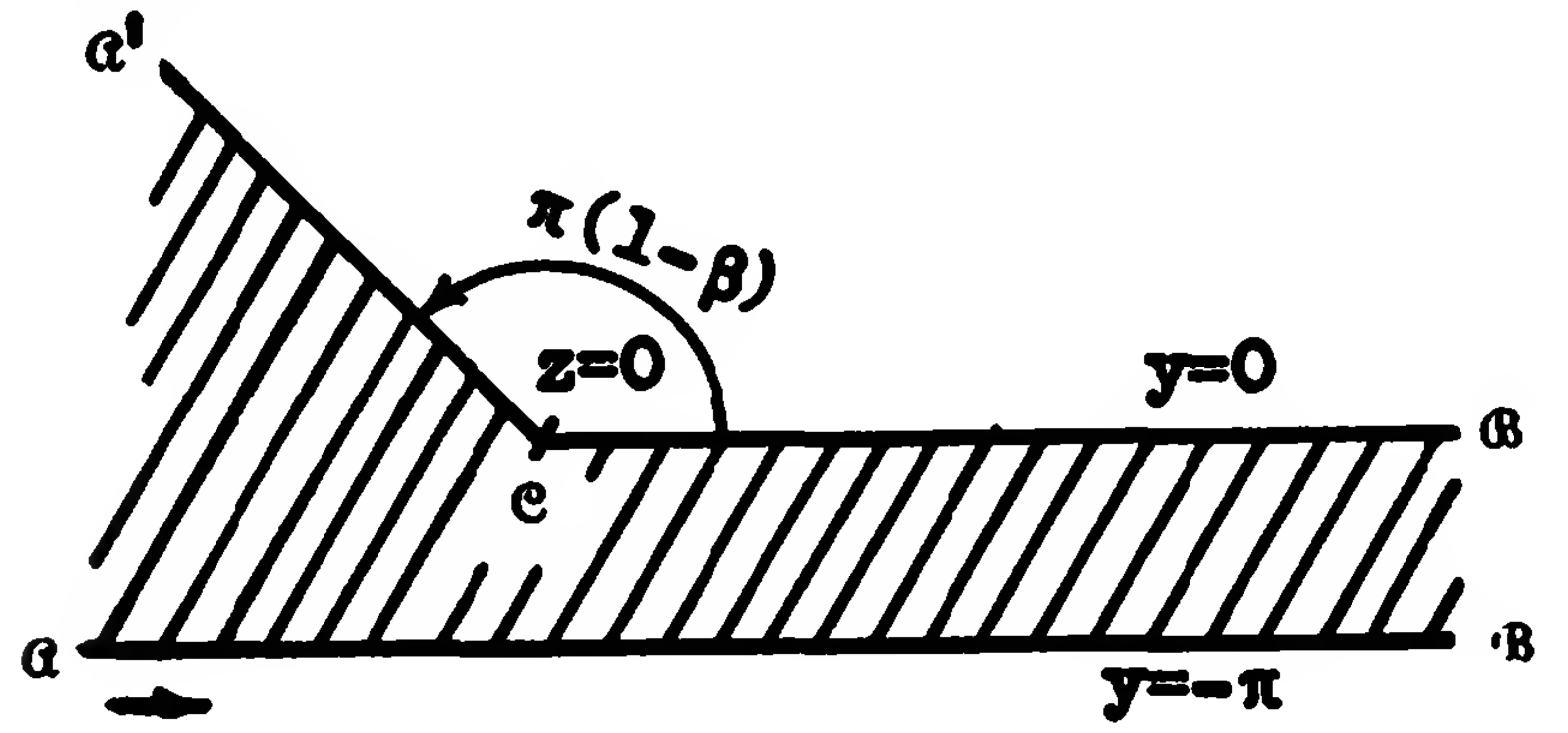
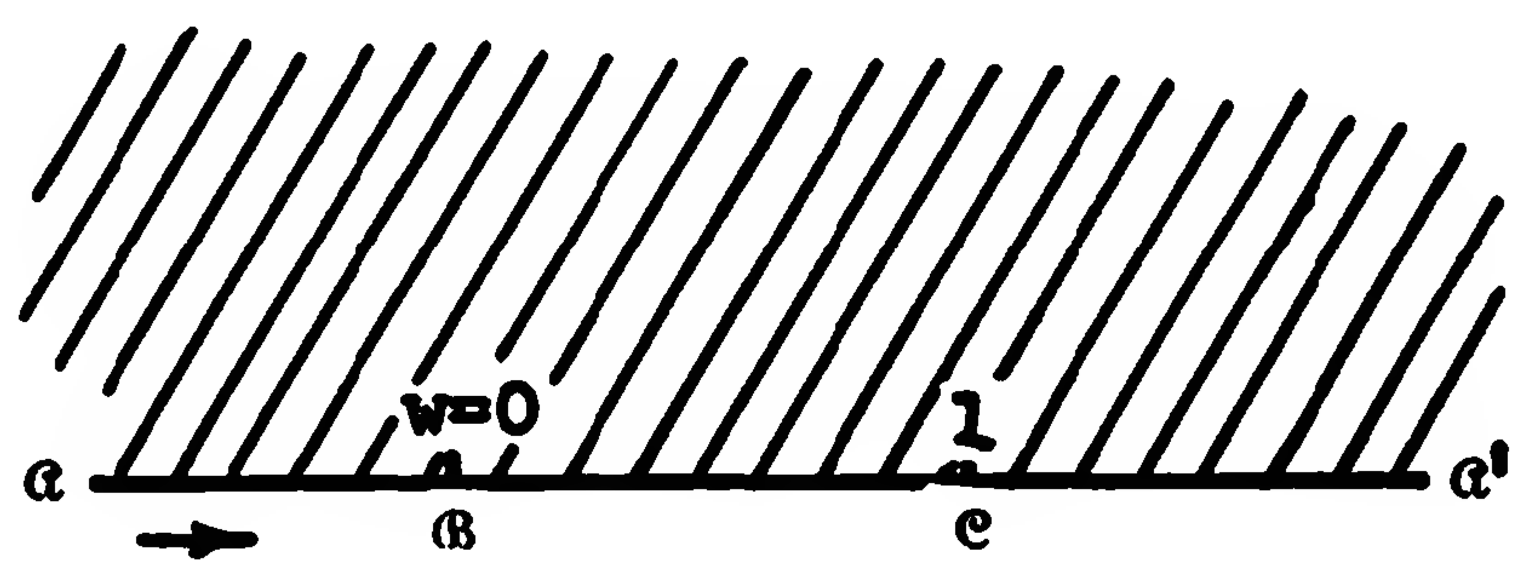
$$\boxed{\frac{dz}{dw} = \frac{w-a}{w^2-1}}\;, \quad -1 < a < 1 \quad (\text{for } a = 0 \text{ cf. } §10.4).$$

$$z = \frac{1-a}{2}\log(w-1) + \frac{1+a}{2}\log(w+1)$$

z - plane	w - plane
points $z_o = \frac{i\pi}{2}(1-a);\ \frac{i\pi}{2}(1-a) + \frac{1-a}{2}\log(1-a) + \frac{1+a}{2}\log(1+a) = z_a$	points $w = 0;\ a$
strip $0 < y < \pi$, with slit	half-plane $v > 0$
	

12.6 Regions bounded by the arms of two or more angles.

$$\frac{dz}{dw} = -\frac{(1-w)^{\beta}}{w}\;, \quad 0 < \beta < 1; \quad z = \int_w^1 \frac{ds(1-s)^{\beta}}{s}\;.$$

z - plane	w - plane
point $z = 0$	w - plane
	

Representable in terms of elementary functions if $\beta = p/q$, where $0 < p < q$, p and q integers: $z = q\int_0^{(1-w)^{1/q}} \frac{t^{p+q-1}\,dt}{1-t^q}$.

Example (i) $\beta = \frac{1}{3}$. $z = -3(1-w)^{1/3} - \frac{3}{2}\log\left\{1-(1-w)^{1/3}\right\}$

$$+ \frac{1}{2}\log w + \sqrt{3}\tan^{-1}\frac{\sqrt{3}}{1+2(1-w)^{-1/3}}.$$

Example (ii) $\beta = \frac{1}{2}$. $\boxed{\frac{dz}{dw} = -\frac{h}{\pi}\frac{(1-w)^{1/2}}{w}}$;

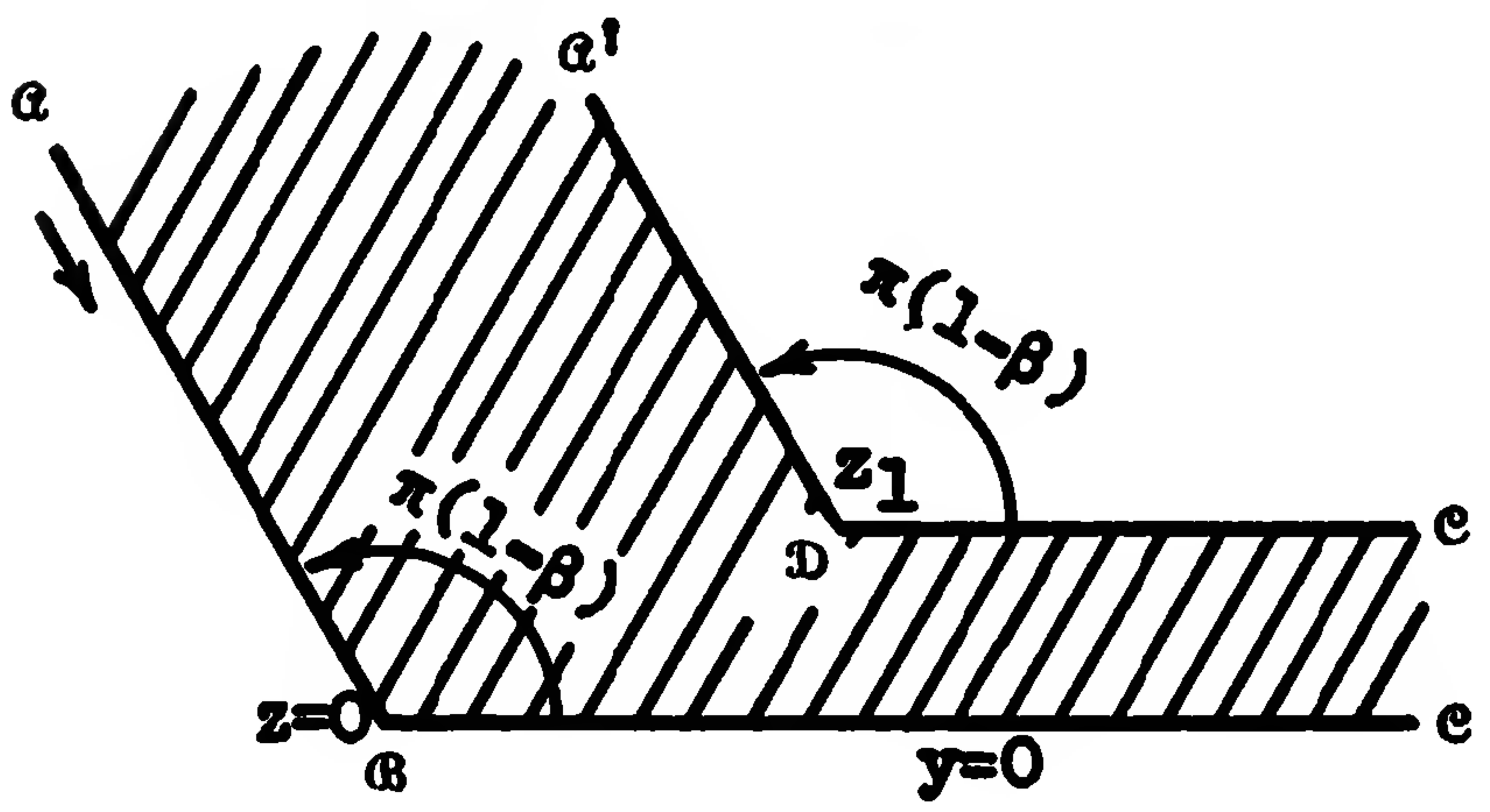
$$z = \frac{2h}{\pi}\left\{\tanh^{-1}\sqrt{1-w} - \sqrt{1-w}\right\},$$

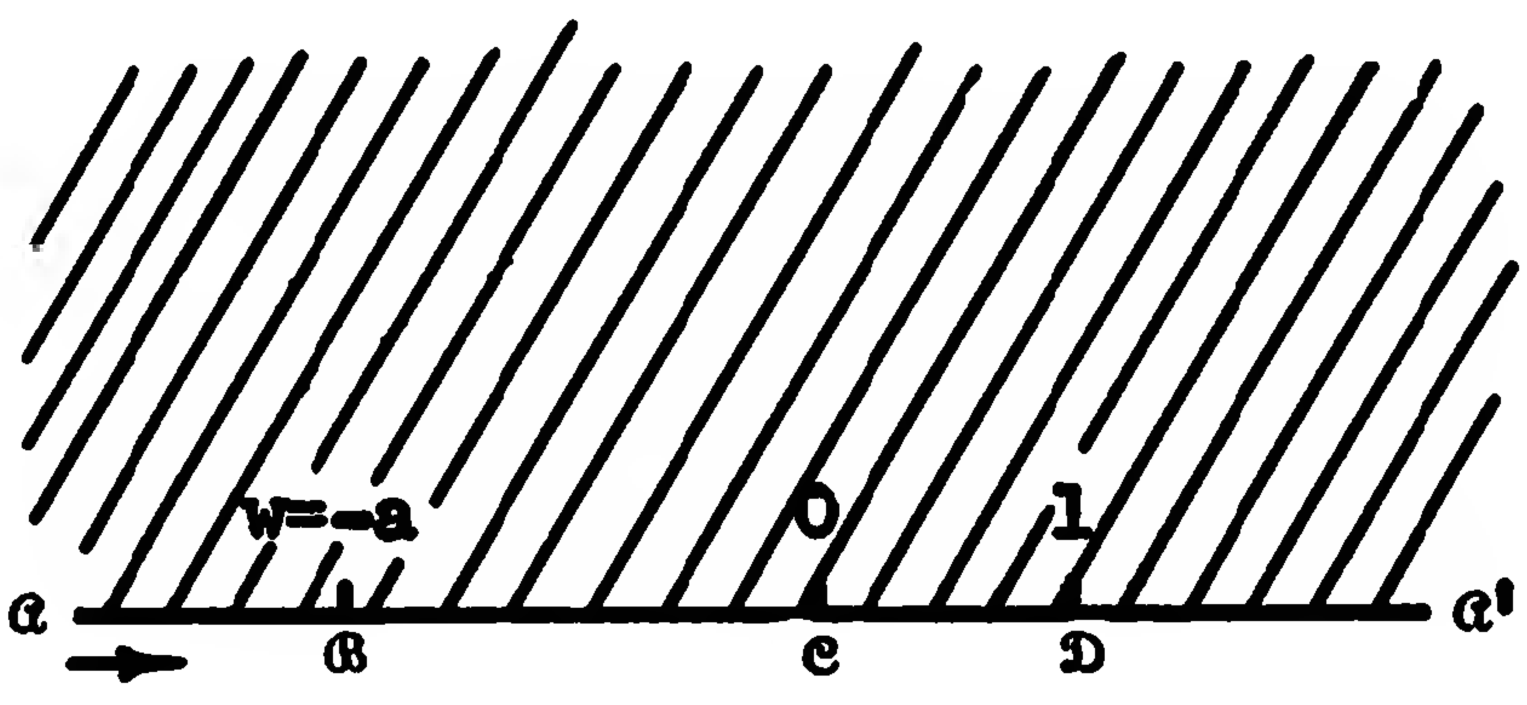
$$= \frac{h}{\pi}\left\{\cosh^{-1}\left(\frac{2}{w}-1\right) - 2\sqrt{1-w}\right\};\quad h > 0.$$

Figures as above; but on the left, take $\angle C = \frac{1}{2}\pi$, and $y = -h$ instead of $y = -\pi$.

$\boxed{\frac{dz}{dw} = -\frac{1}{w}\left(\frac{1-w}{a+w}\right)^\beta}$; $0 < \beta < 1$, $a > 0$; $z = -\int_{-a}^{w}\frac{ds}{s}\left(\frac{1-s}{a+s}\right)^\beta$.

z - plane	w - plane
points $z = 0$; $z_1 = h + ik = i\pi a^{-\beta} + \lim_{\varepsilon\to 0}\left\{\int_\varepsilon^a \frac{dt}{t}\left(\frac{1+t}{a-t}\right)^\beta - \int_\varepsilon^1 \frac{dt}{t}\left(\frac{1-t}{a+t}\right)^\beta\right\}$	points $w = -a$; $w_1 = 1$





Representable in terms of elementary functions if $\beta = p/q$, where $0 < p < q$, p and q integers:

$$z = q(a+1)\int_{\left(\frac{1-w}{a+w}\right)^{1/q}}^{\infty} \frac{dt\ t^{p+q-1}}{(-1+at^q)(1+t^q)} .$$

Example (i) $\beta = \frac{1}{3}$, $a = 1$; $z = \log \frac{2\sqrt{-w}}{w+1} - \frac{3}{2}\log\left\{-1+\left(\frac{1-w}{1+w}\right)^{2/3}\right\}$

$$+ \sqrt{3}\tan^{-1}\left\{\frac{\sqrt{3}}{1+2\left(\frac{1-w}{1+w}\right)^{2/3}}\right\}.$$

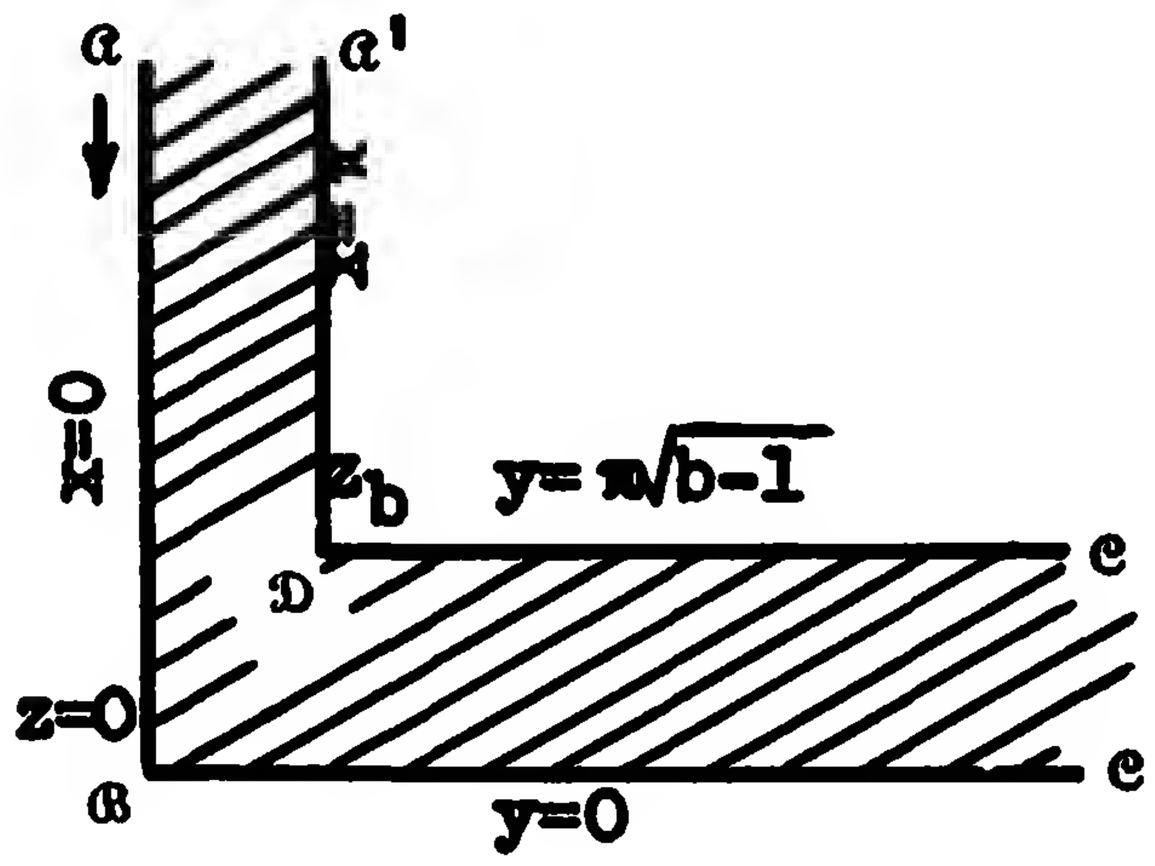
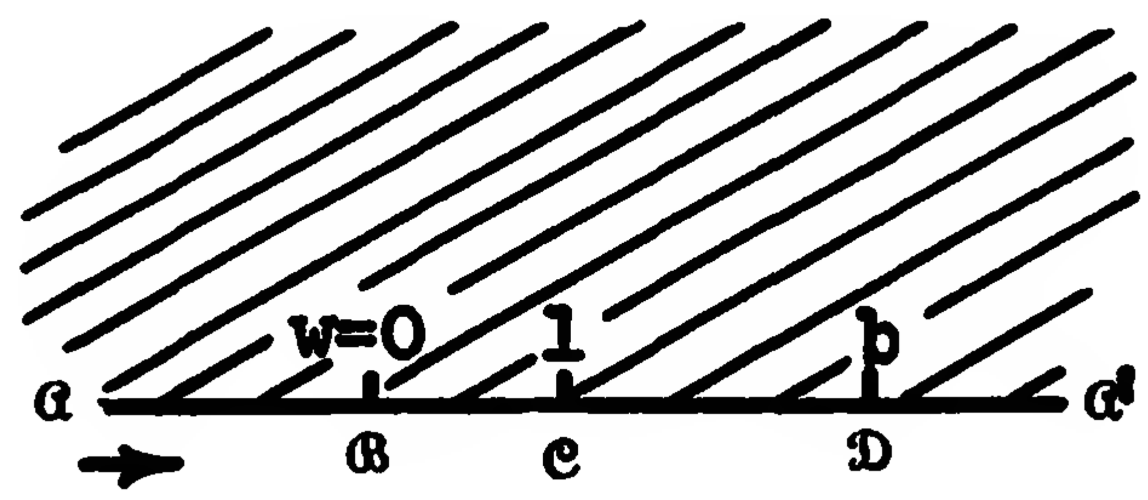
$$z_1 = \frac{\pi}{\sqrt{3}} + i\pi, \text{ i.e. } \mathcal{B}\mathcal{D} \text{ bisects } \angle\mathcal{B} = \frac{2\pi}{3} .$$

Example (ii) $\beta = \frac{1}{2}$. $z = -\cos^{-1}\frac{a-1+2w}{a+1} + \pi - \frac{1}{\sqrt{a}}\cosh^{-1}\frac{aw-w-2a}{w(a+1)}$;

a similar transformation is

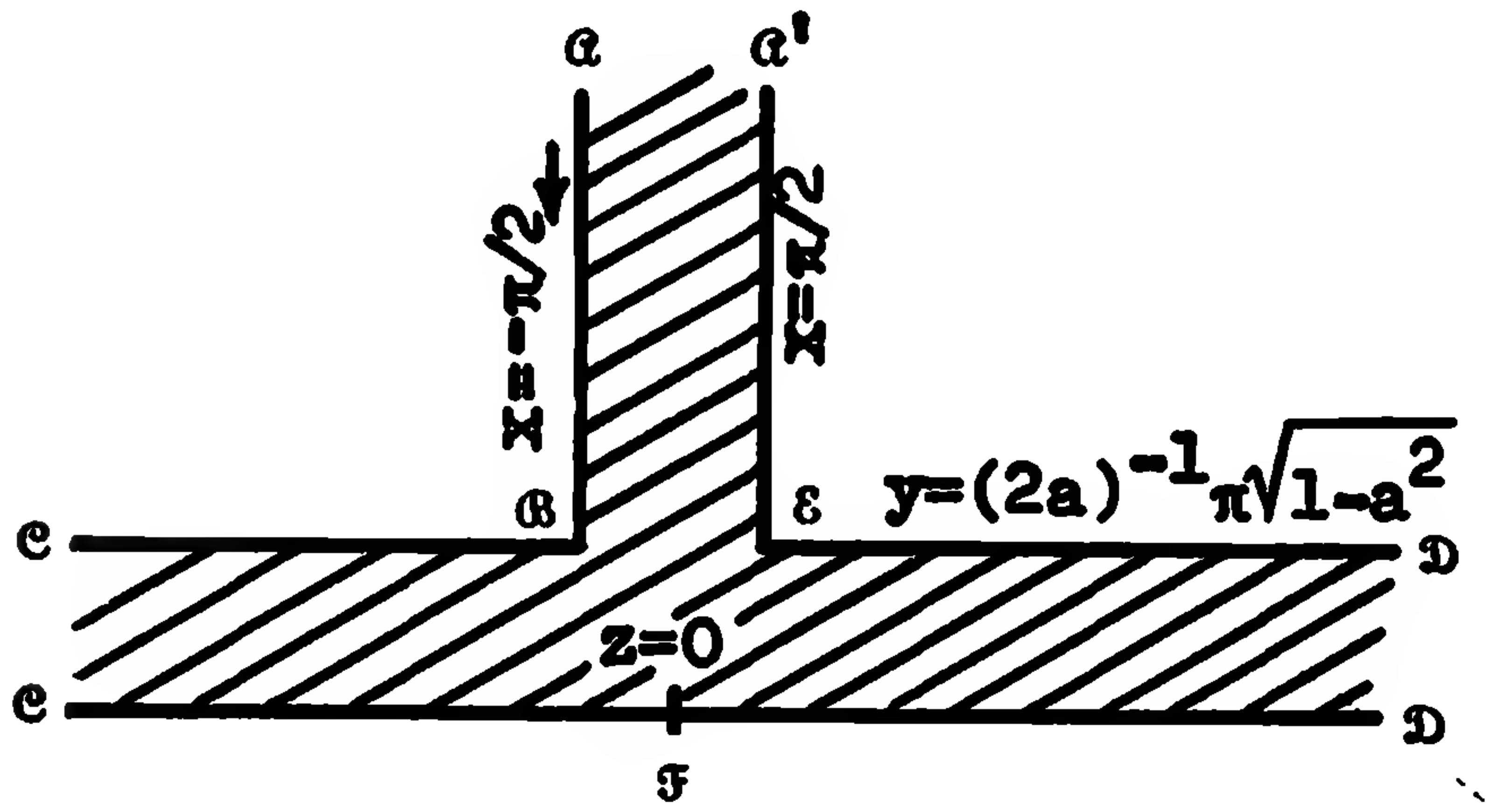
$$\boxed{\frac{dz}{dw} = \frac{\sqrt{(b-w)}}{\sqrt{w(1-w)}}} , \quad b > 1;$$

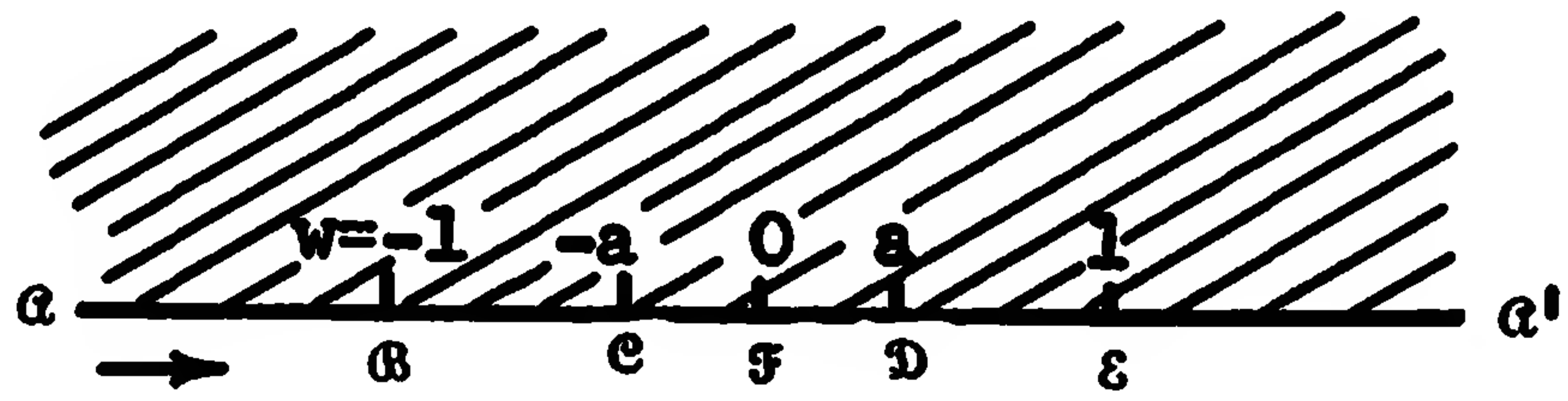
$$z = \cos^{-1}\frac{b-2w}{b} + \sqrt{b-1}\cosh^{-1}\frac{bw+b-2w}{b(1-w)} .$$

z - plane	w - plane
points $z = 0$; $\pi + i\pi\sqrt{b-1} = z_b$	points $w = 0$; b
	

$$\boxed{\frac{dz}{dw} = -\frac{(1-w^2)^{1/2}}{w^2 - a^2}}\ , \quad 0 < a < 1; \quad z = \sin^{-1} w + \frac{\sqrt{(1-a^2)}}{2a} \cosh^{-1} \frac{w^2-2a^2w^2+a^2}{a^2-w^2}.$$

z - plane	w - plane
point $z = 0$	point $w = 0$



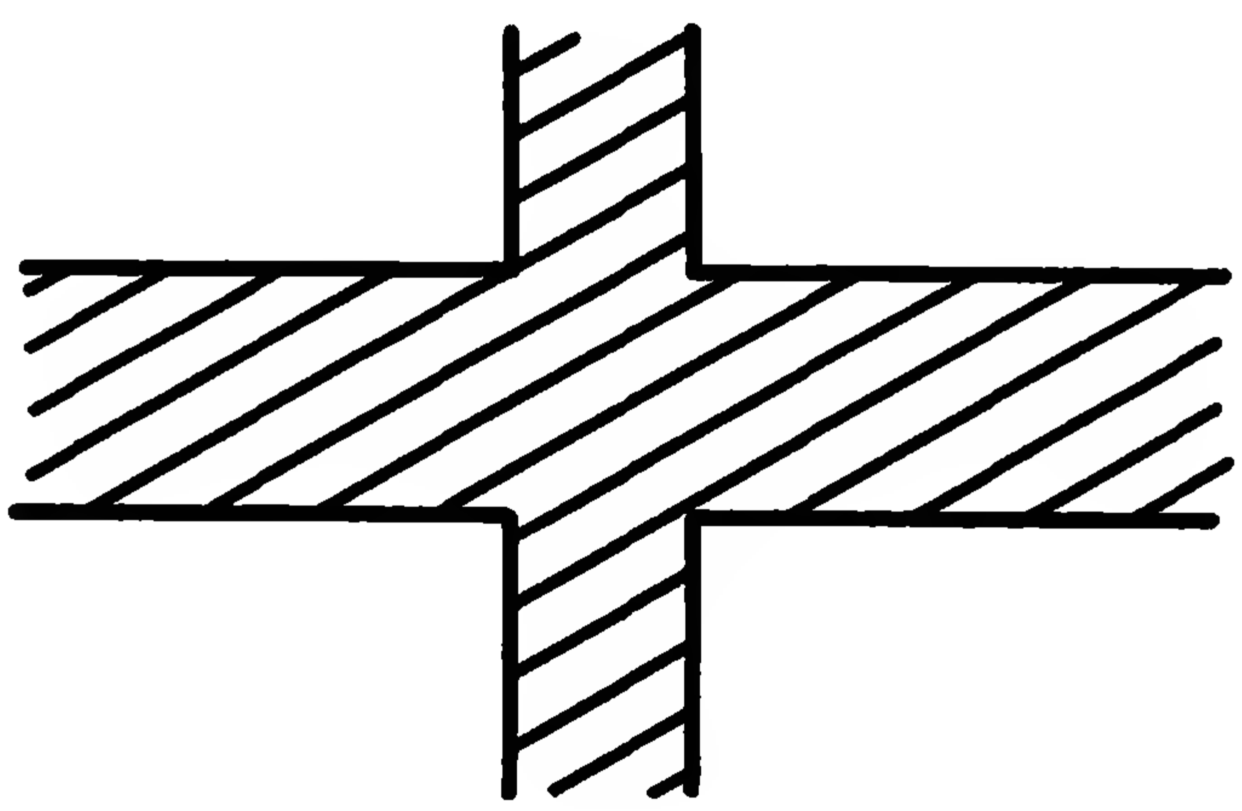


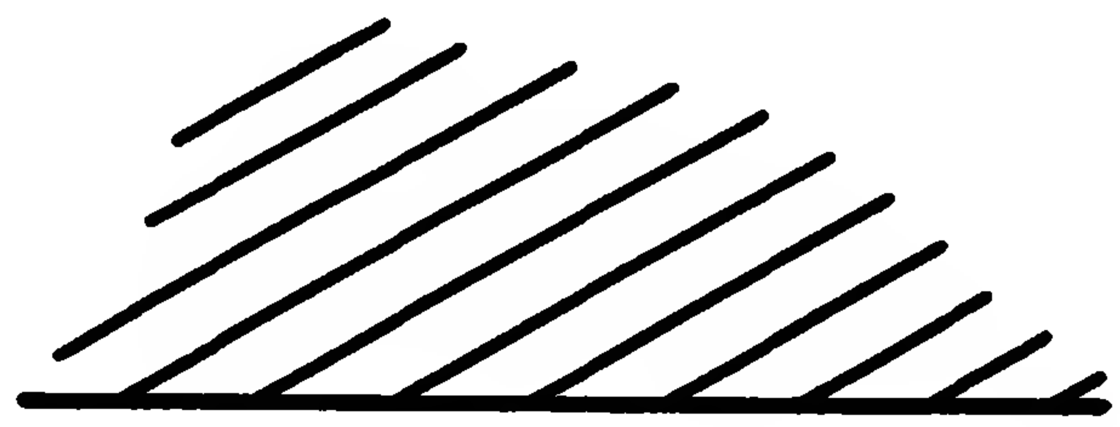
$\mathcal{E}:\ \frac{\pi}{2} + i\,\frac{\pi}{2a}\sqrt{(1-a^2)}$

$\mathcal{B}:\ -\frac{\pi}{2} + \frac{i\pi}{2a}\sqrt{(1-a^2)}$

Combining this with $w = \frac{a}{2}(\xi + \frac{1}{\xi})$: $z = \sin^{-1}\left\{\frac{a}{2}(\xi + \frac{1}{\xi})\right\} + \frac{\sqrt{(1-a^2)}}{2a} \cosh^{-1}\left\{2a^2 - 1 + \frac{8(a^2-1)}{(\xi - \frac{1}{\xi})^2}\right\}$

z - plane	w - plane



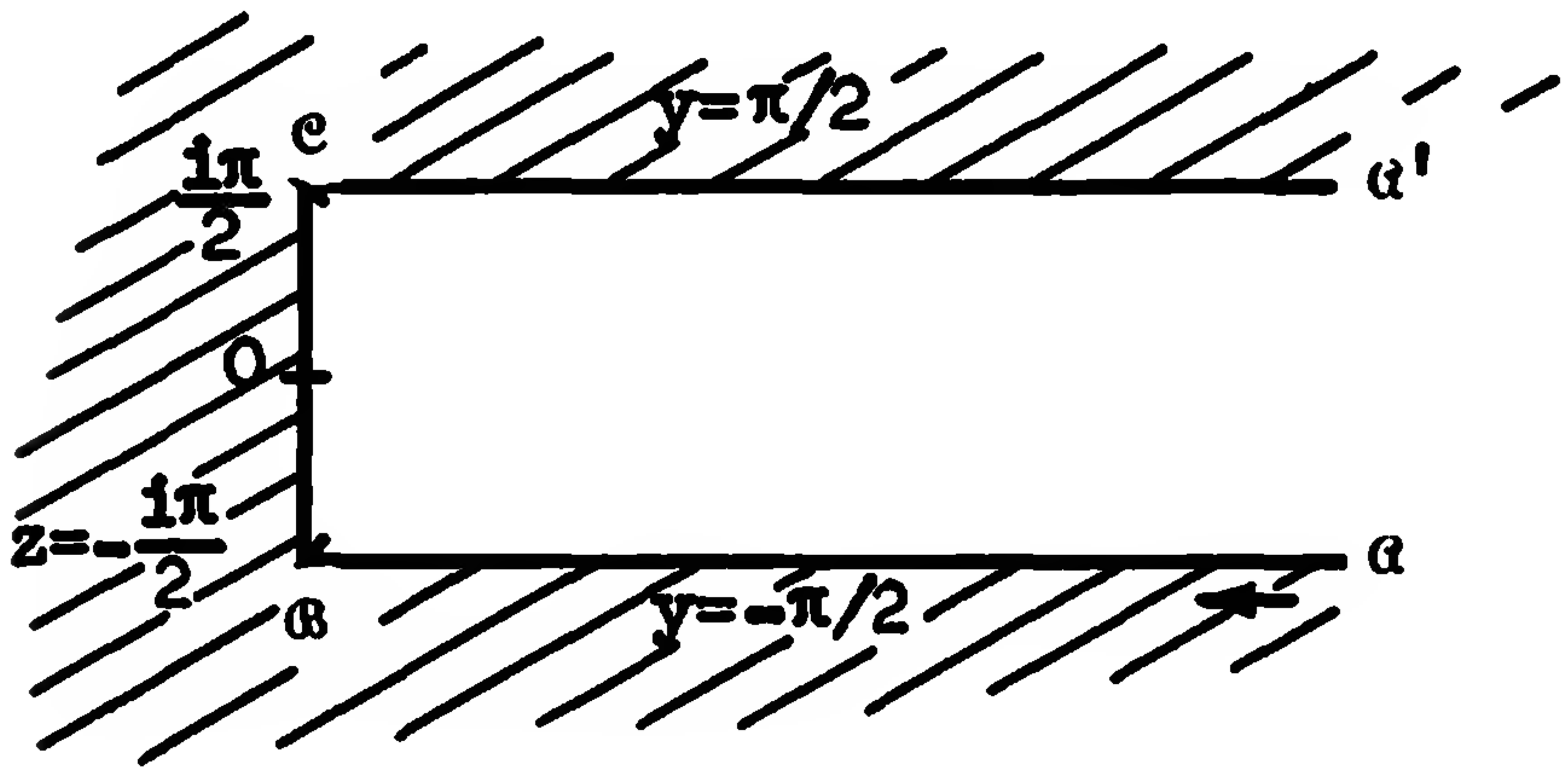


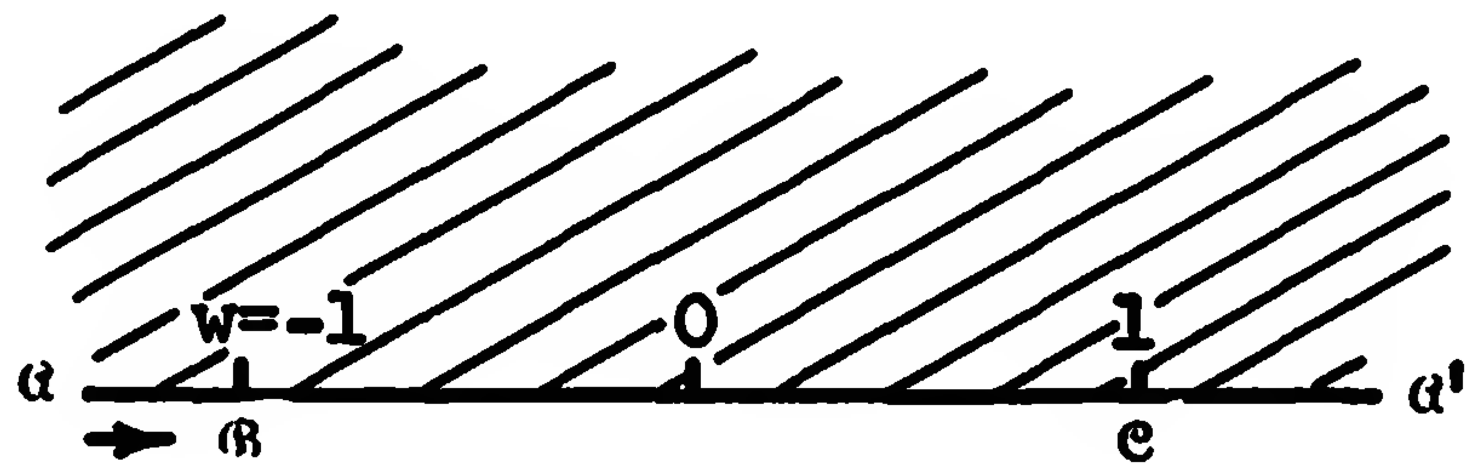
For equipotentials, see W. Göhre.

12.7 Some regions exterior to a semi-infinite strip.

$$\boxed{\frac{dz}{dw} = 2\sqrt{(w^2-1)}} \quad ; \quad z = w\sqrt{(w^2-1)} - \cosh^{-1} w + \tfrac{1}{2} i\pi.$$

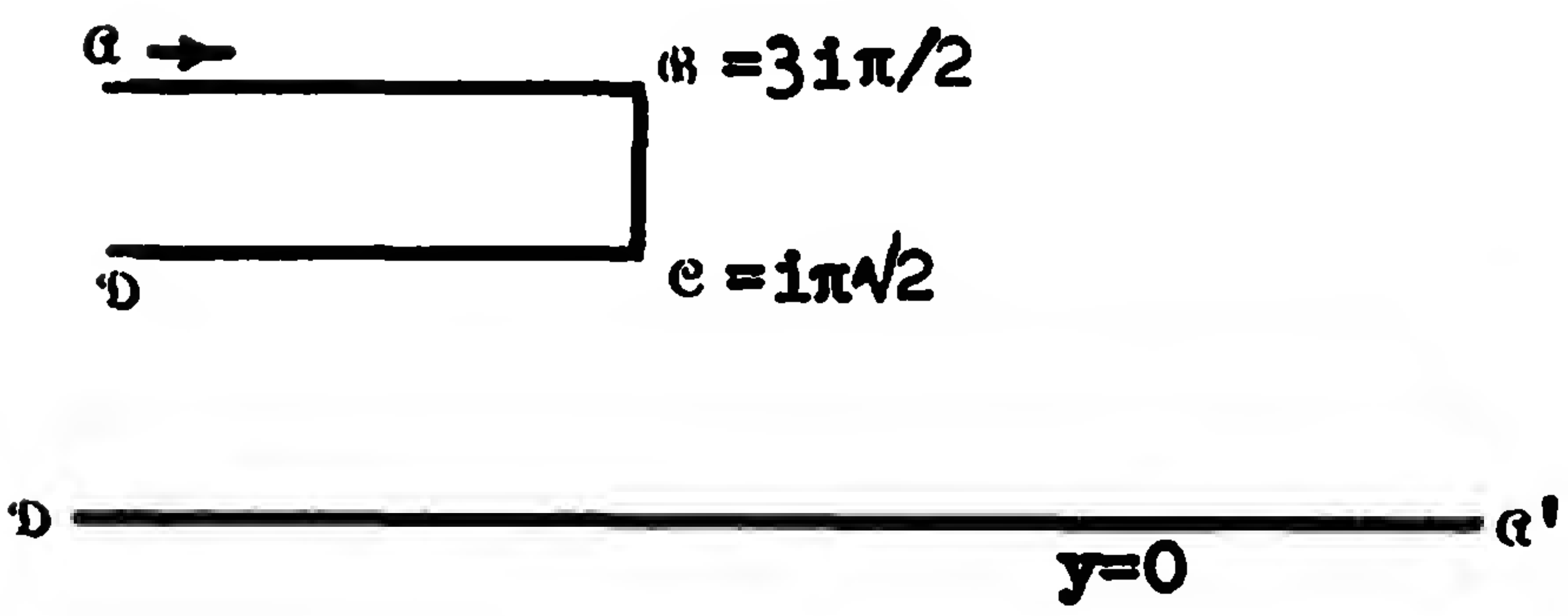
z - plane	w - plane
points $z = -\frac{i\pi}{2};\ 0;\ \frac{i\pi}{2}$	points $w = -1;\ 0;\ 1$

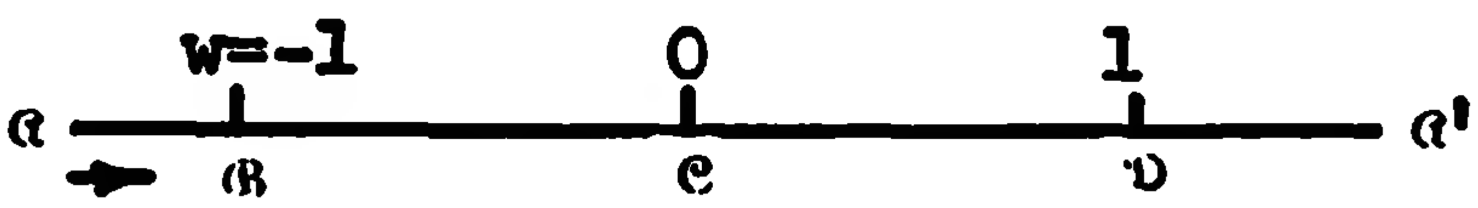




$$\boxed{\frac{dz}{dw} = \frac{w^{1/2}(w+1)^{1/2}}{w-1}} \quad ; \quad z = w^{1/2}(w+1)^{1/2} + \frac{3}{2}\cosh^{-1}(2w+1) - \sqrt{2}\cosh^{-1}\frac{3w+1}{w-1}$$

z - plane	w - plane
points $z = 3i\pi/2;\ i\pi\sqrt{2}$ half-plane, excluding the semi-infinite strip A B C D	points $w = -1;\ 0$ half-plane $v > 0$

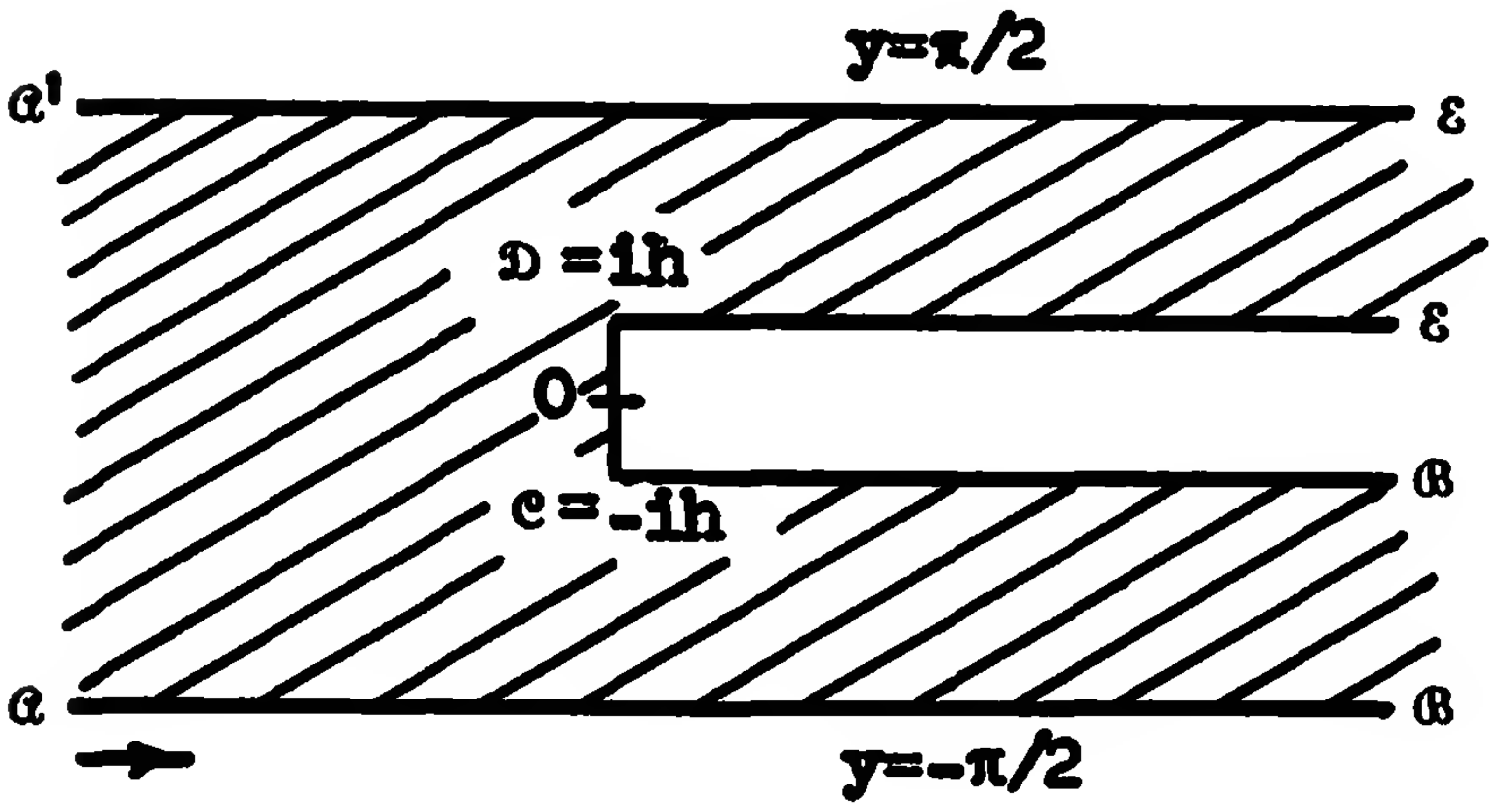


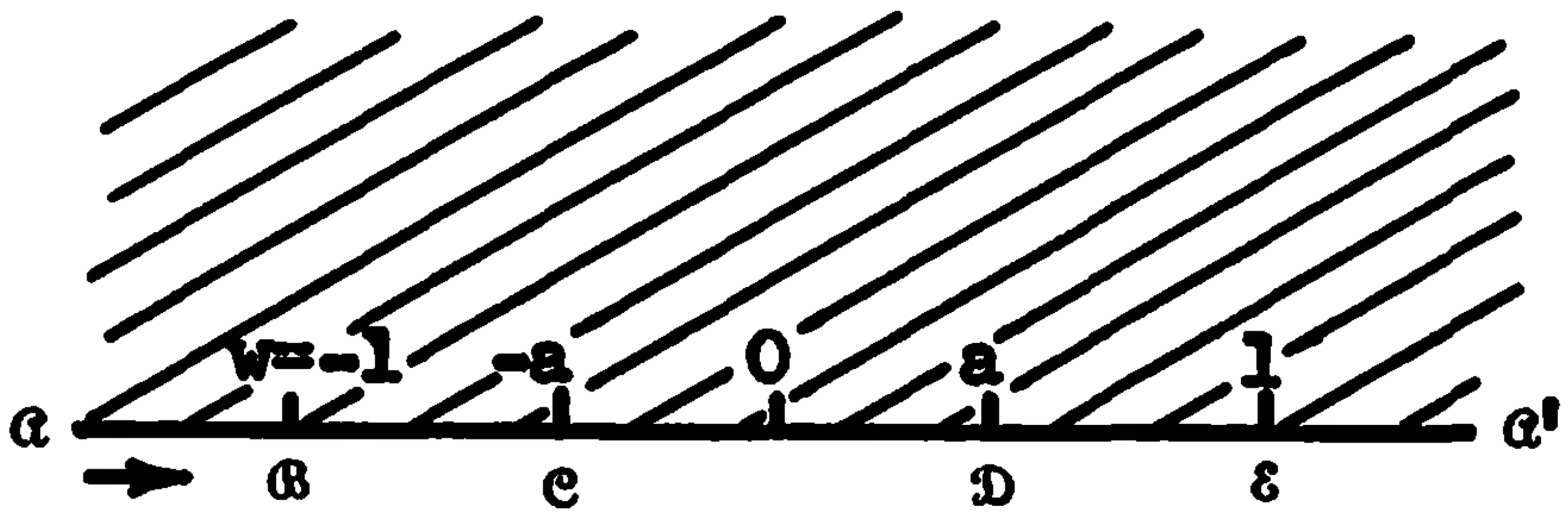


$$\boxed{\frac{dz}{dw} = -\frac{\sqrt{(w^2-a^2)}}{w^2-1}}\ , \quad 0 < a < 1.$$

$$z = \frac{i\pi}{2} - \cosh^{-1}\frac{w}{a} + \frac{(1-a^2)^{1/2}}{2}\cosh^{-1}\frac{w^2(2-a^2)-a^2}{a^2(w^2-1)} .$$

z - plane	w - plane
points $z = 0;\ -ih;\ ih$	points $w = 0;\ -a;\ a$



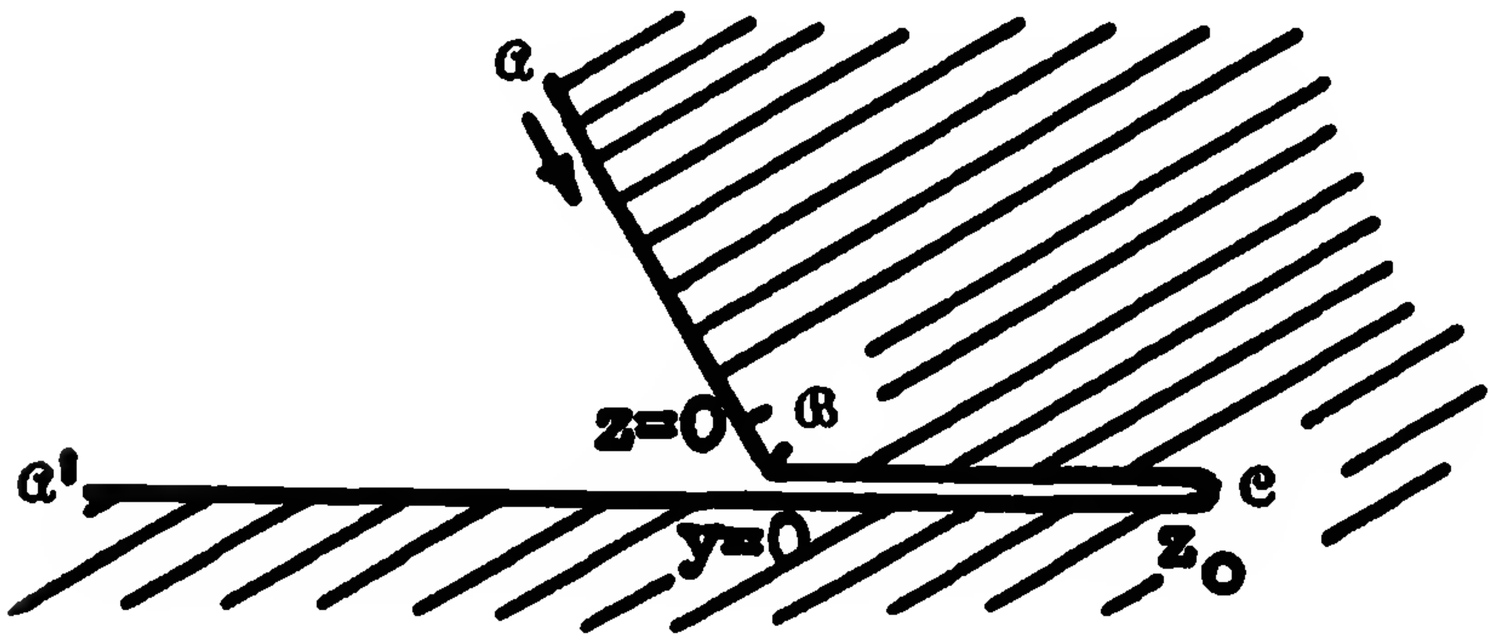


$$a = \frac{2}{\pi}\sqrt{(\pi h-h^2)}, \quad 0 < h = \frac{1}{2}\pi - \frac{1}{2}\pi\sqrt{(1-a^2)} < \frac{1}{2}\pi$$

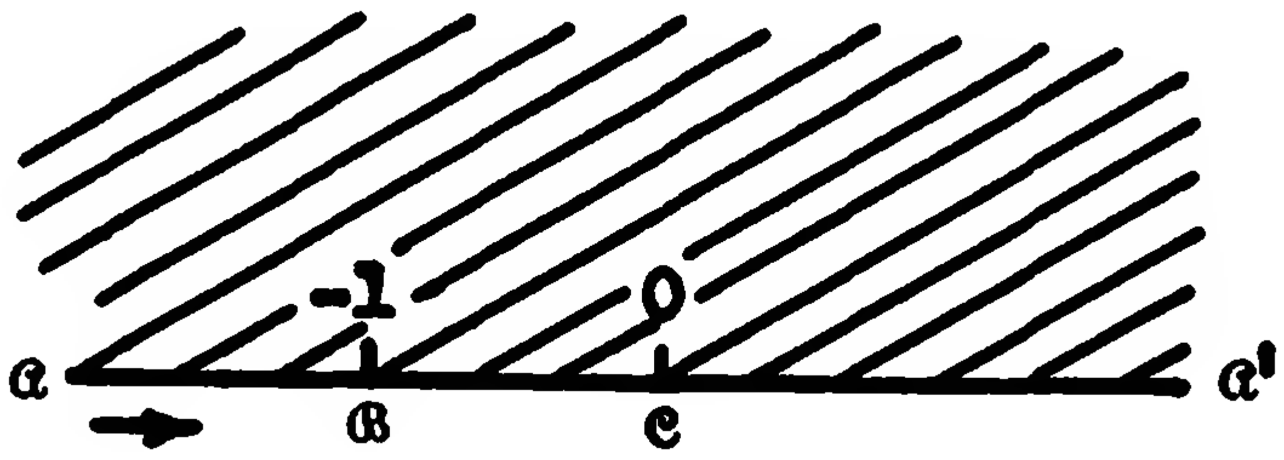
12.8 Further transformations.

$$\boxed{\frac{dz}{dw} = -\frac{w}{(w+1)^{\beta}}}\ , \ 0 < \beta < 1; \ z = -\frac{1}{2-\beta}(w+1)^{2-\beta} + \frac{1}{1-\beta}(w+1)^{1-\beta}.$$

z - plane	w - plane
points $z = 0;\ \left\{(1-\beta)(2-\beta)\right\}^{-1} = z_0$	points $w = -1;\ 0$

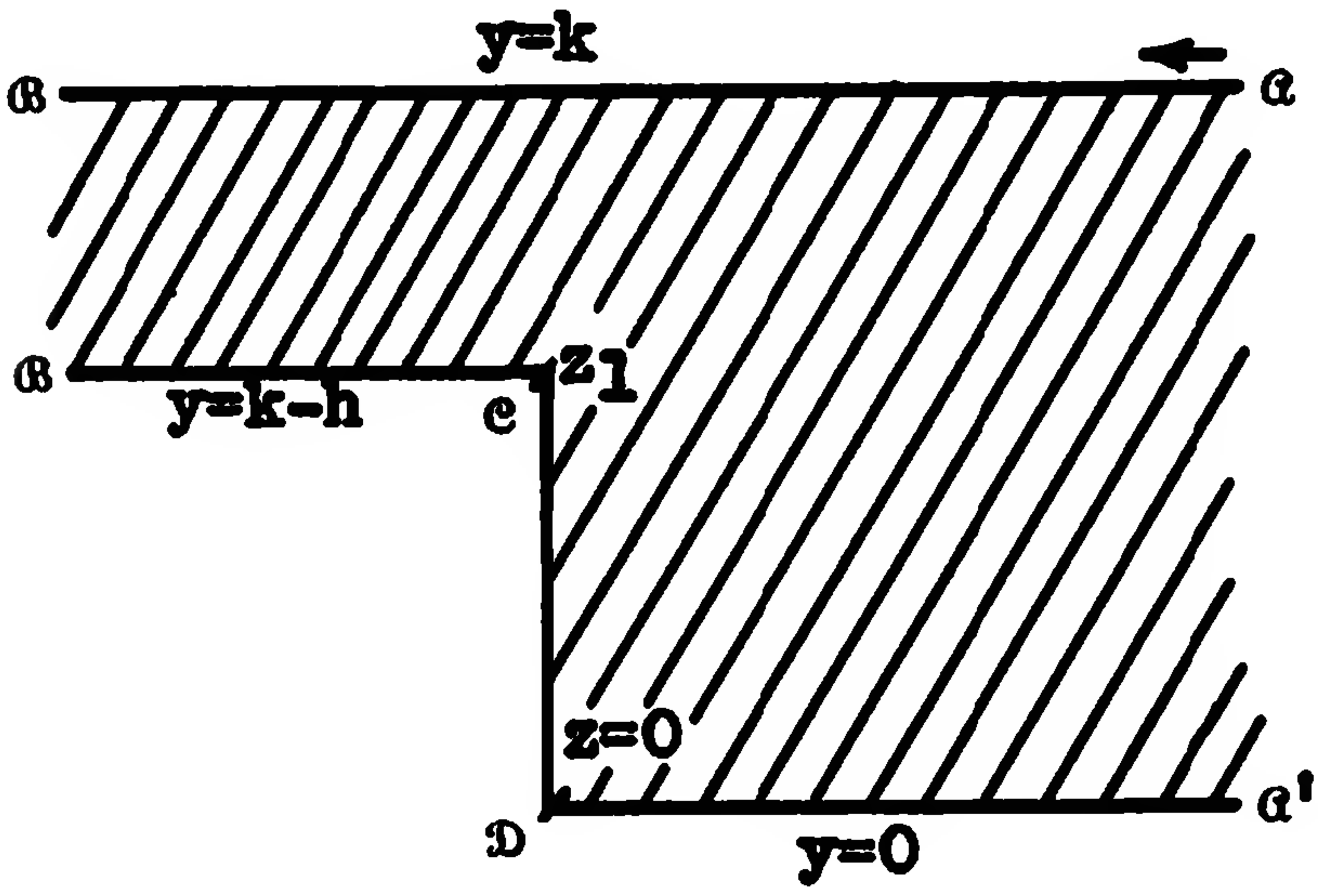
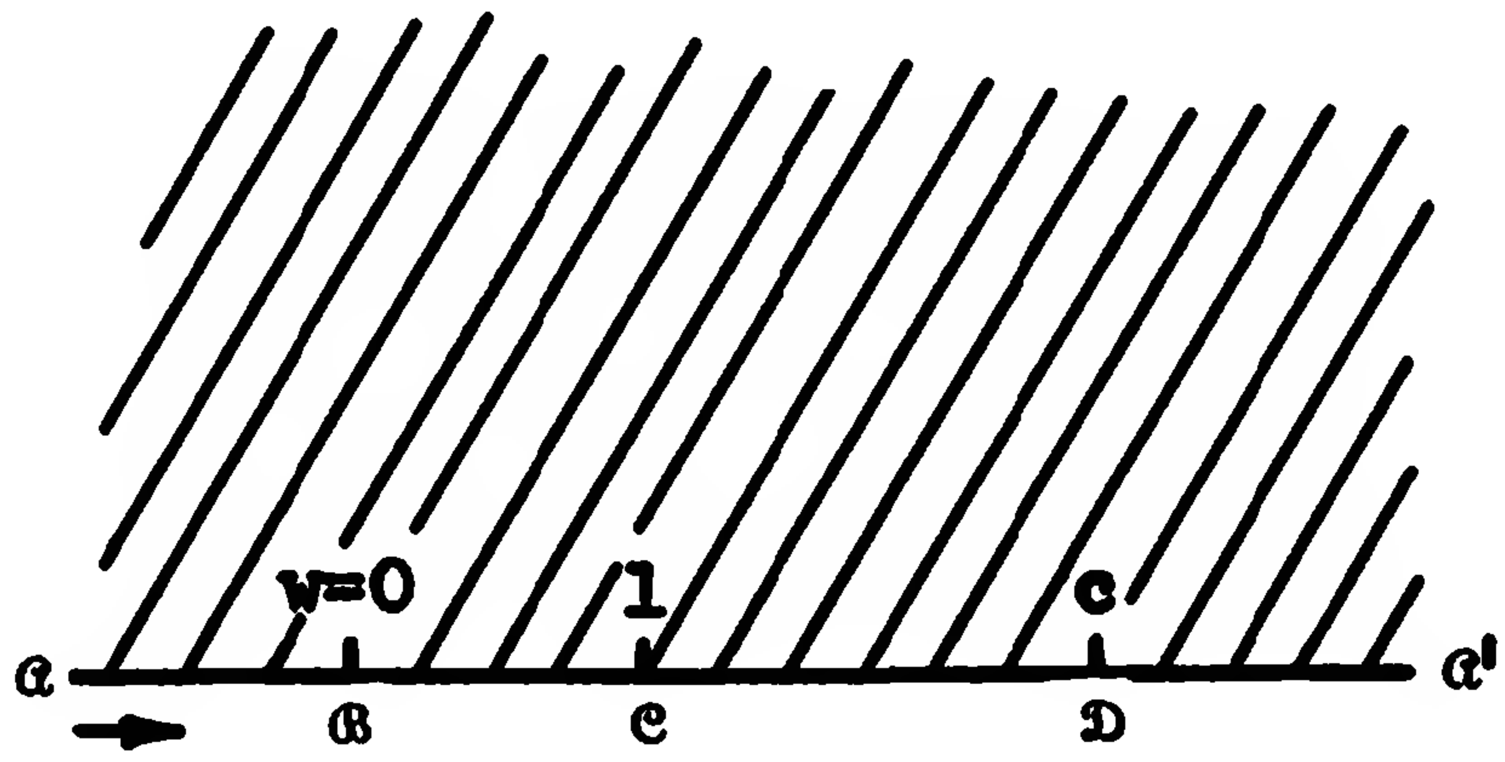


$\angle CBA = \pi - \beta\pi$



$$\boxed{\frac{dz}{dw} = \frac{k(w-1)^{1/2}}{\pi w(w-c)^{1/2}}} \quad , \; c > 1; \quad k > 0, \; kc^{-1/2} = h$$

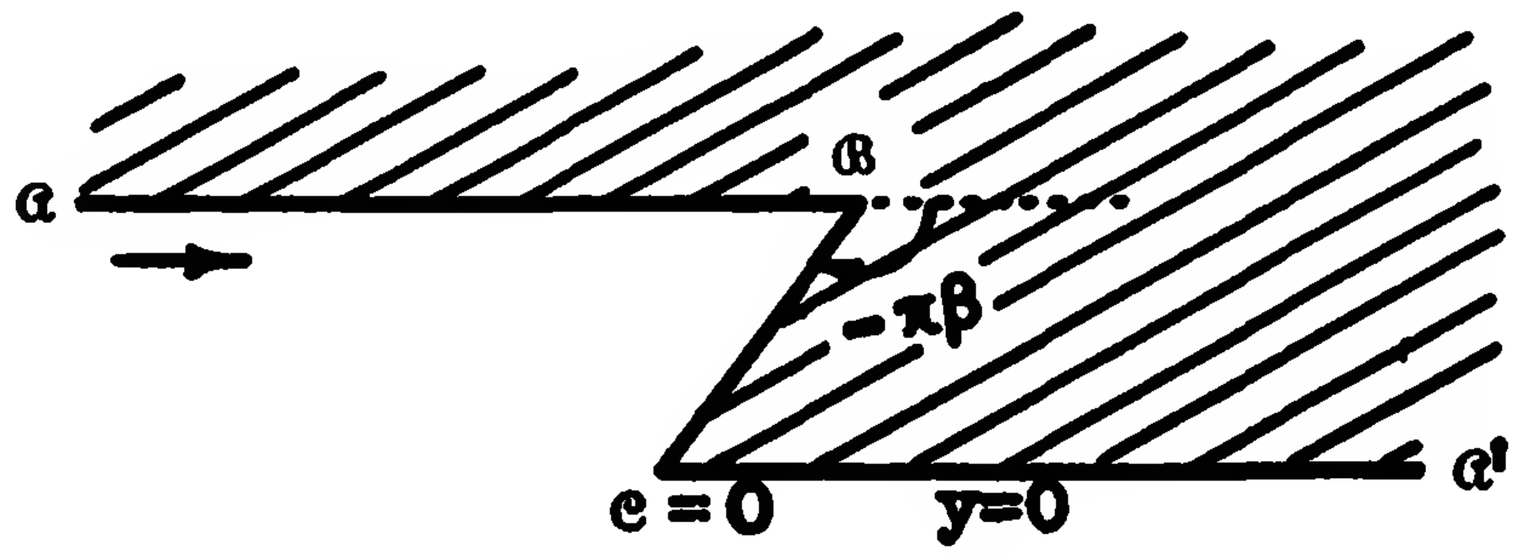
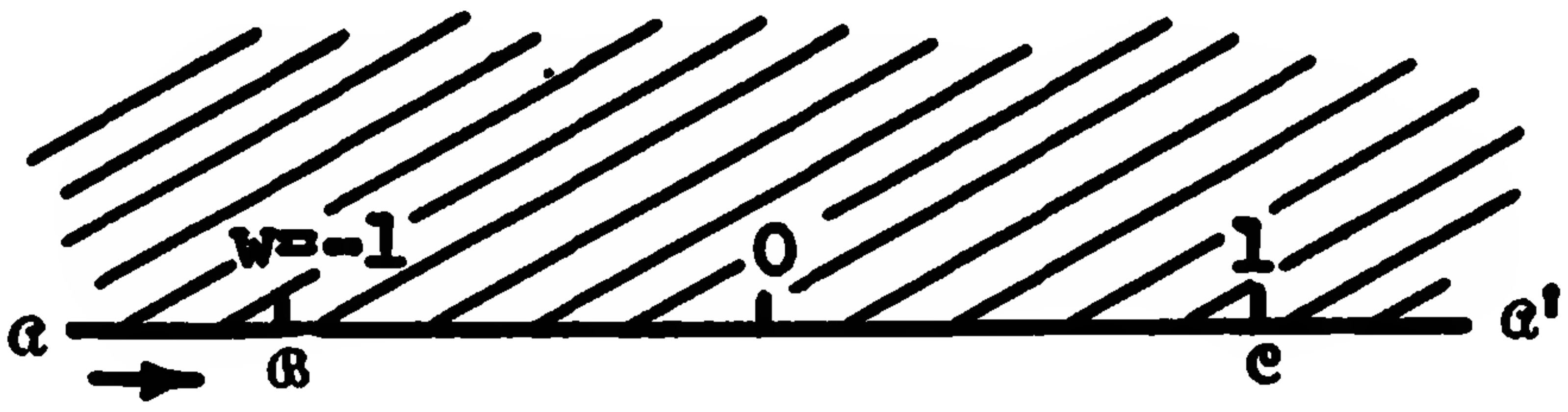
$$z = \frac{k}{\pi} \cosh^{-1}\left(\frac{2w-c-1}{c-1}\right) - \frac{k}{\pi\sqrt{c}} \cosh^{-1}\left(\frac{(c+1)w-2c}{(c-1)w}\right)$$

z - plane	w - plane
points $i(k-h) = z_1$; 0	points $w = 1$; c
	

$$\boxed{\frac{dz}{dw} = \left(\frac{w+1}{w-1}\right)^{\beta}} \quad , \quad 0 < \beta < 1.$$

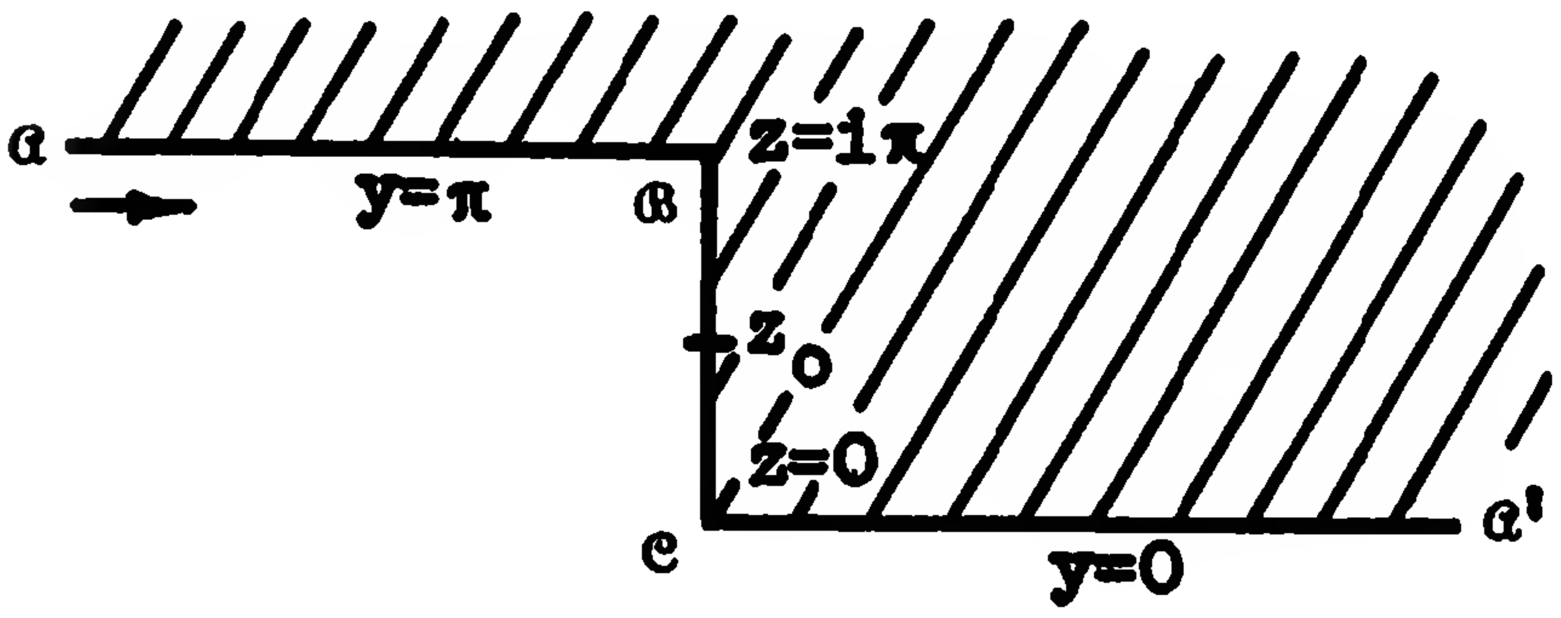
Representable in terms of elementary functions, if $\beta = p/q$ (p, q integers):

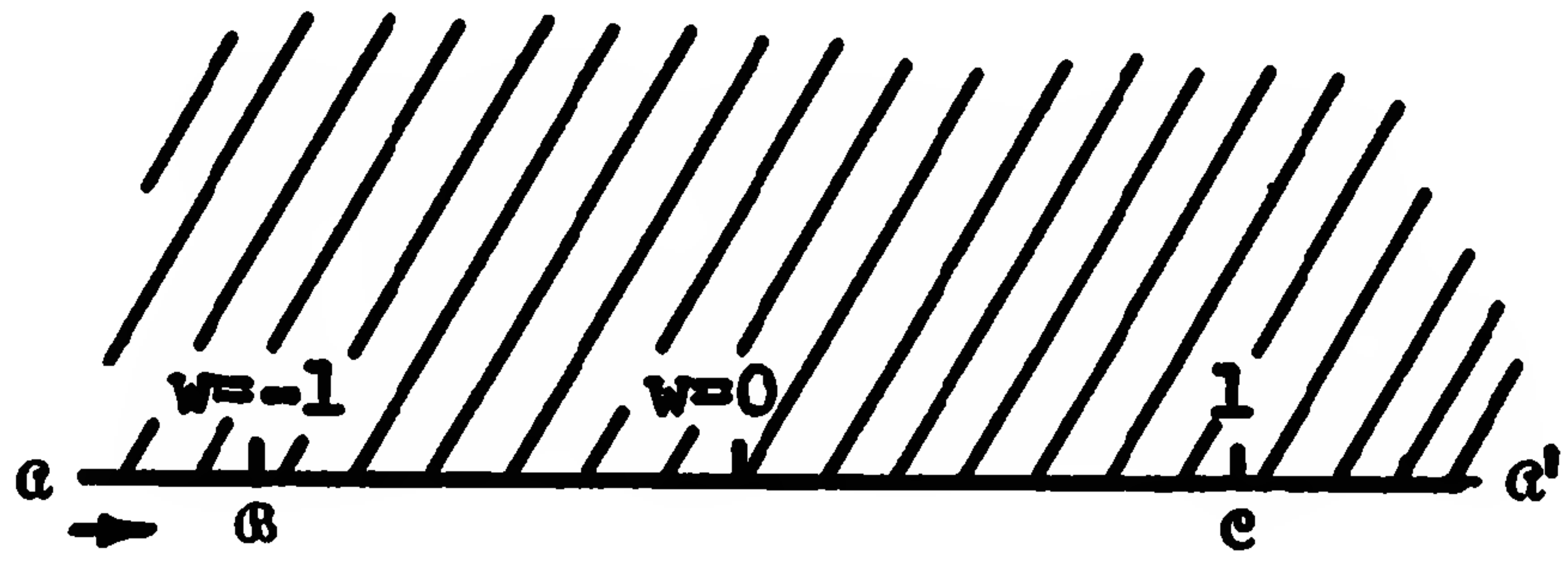
$$z = 2q \int_{\zeta}^{\infty} t^{p+q-1}(t^q-1)^{-2}\, dt, \text{ where } \zeta = (w+1)^{1/q}(w-1)^{-1/q}.$$

z - plane	w - plane
	

Example: $p = 1$, $q = 2$. $\boxed{\frac{dz}{dw} = \sqrt{\left(\frac{w+1}{w-1}\right)}}$; $z = \sqrt{(w^2-1)} + \cosh^{-1} w$

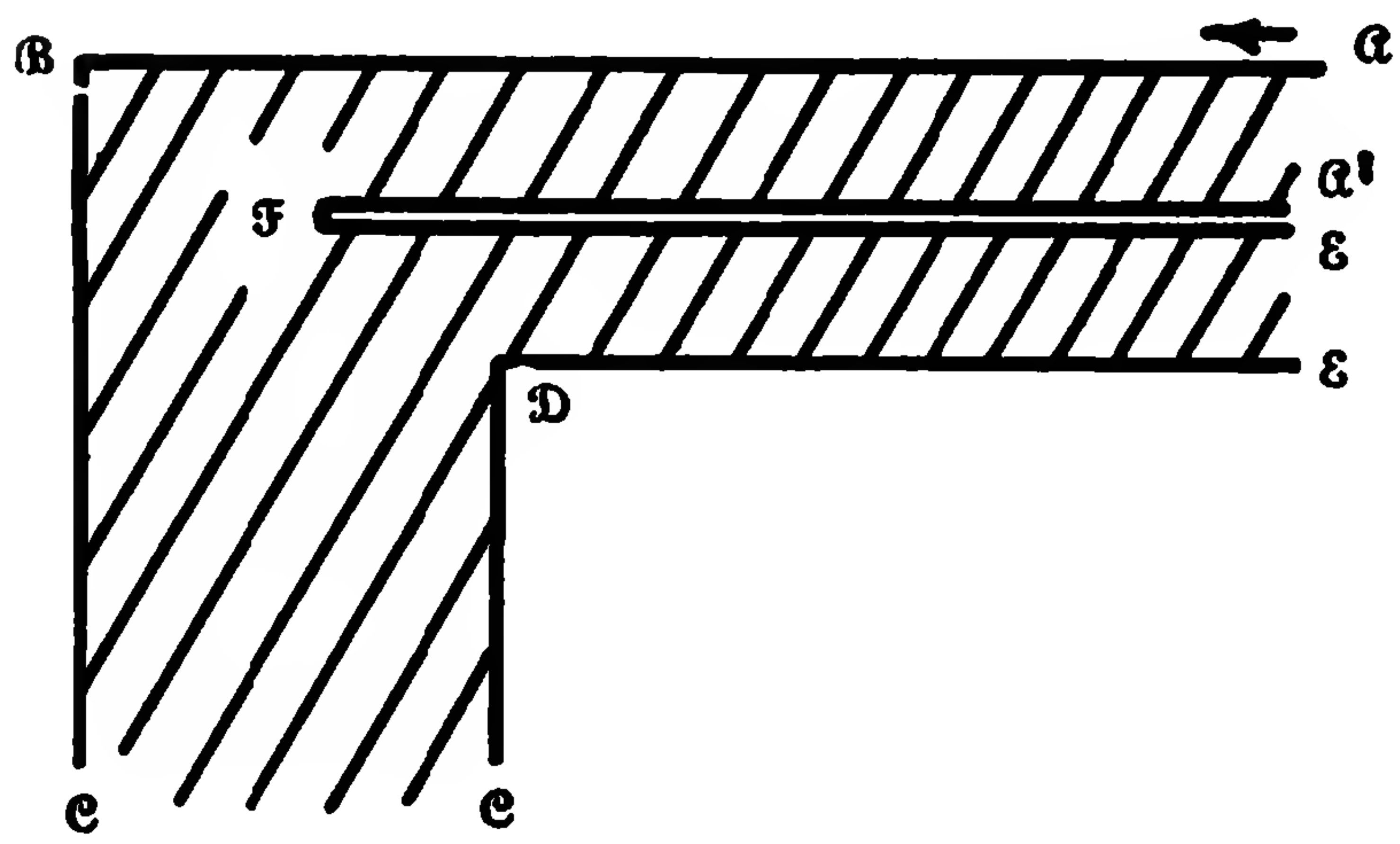
z - plane	w - plane
points $z = i\pi$; $i(1 + \frac{\pi}{2}) = z_o$; 0	points $w = -1$; 0; 1

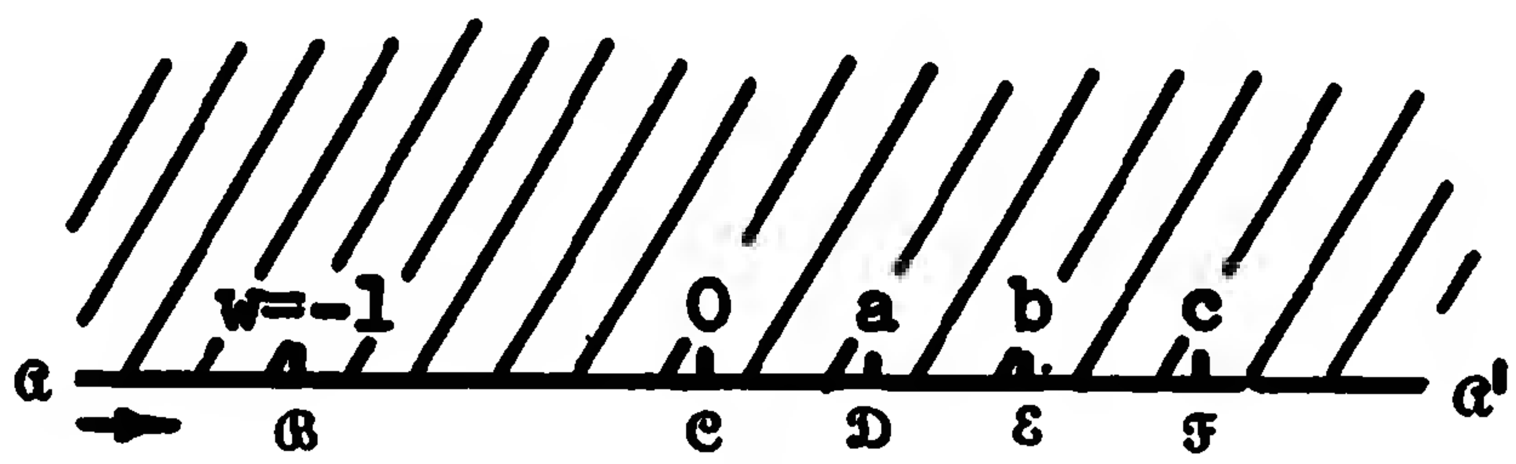




$\boxed{\frac{dz}{dw} = C\frac{(w-c)\ \sqrt{(w-a)}}{w(w-b)\ \sqrt{(w+1)}}}$, $0 < a < b < c$

z - plane	w - plane

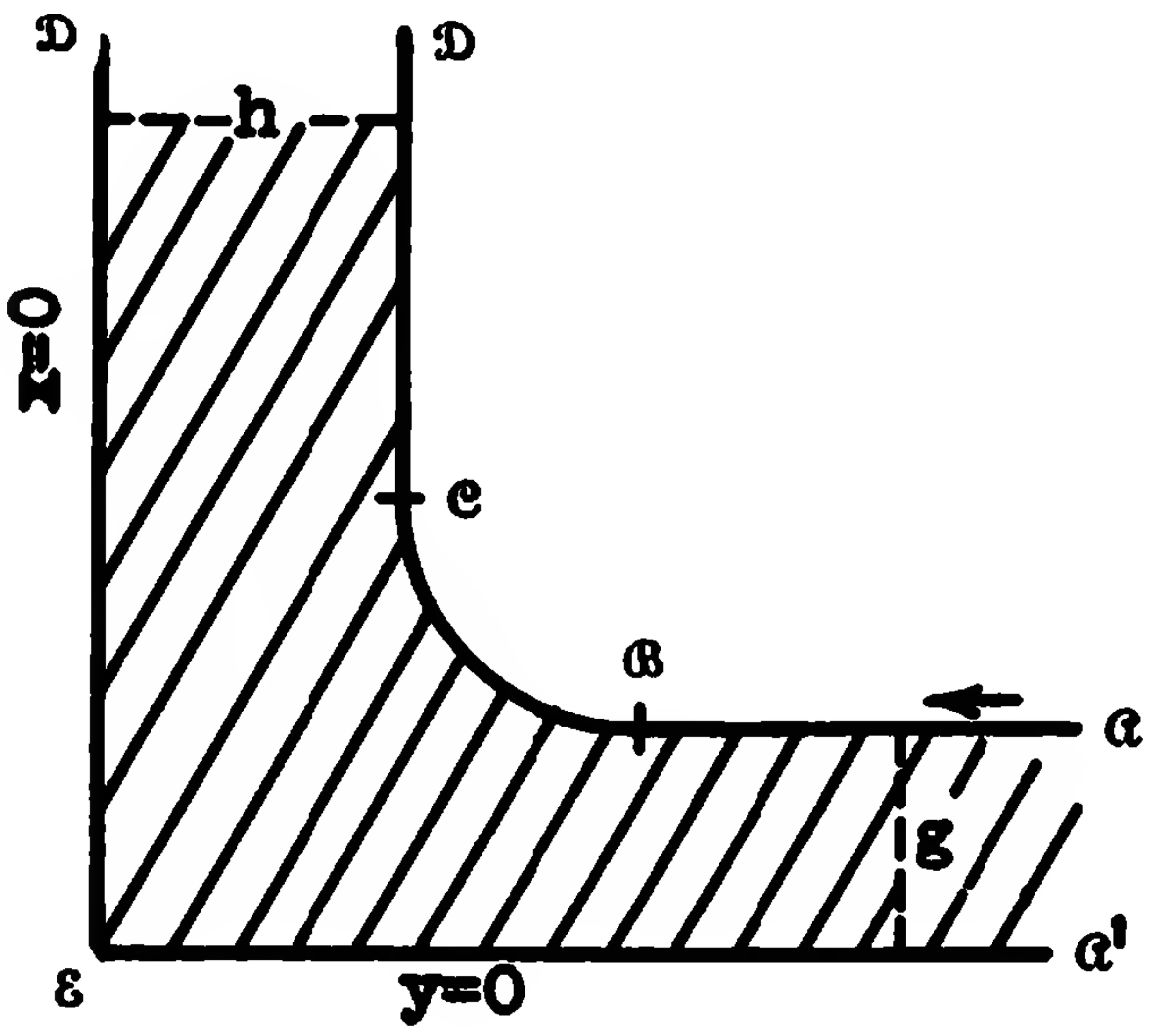


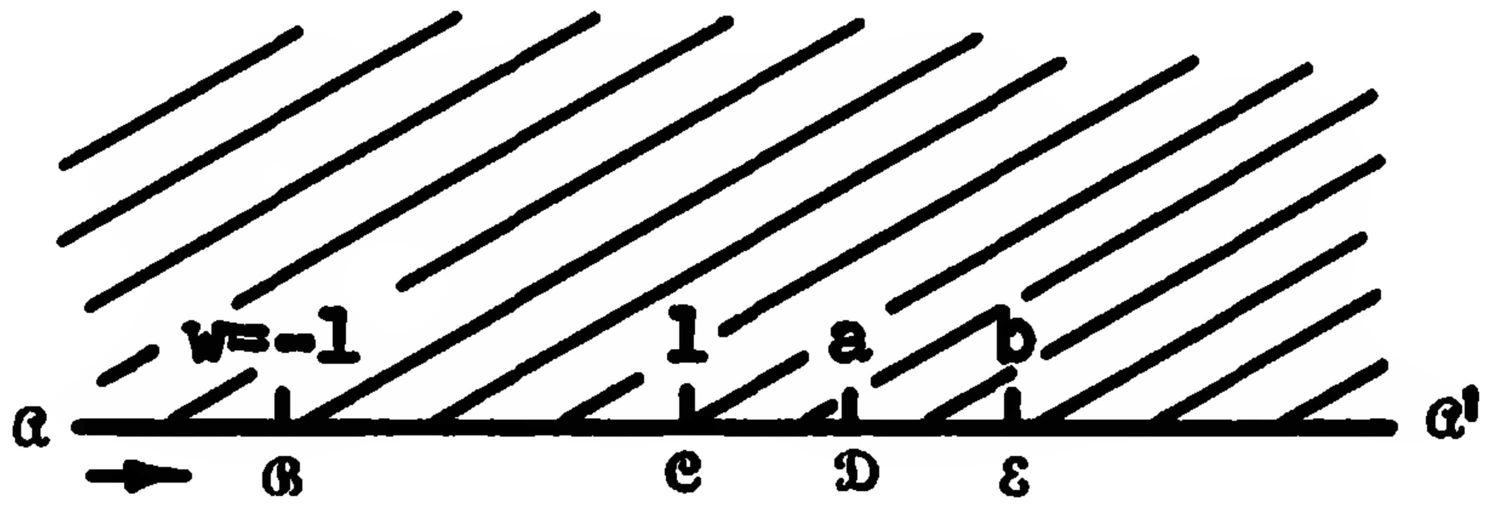


12.9 Curved boundary-line.

$$\boxed{\frac{dz}{dw} = \frac{\sqrt{(w+1)} + c\,\sqrt{(w-1)}}{(w-a)\,\sqrt{(w-b)}}} \quad ; \quad b > a > 1, \quad c = \sqrt{\left(\frac{b+1}{b-1}\right)} > 0.$$

z - plane	w - plane



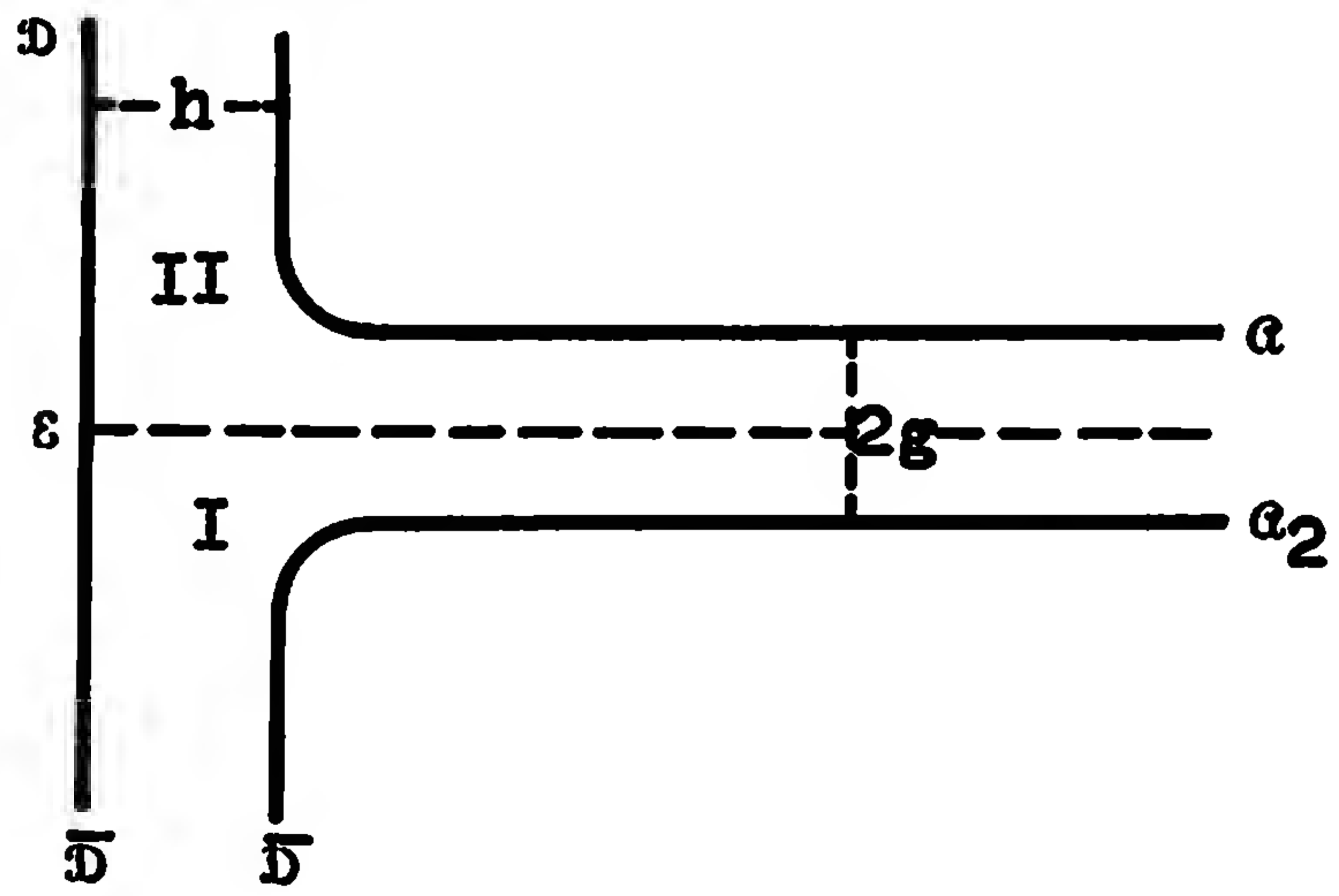


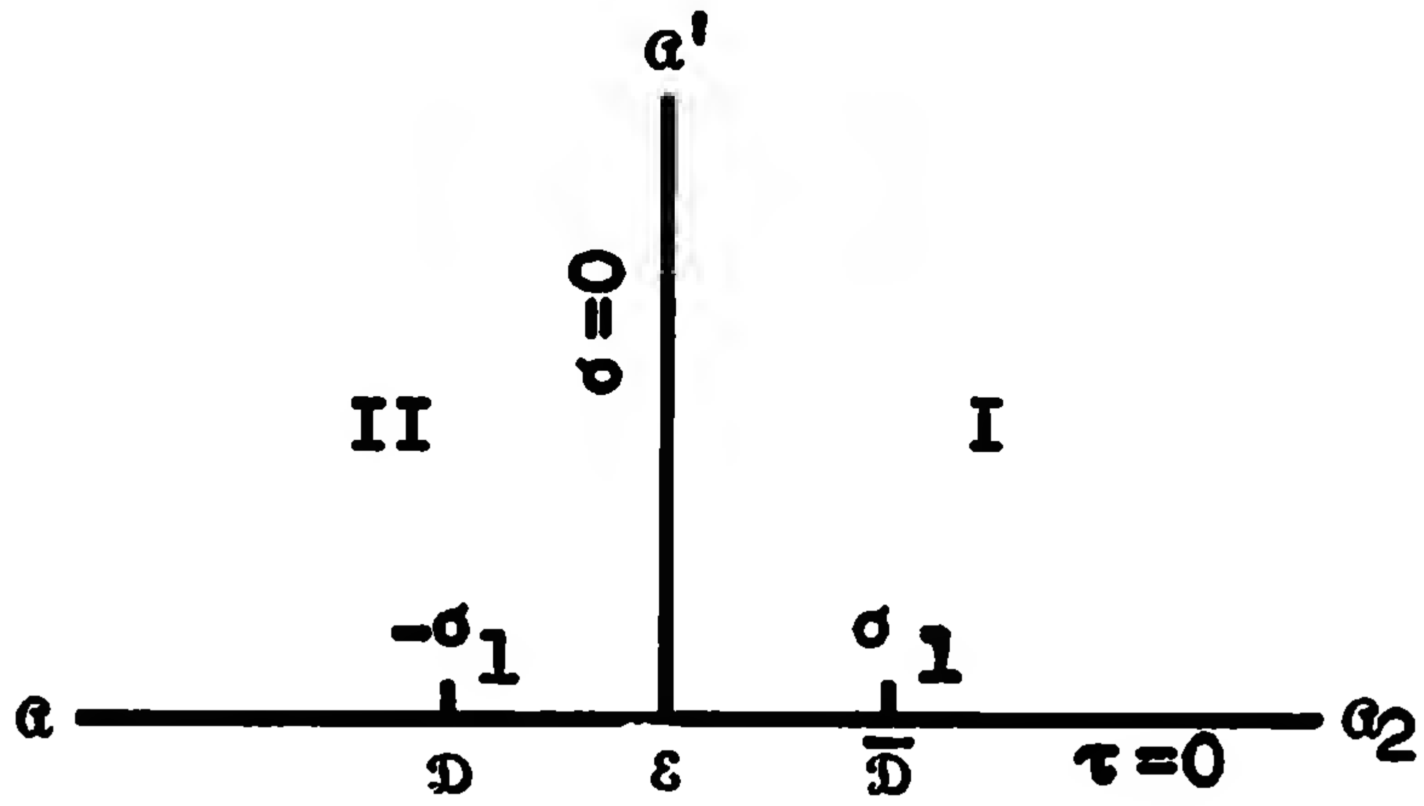
$g = \pi(1+c)$; $h = \pi\left(\sqrt{\frac{a+1}{b-a}} + c\sqrt{\frac{a-1}{b-a}}\right)$; curve BC is approximately a quarter of a circle.

The figure is reprinted by permission from "The effect of curved boundaries on the distribution of electrical stress round conductors", by J. D. Cockroft, Journal Electr. Eng. 66 (1927), p. 385 etc. Details are given in that article.

Combining this transformation with $w = b + e^{-i\pi}\zeta^2$:

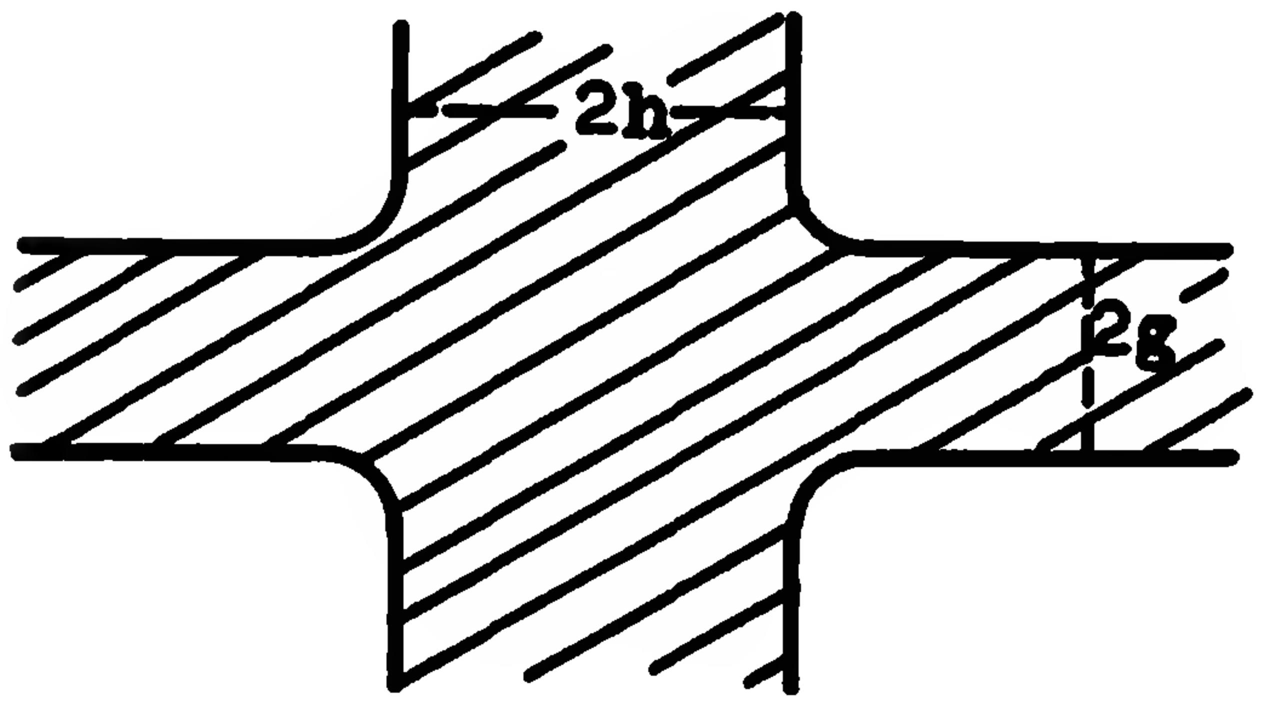
z - plane	ζ - plane ($\zeta = \sigma + i\tau$)
Region, bounded by $x = 0$ and the two curves	Half-plane $\tau > 0$





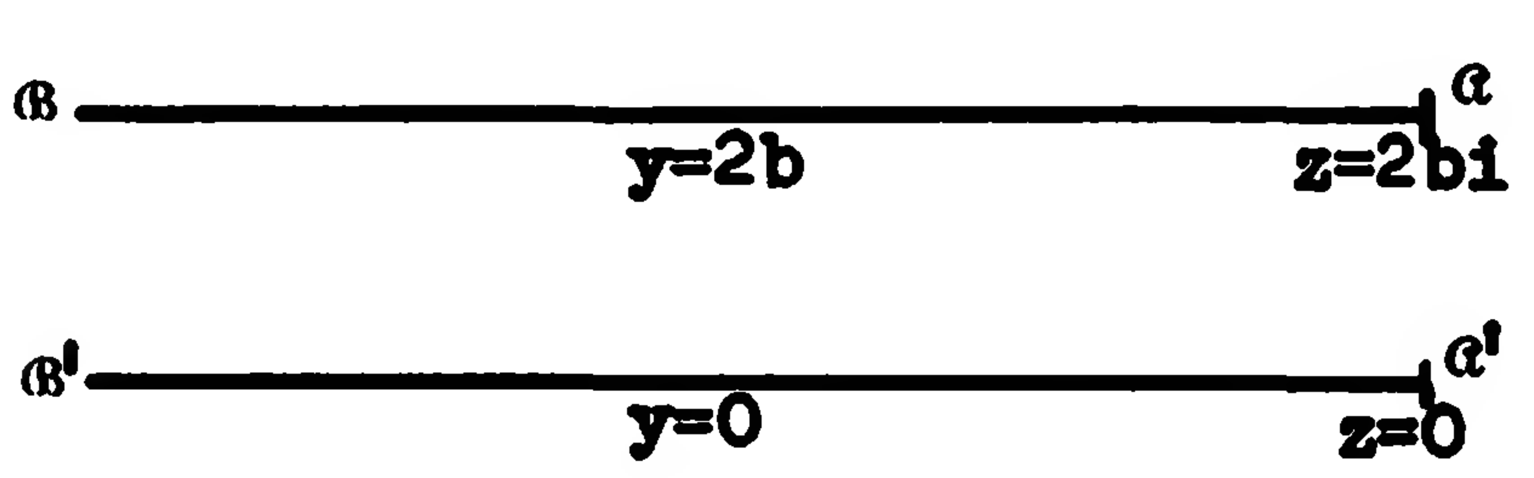
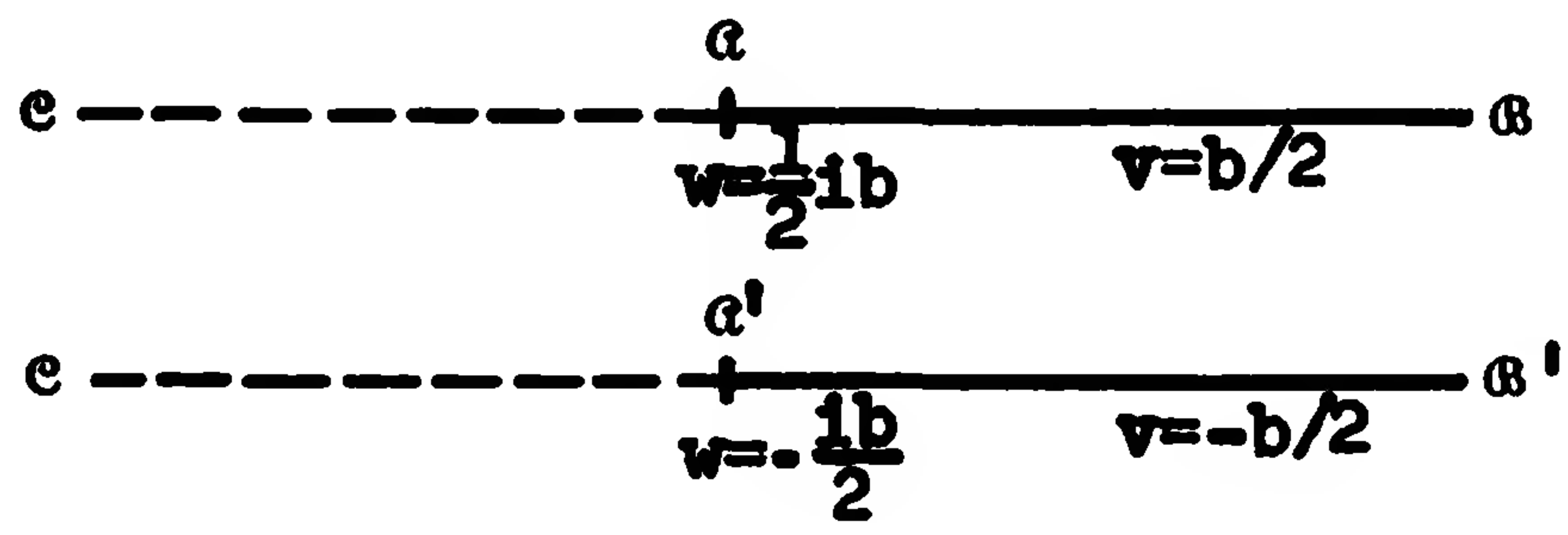
$\sigma_1 = \sqrt{(b-a)} > 0$

Combining it with $w = b + \frac{a-b}{4}(\zeta + \frac{1}{\zeta})^2$:

z - plane	ζ - plane
shaded region 	half-plane $\Im(\zeta) > 0$

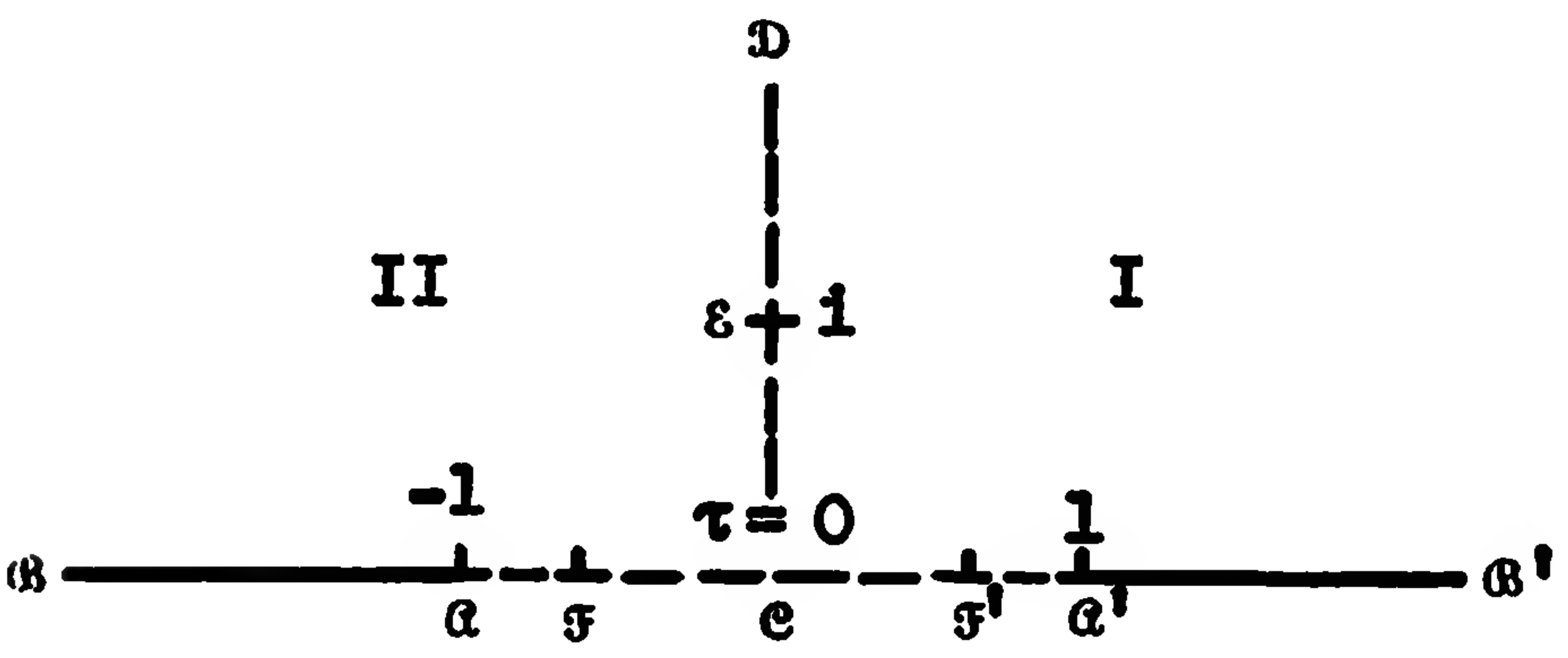
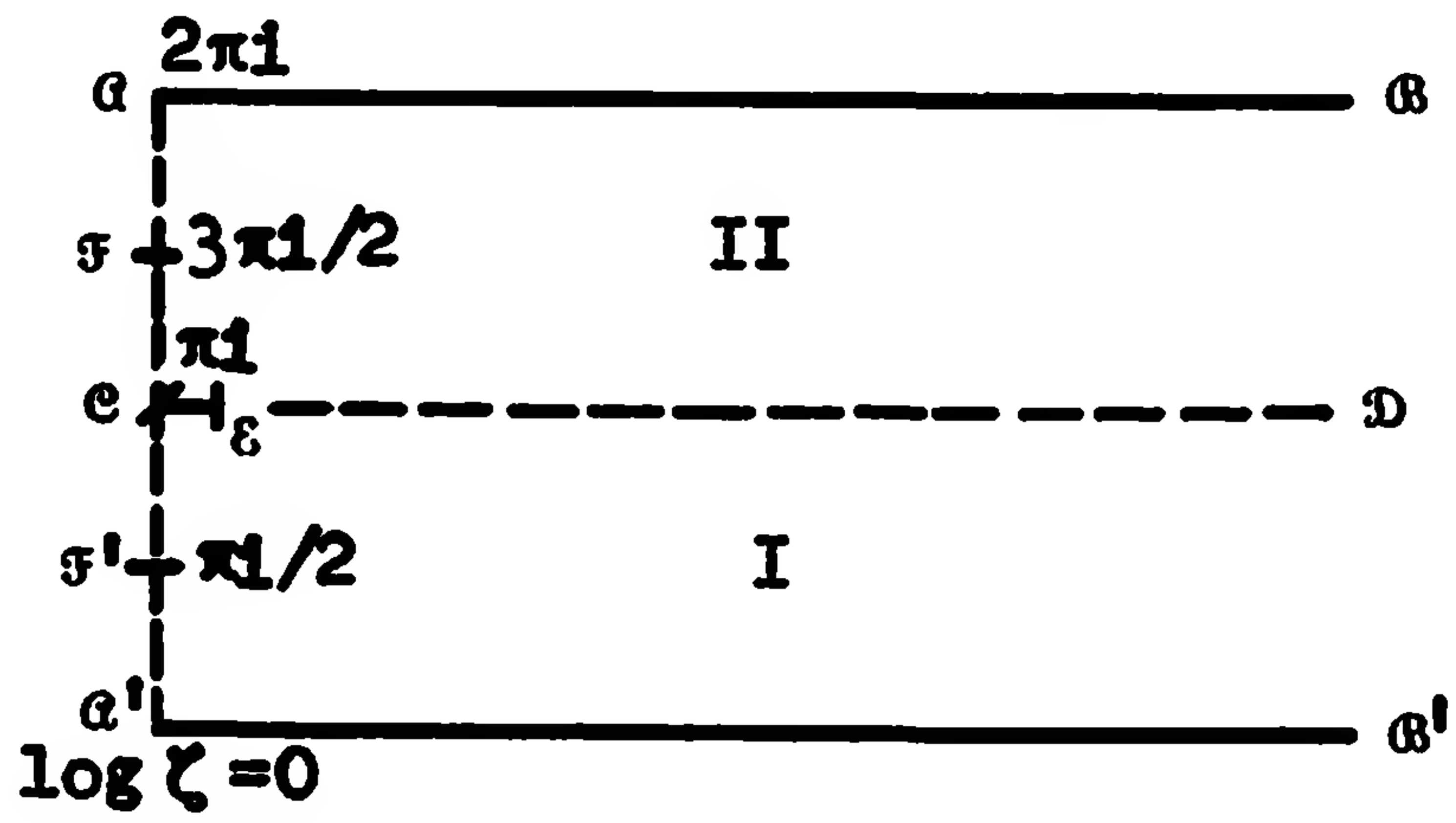
12.10 <u>Combination, using a differential equation</u> (see pp. 144, 145).

Example (i)‡ Given the half-lines $\mathcal{A}\mathcal{B}$, $\mathcal{A}'\mathcal{B}'$ as shown ($b > 0$).

z - plane	w - plane
	

To map $\mathcal{A}\mathcal{B}$ on $\mathcal{A}\mathcal{B}$, $\mathcal{A}'\mathcal{B}'$ on $\mathcal{A}'\mathcal{B}'$ so that (i) dz/dw is real and negative along $\mathcal{A}\mathcal{B}$ and $\mathcal{A}'\mathcal{B}'$, (ii) $dz/dw = -1$ at $\mathcal{A}$ and $\mathcal{A}'$; to find the curves corresponding to $\mathcal{A}\mathcal{C}$ and $\mathcal{A}'\mathcal{C}$ in the z-plane ("Borda's mouthpiece").

$$\zeta = -\frac{dz}{dw}$$

τ - plane	log ζ - plane
	

$\mathcal{F}$: $\tau = -2^{-1/2}$; $\mathcal{F}'$ = : $\tau = 2^{-1/2}$

Transformation: $\log \zeta = 2\cosh^{-1} \tau$

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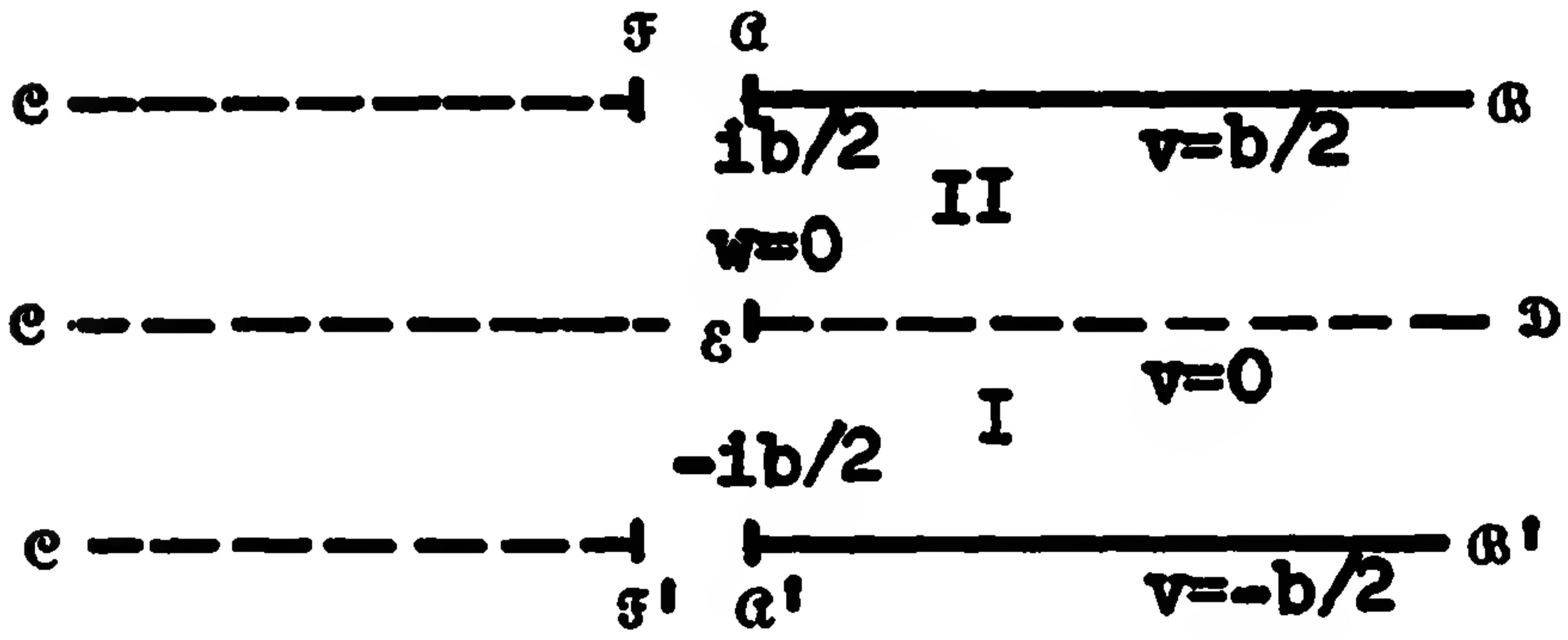
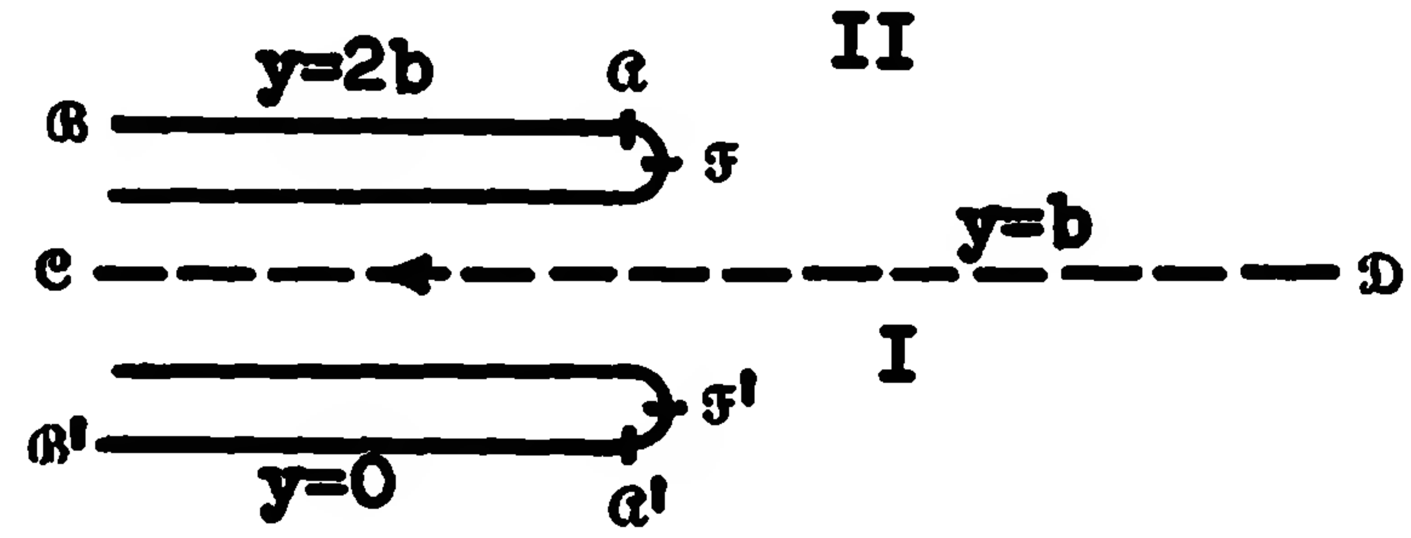
‡ The figures in the examples (i) and (ii) are reprinted by permission from HYDRODYNAMICS by H. Lamb, published by the Cambridge University Press (1932); see p.p. 97, 99. For details of the transformations in (i), (ii) and (iii), and for the figures to (iii), see Lamb also.

w - plane	z - plane
	

w - plane

$\mathcal{F}'$: $w = \frac{b}{2\pi}\log 2 - \frac{bi}{2}$

$\mathcal{F}$: $w = \frac{b}{2\pi}\log 2 + \frac{bi}{2}$

On $\mathcal{A}'\mathcal{C}$: $1 \geq \tau > 0$

On $\mathcal{A}\mathcal{C}$: $-1 \leq \tau < 0$

z - plane

$\mathcal{A}'$: $z = 0$. $\mathcal{A}$: $z = 2bi$.

$\mathcal{F}'$: $\frac{b}{2\pi}(1 - \log 2 + \frac{\pi i}{2} - i)$

$\mathcal{F}$: $\frac{b}{2\pi}(1 - \log 2 + \frac{7\pi i}{2} + i)$

I = region bounded by $\mathcal{B}'\mathcal{A}'\mathcal{F}'\mathcal{C}$ and $y = b$.

II = region bounded by $\mathcal{B}\ \mathcal{A}\ \mathcal{F}\ \mathcal{C}$ and $y = b$.

Transformations:

$$\boxed{w = \frac{b}{\pi}\log\tau - \frac{ib}{2}}$$

$$\boxed{z = -\frac{b}{\pi}\left\{\tau^2 - \log\tau + \tau(\tau^2-1)^{1/2} - \cosh^{-1}\tau - 1\right\}}$$

Curve $\mathcal{A}'\mathcal{F}'\mathcal{C}$ in z-plane: $x = \frac{b}{\pi}(\sin^2\frac{1}{2}\theta + \log\cos\frac{1}{2}\theta)$,

$$y = \frac{b}{2\pi}(\theta - \sin\theta);\ 0 \leq \theta < -\pi;$$
$$\frac{\theta}{2} = \cos^{-1}\tau,\ 1 \geq \tau > 0.$$

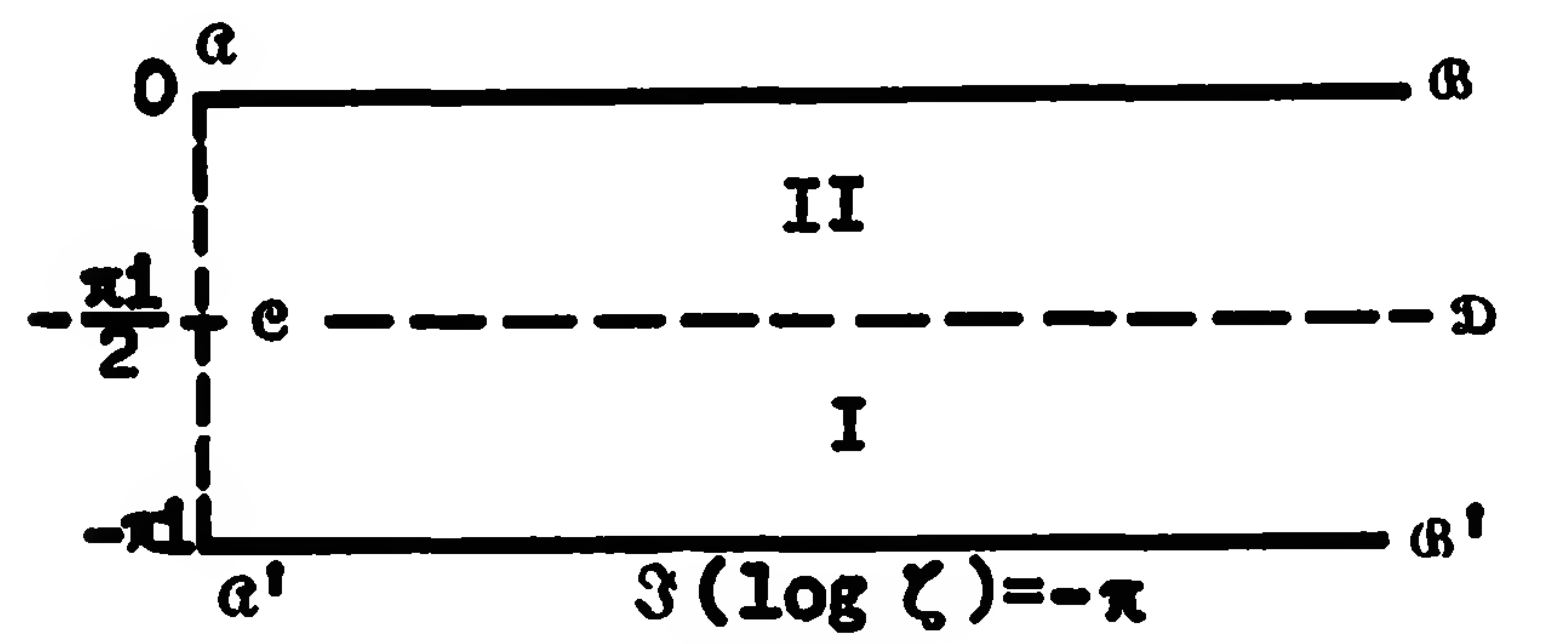
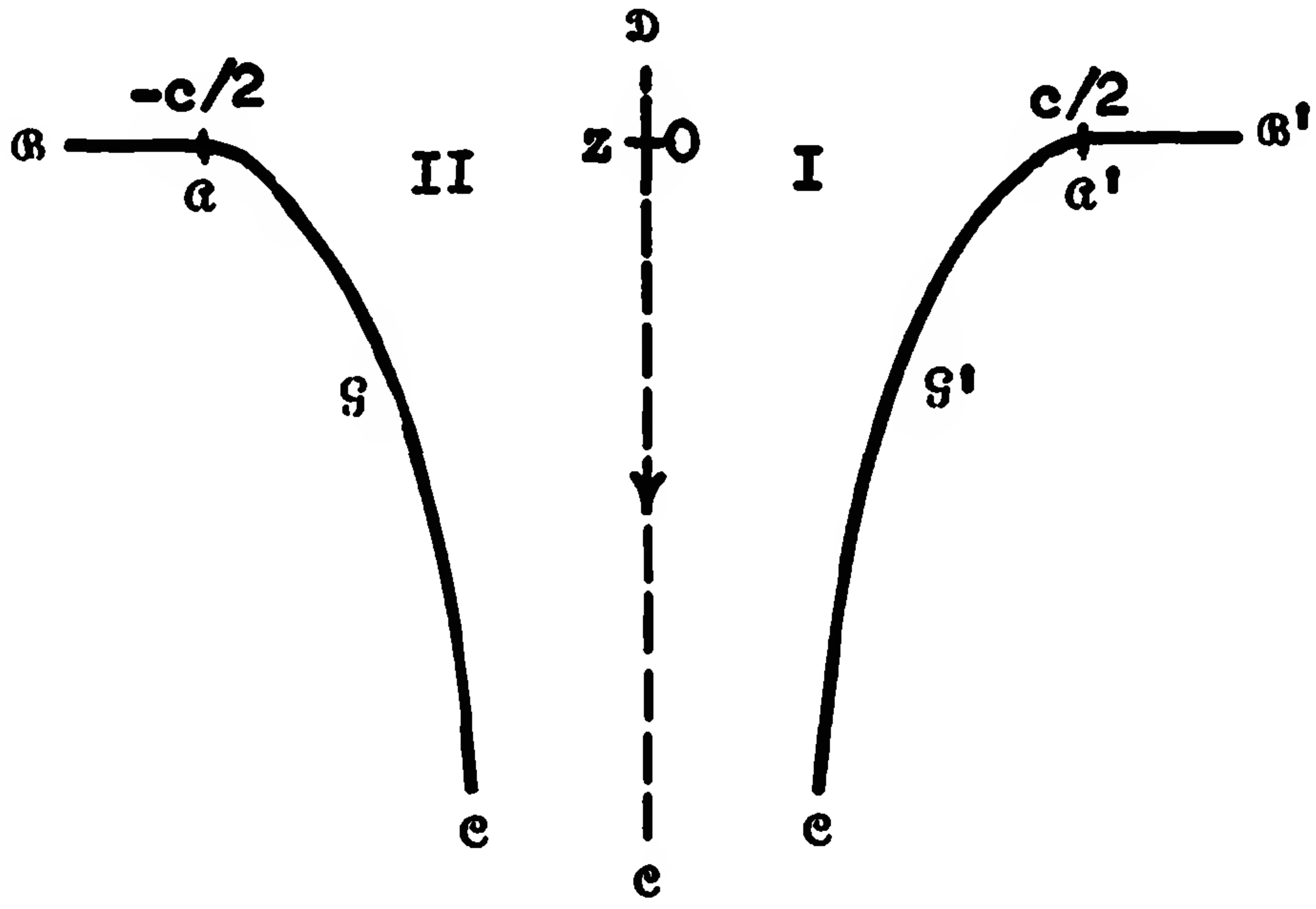
Curve $\mathcal{A}\ \mathcal{F}\ \mathcal{C}$ in z-plane: x as before, $y = 2b + \frac{b}{2\pi}(\theta - \sin\theta)$, $0 \geq \theta > -\pi$.

Asymptotic distance between curves (at $\mathcal{C}$) = b.

Example (ii) Configuration in z-plane: $\mathcal{B}$ ———| $\mathcal{A}$ $\mathcal{A}'$ |——— $\mathcal{B}'$

w-plane, τ-plane as in previous example, but omit $\mathcal{F}$, $\mathcal{F}'$;

$\boxed{w = (b/\pi)\log \tau - ib/2}$ (simple jet).

log ζ - plane	z - plane; $c > 0$, $b = c\pi(\pi+2)^{-1}$
	

Transformations:

$$\log \zeta = \cosh^{-1} \tau - \pi i$$

$$\boxed{z = \frac{b}{\pi}\left(\tau + (\tau^2-1)^{1/2} - \cos^{-1}\frac{1}{\tau} + \frac{\pi}{2}\right)}$$

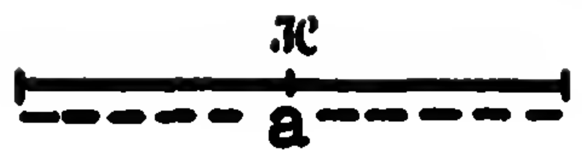
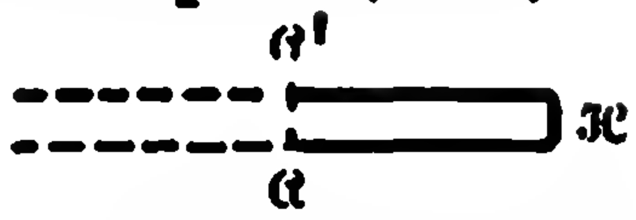
Curve $\mathcal{A}\,\mathcal{G}\,\mathcal{C}$ in z-plane: $x = \frac{2b}{\pi}\sin^2 \frac{1}{2}\theta - \frac{c}{2}$,

$$y = \frac{b}{\pi}\left\{\log \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) - \sin\theta\right\}, \quad 0 \geq \theta > -\frac{\pi}{2};$$

Curve $\mathcal{A}'\,\mathcal{G}'\,\mathcal{C}$ in z-plane: $x = -\frac{2b}{\pi}\sin^2 \frac{1}{2}\theta + \frac{c}{2} = \frac{b}{\pi}\cos\theta + \frac{b}{2}$,

$$y = \frac{b}{\pi}\left\{\log \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \sin\theta\right\}, \quad 0 \leq \theta < \frac{\pi}{2};$$

$\cos\theta = \tau$, $1 \geq \tau > 0$; asymptotic distance between curves (at $\mathcal{C}$) = b.

Example (iii) Configuration in z-plane:  α, in w-plane:  :plane barrier

Transformation $w = f(z)$:

$$\frac{z}{b} = \cos^{-1}\frac{1}{\tau} - \frac{(\tau^2-1)^{1/2}}{\tau^2} - \frac{2}{\tau} - \frac{\pi}{2}\,; \qquad w = -\frac{b}{\tau^2}, \text{ where } b(\pi+4) = a > 0.$$

z - plane	w - plane
points $z = 0$ (i.e. $\varkappa$); $\frac{1}{2}a$ (α); $-\frac{1}{2}a$ (α')	points $w = 0$; $-b$; $-b$.
(1) half-line $x = 0$, $\infty > y \geq 0$	half-line $v = 0$, $\infty > u \geq 0$
(2) line-segment $y = 0$, $0 \leq x \leq \frac{1}{2}a$ (2') line-segment $y = 0$, $0 \geq x \geq -\frac{1}{2}a$	line-segment $v = 0$, $0 \geq u > -b$.
(3) curve $x = 2b(\operatorname{cosec}\theta + \frac{1}{4}\pi) = x(\theta)$, $y = -b(\cos\theta\operatorname{cosec}^2\theta + \log\tan\frac{1}{2}\theta) = y(\theta)$ (3') curve $x = -x(\theta)$, $y = y(\theta)$, curves start at α or α', respectively; $\frac{1}{2}\pi \geq \theta > 0$; $\sin\theta = \mp\tau$	half-line $v = 0$, $-b \geq u > -\infty$.
region bounded by (1), (2), and (3)	half-plane $v < 0$.
region bounded by (1), (2') and (3')	half-plane $v > 0$.

PART FIVE

HIGHER TRANSCENDENTAL FUNCTIONS

The number of transformations which are representable in terms of higher transcendental functions is enormous. Only some fundamental transformations of this kind are discussed in part five.‡

Notations

The same notations are used as by E. Jahnke and F. Emde; except for the following notations:

1) The inverse function of $w = \wp z = \wp(z;g_2,g_3)$ is, according to the classical notation,

$$z = \int_{\infty}^{w} \frac{ds}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}} = \int_{\infty}^{w} \frac{ds}{\sqrt{4s^3-g_2s-g_3}} .$$

2) The periods of $\wp z$ are written $2\omega_1$, $2\omega_2$; while $\omega_3 = -\omega_1-\omega_2$; so that $\wp\omega_1 = e_1$, $\wp\omega_2 = e_2$, $\wp\omega_3 = e_3$.

3) The notation $\zeta(z)$ is used instead of ζz.

4) $\vartheta_0(z|\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \cos 2\pi nz$, where $q=e^{i\pi\tau}$, $\Im(\tau) > 0$.

Similar are the notations $\vartheta_1(z|\tau)$, $\vartheta_2(z|\tau)$, $\vartheta_3(z|\tau)$.

5) $\Theta_0(\tau) = \vartheta_0(0|\tau)$, $\Theta_2(\tau) = \vartheta_2(0|\tau)$, $\Theta_3(\tau) = \vartheta_3(0|\tau)$.

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‡A list of further transformations is given in the Hanson thesis which Professor L. Rosenhead was kind enough to lend to the author.

For altitude Charts of the Bessel function $J_\nu(z)$ for some real values of ν see J. Lense.

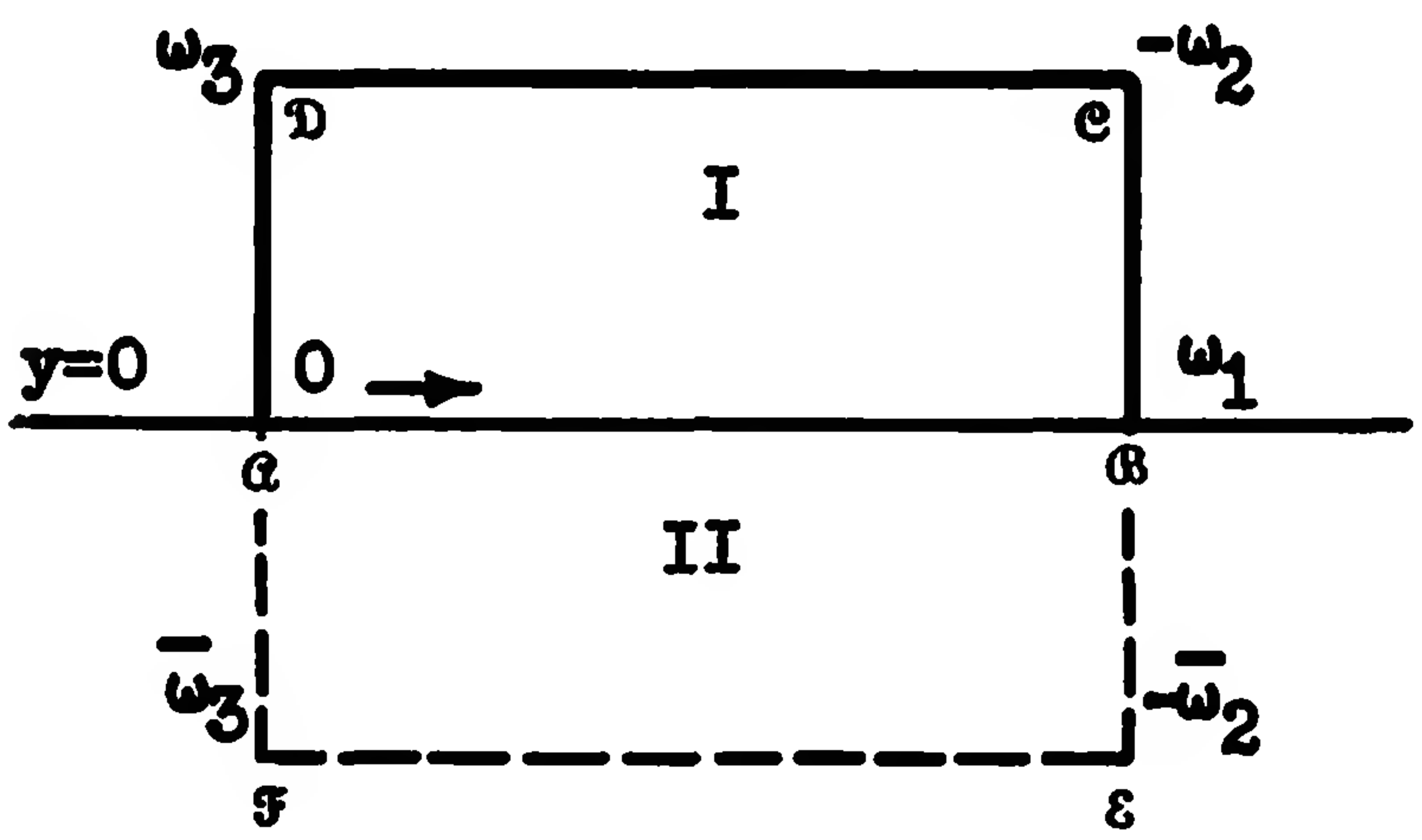
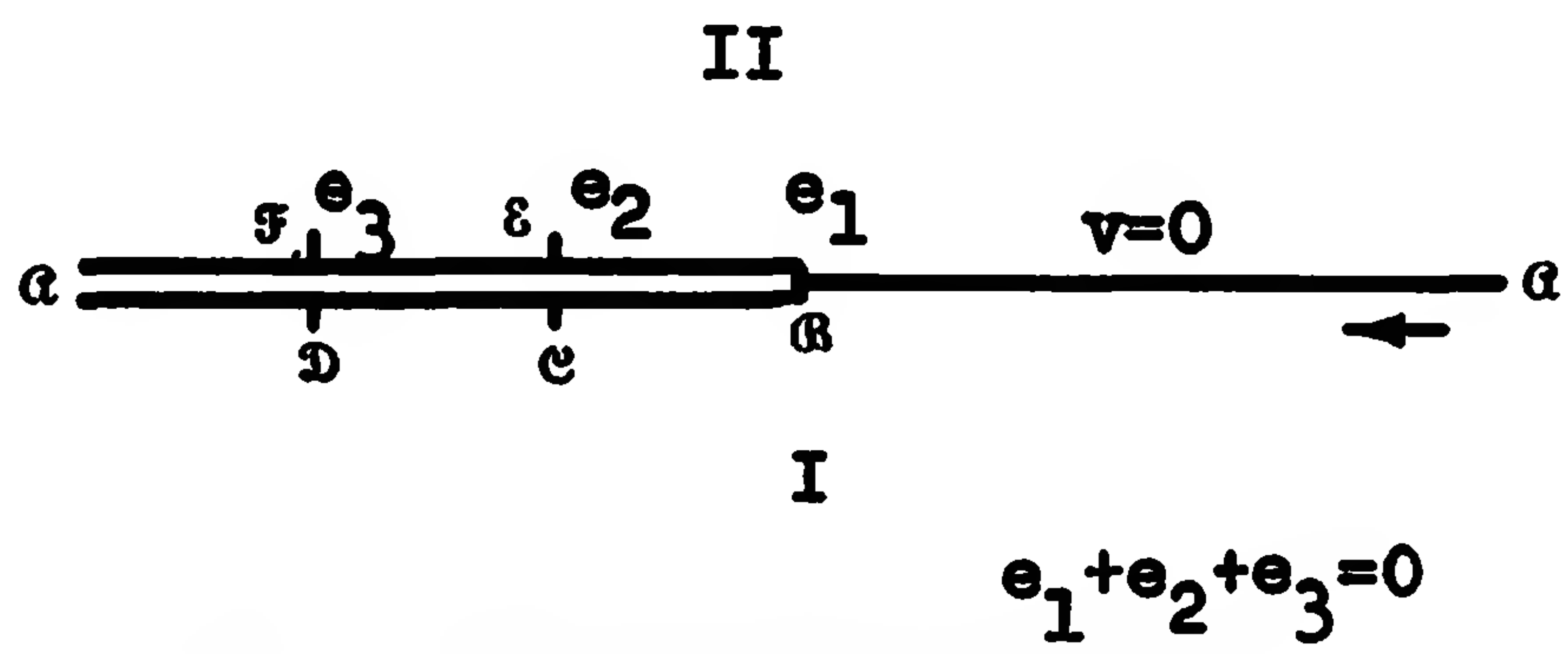
13. ELLIPTIC FUNCTIONS.

13.1 Interior of rectangle on half-plane.

$$\boxed{w = \wp z} = e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2(z\sqrt{e_1-e_3}, k)} \qquad \left[k^2 = \frac{e_2-e_3}{e_1-e_3}\right]$$

For the relief of $w = \wp z$, with $\omega_1 = 2$, $\omega_3 = 1.751$, see Jahnke-Emde, p. 99; for the equianharmonic case $w = \wp(z;0,1)$ see pp. 100-104, for the lines u = constant, v = constant see Tricomi p. 31, fig. 8.

(1) Given: Rectangle, with vertices at $0, \omega_1, -\omega_2, \omega_3$; $\omega_1 > 0$, $\frac{\omega_3}{i} > 0$,
$\omega_1 + \omega_2 + \omega_3 = 0$.

z - plane	w - plane
	

Transformation $w = \wp z$ represented in the Schwarz-Christoffel form:

$$\boxed{z = \int_\infty^w \frac{dt}{\sqrt{4(t-e_1)(t-e_2)(t-e_3)}} = \int_\infty^w \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}}}, \quad \text{where}$$

$$e_1 = \frac{1}{3}\left(\frac{\pi}{2\omega_1}\right)^2 (\Theta_3^4(\tau) + \Theta_0^4(\tau)), \qquad e_2 = \frac{1}{3}\left(\frac{\pi}{2\omega_1}\right)^2 (\Theta_2^4(\tau) - \Theta_0^4(\tau)),$$

$$e_3 = -\frac{1}{3}\left(\frac{\pi}{2\omega_1}\right)^2 (\Theta_2^4(\tau) + \Theta_3^4(\tau)); \qquad \tau = \frac{\omega_3}{\omega_1}\,[\, i\tau < 0;\ -\infty < e_3 < e_2 < e_1 < \infty].$$

$$g_2 = \frac{2}{3}\left(\frac{\pi}{2\omega_1}\right)^4 (\Theta_2^8(\tau) + \Theta_3^8(\tau) + \Theta_0^8(\tau)), \quad g_3 = 4e_1e_2e_3.$$

Transformation represented <u>in terms of θ-series</u>:

$$w = \wp z = e_1 + (\mu\vartheta_2(\xi|\tau))^2 = e_2 + (\mu\vartheta_3(\xi|\tau))^2 = e_3 + \left(\mu \frac{\Theta_2(\tau)}{\Theta_0(\tau)} \vartheta_0(\xi|\tau)\right)^2,$$

where $\tau = \frac{\omega_3}{\omega_1}$, $v = \frac{z}{2\omega_1}$, $\mu = \frac{\vartheta_1'(\tau)}{2\omega_1\Theta_2(\tau)\vartheta_1(\xi|\tau)}$.

Or

(2) <u>Given</u>: e_1, e_2, e_3 (real; $e_3 < e_2 < e_1$, $e_1 + e_2 + e_3 = 0$).

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad k'^2 = \frac{e_1 - e_2}{e_1 - e_3};$$

$$\omega_1 = \frac{K}{\sqrt{e_1 - e_3}} \ (\omega_1 > 0); \quad \omega_3 = \frac{iK'}{\sqrt{e_1 - e_3}} \ \left(\frac{\omega_3}{i} > 0\right),$$

where

$$K = \int_0^1 \frac{dt}{\sqrt{1-t^2}\ \sqrt{1-k^2t^2}} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

$$= \frac{\pi}{2}\left\{1 + \sum_{n=1}^{\infty} \left(\frac{1.3.5 \ldots (2n-1)}{2.4.6 \ldots 2n}\right)^2 k^{2n}\right\} = F(k, \tfrac{\pi}{2}).$$

$$K' = \int_1^{1/k} \frac{dt}{\sqrt{(-1+t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k'^2\sin^2\varphi}}$$

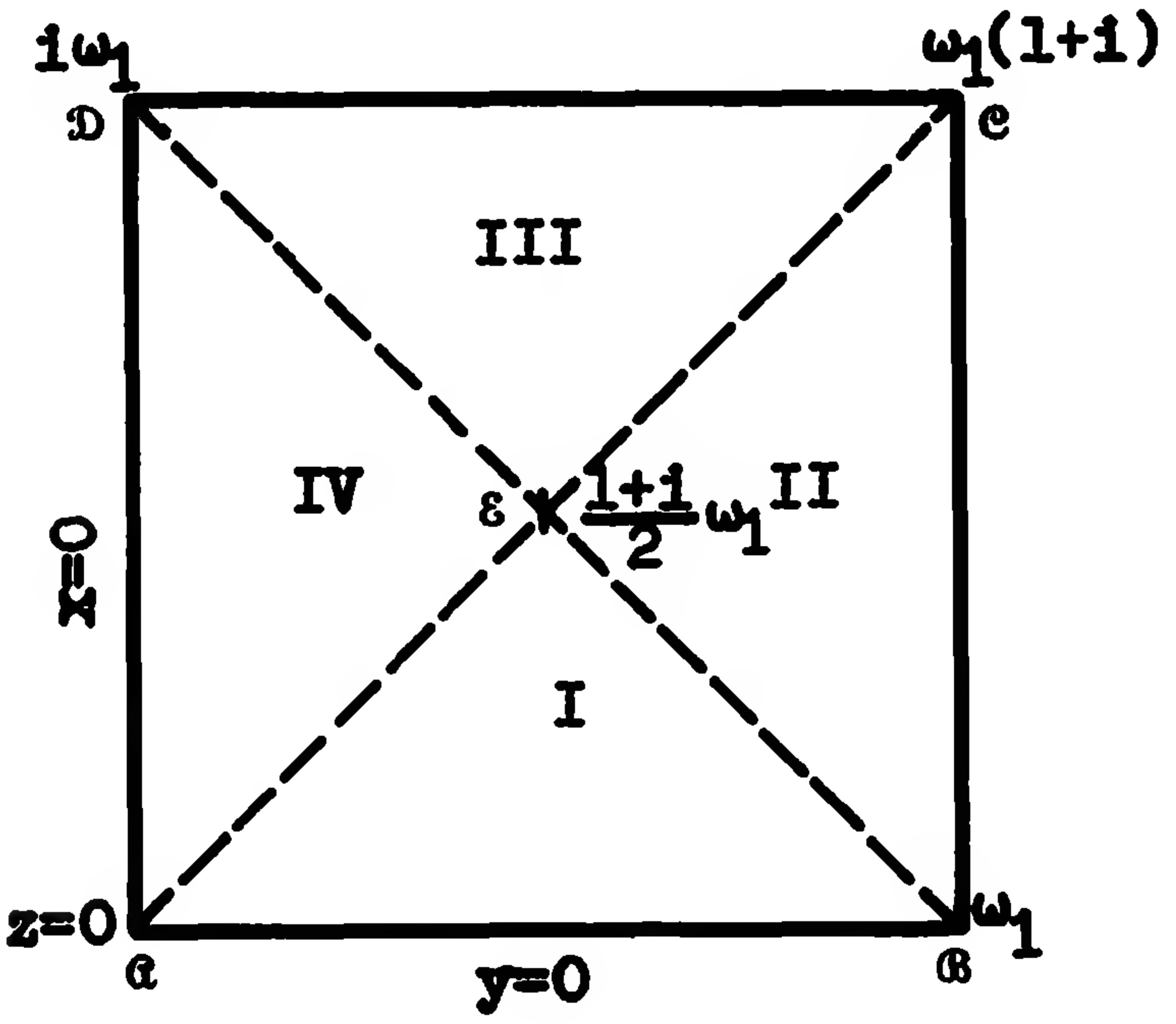
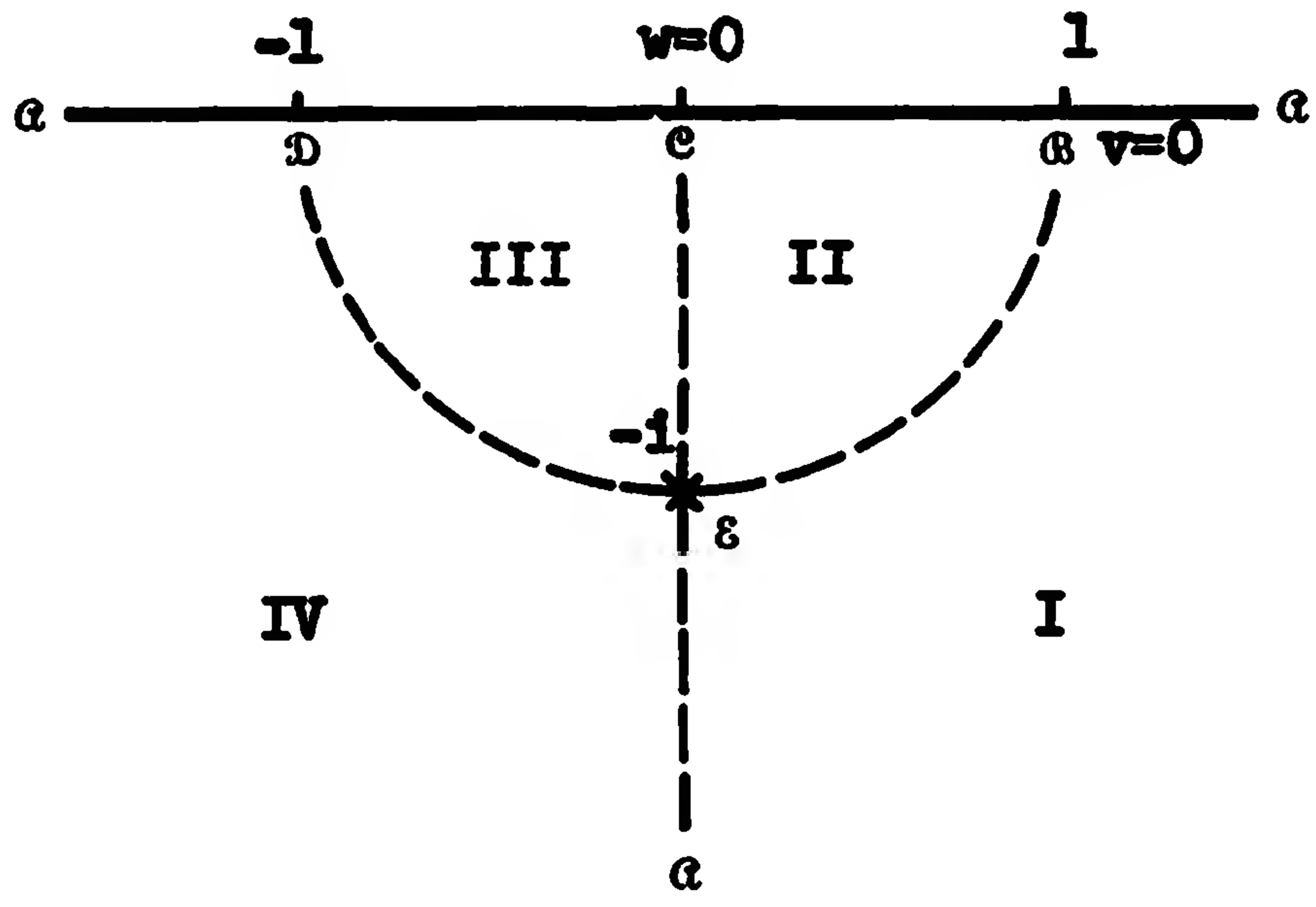
$$= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}} = F(k', \tfrac{\pi}{2}).$$

Transformation as above.

<u>Square on half-plane; triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{4}$.</u>

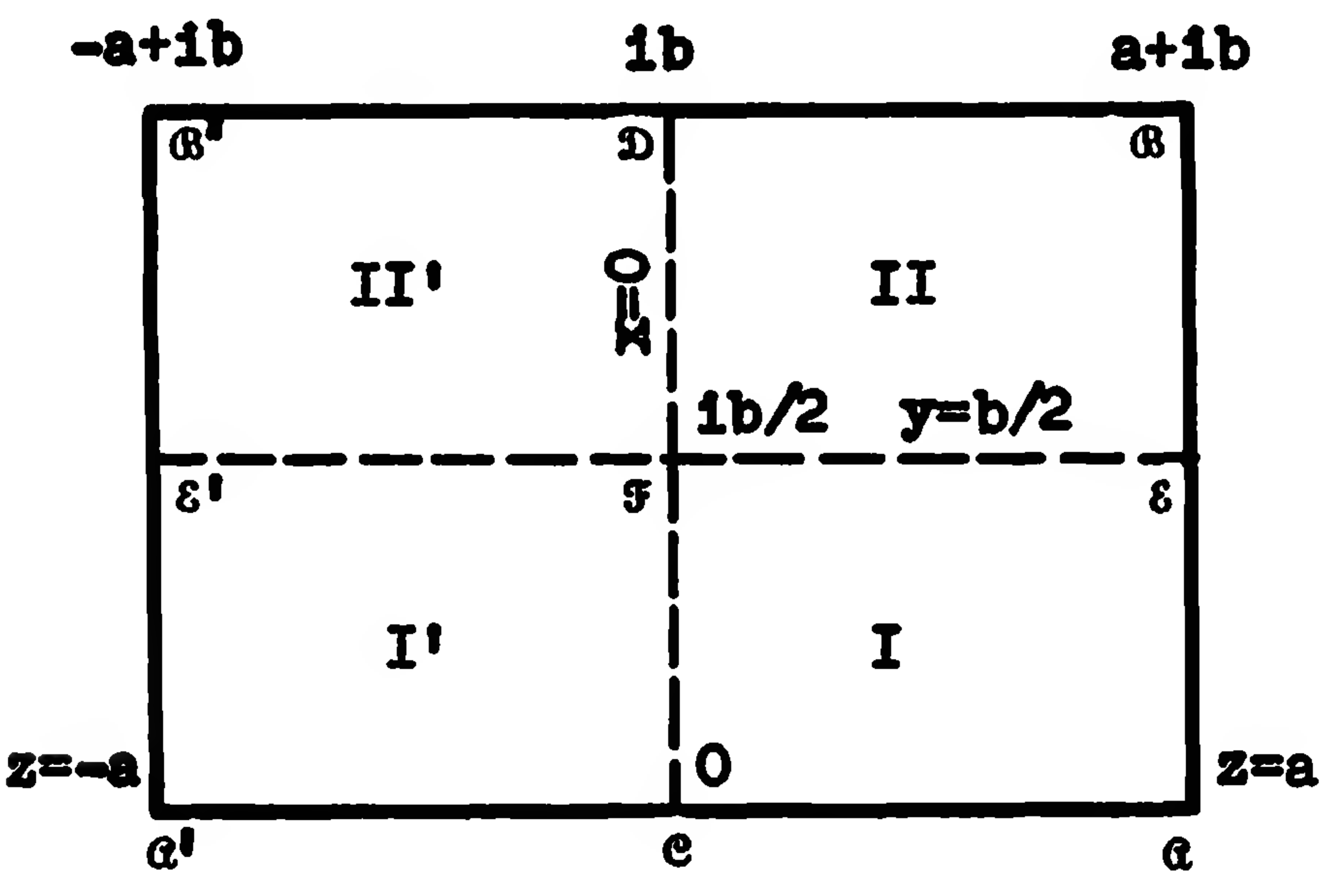
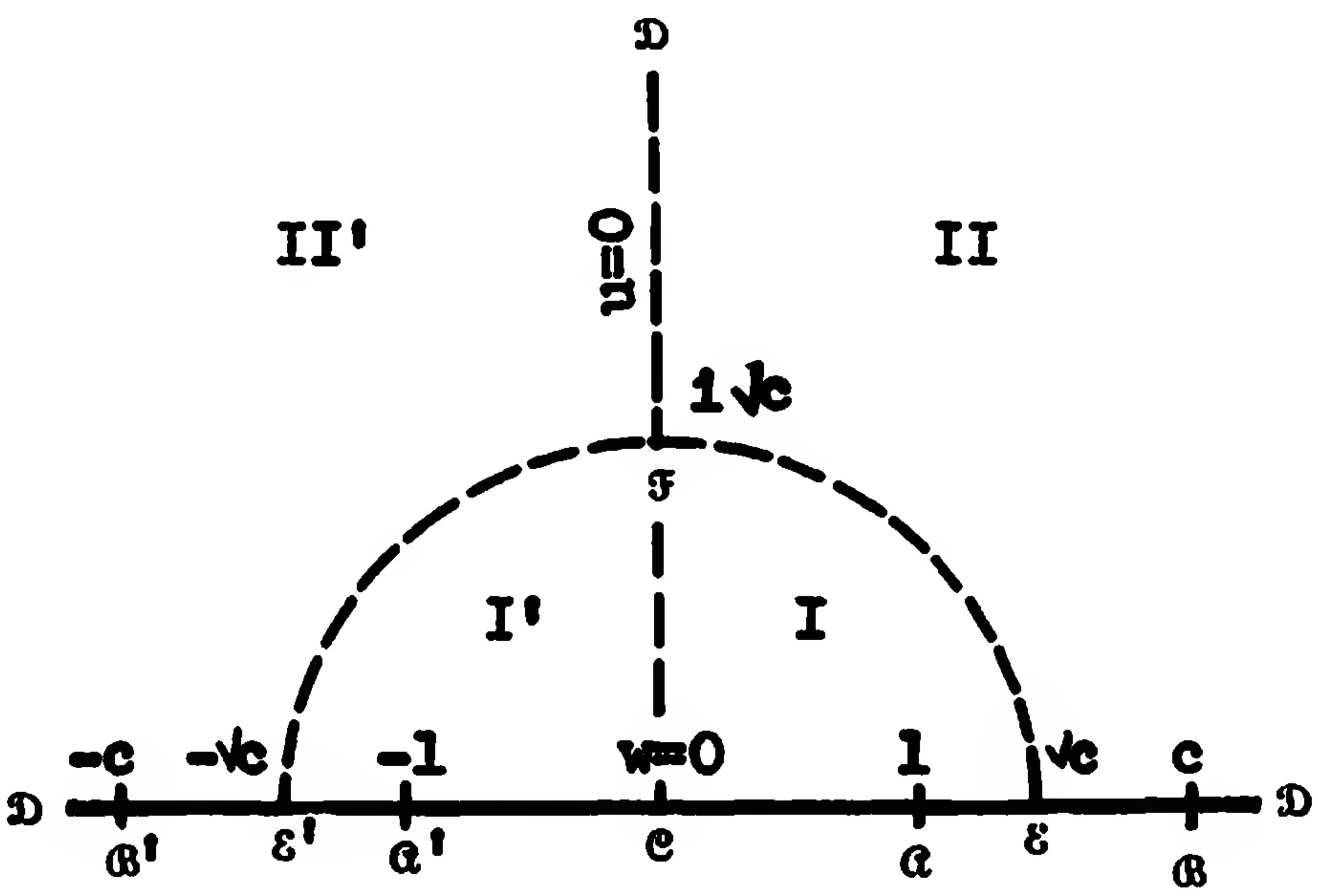
$w = \wp(z;4,0)$, i.e. $g_2 = 4$, $g_3 = 0$; $e_1 = -e_3 = 1$, $e_2 = 0$,

$$\omega_1 = \int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{\pi}{2\sqrt{2}} \left\{1 + \sum_{n=1}^{\infty} \left(\frac{1.3 \ldots (2n-1)}{2.4 \ldots 2n}\right)^2 2^{-n}\right\}, \quad \omega_3 = i\omega_1.$$

z - plane	w - plane
	

The triangles with angles $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}; \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}; \frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{6}; \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$ (cf. §13.7, 13.8, 13.9) represent the only cases where a rectilinear triangle in the z-plane is mapped on a half-plane by an analytic function $w = f(z)$ which has a finite number of values for any z.

13.2 Rectangle on quarter of plane or on half-plane.

z - plane	w - plane
	
interior of rectangle $\mathcal{A}\,\mathcal{B}\,\mathcal{D}\,\mathcal{C}$	quadrant $u > 0$, $v > 0$.
interior of rectangle $\mathcal{A}\,\mathcal{B}\,\mathcal{B}'\,\mathcal{A}'$	half-plane $v > 0$.
line-segment $y = b/2$, $-a \leq x \leq a$	semi-circle $\lvert w \rvert = \sqrt{c}$, $v \geq 0$.

(1) <u>Given: a, b, both positive</u>.

$$\tau = \frac{ib}{a}; \quad k = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)}\right)^2 = \sqrt{\lambda(\tau)} \quad (\text{cf. §14.1}), \quad 0 < k < 1; \quad c = \frac{1}{k}.$$

$$\alpha = \frac{a}{K}, \quad \text{where } K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad (\text{cf. §13.1}).$$

Transformation required:

$$\boxed{z = \alpha \int_0^w \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}}, \qquad \boxed{w = \mathrm{sn}(\tfrac{z}{\alpha}, k) = \mathrm{sn}\,\frac{z}{\alpha}}\ .$$

(2) <u>Given c (c > 1).</u>

$$k = \frac{1}{c}; \quad K \text{ as above}, \quad a = K; \quad b = K' = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}},$$

where $k'^2 = 1 - k^2$.

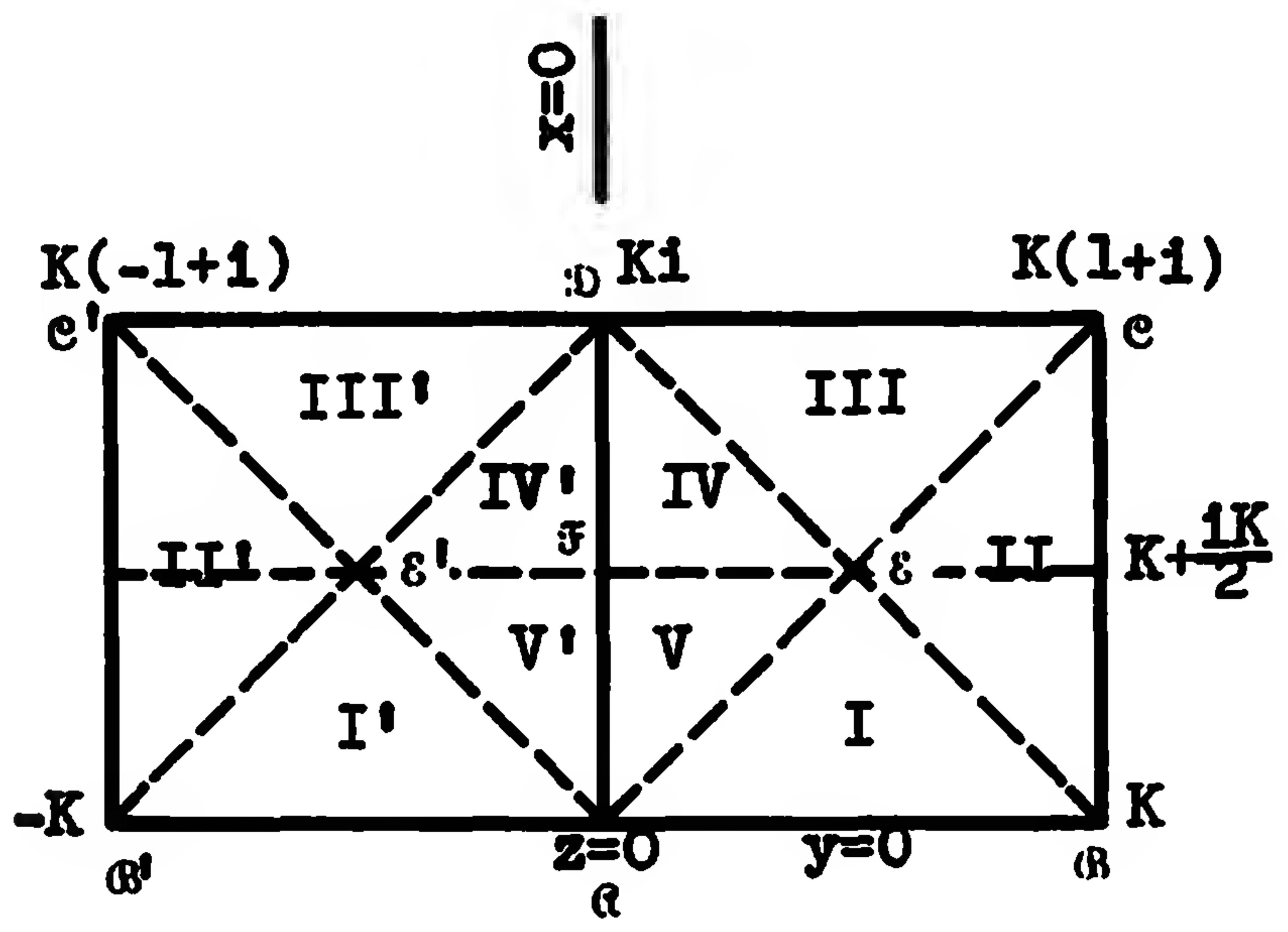
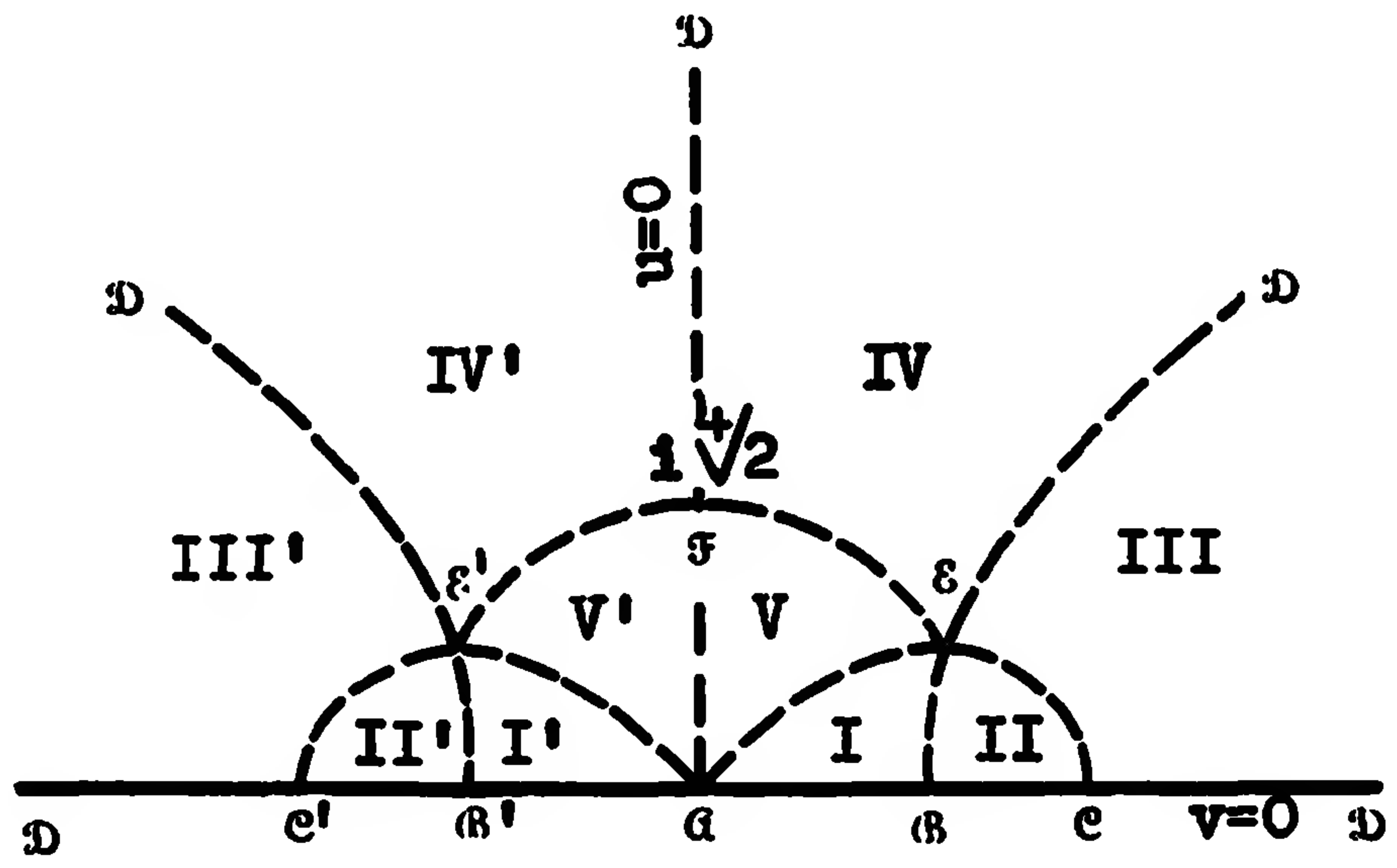
Transformation required: $\boxed{w = \mathrm{sn}(z,k) = \mathrm{sn}\ z.}$

$$\boxed{w = \mathrm{sn}^2(z,k)}$$

z - plane	w - plane
interior of rectangle, with vertices at $z = 0$, K, $K+iK'$, iK'	half-plane $v > 0$.

Square on quarter of plane.

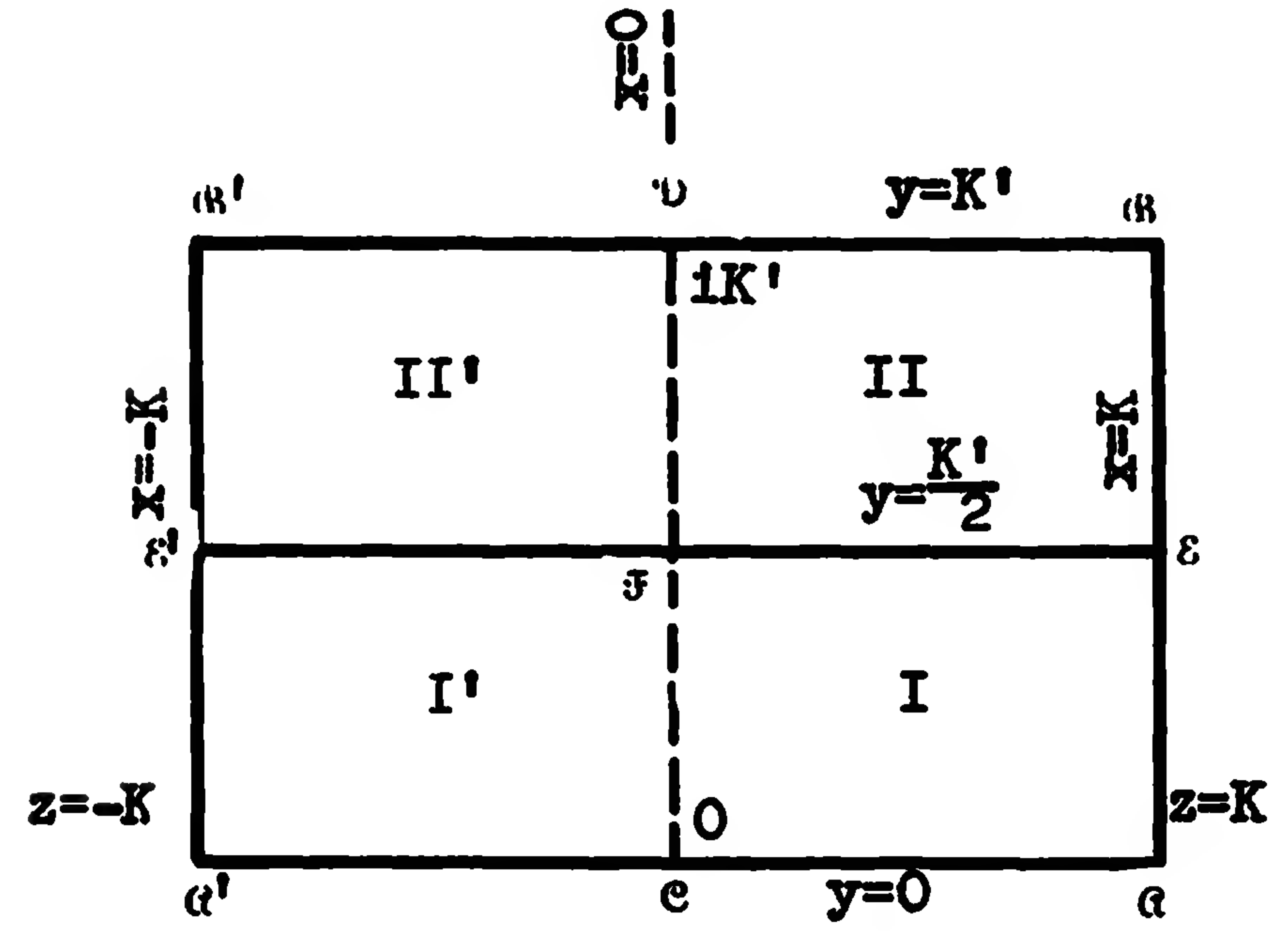
$$w = \mathrm{sn}\left(z, \frac{1}{\sqrt{2}}\right); \quad K = K' = 1.8541.$$

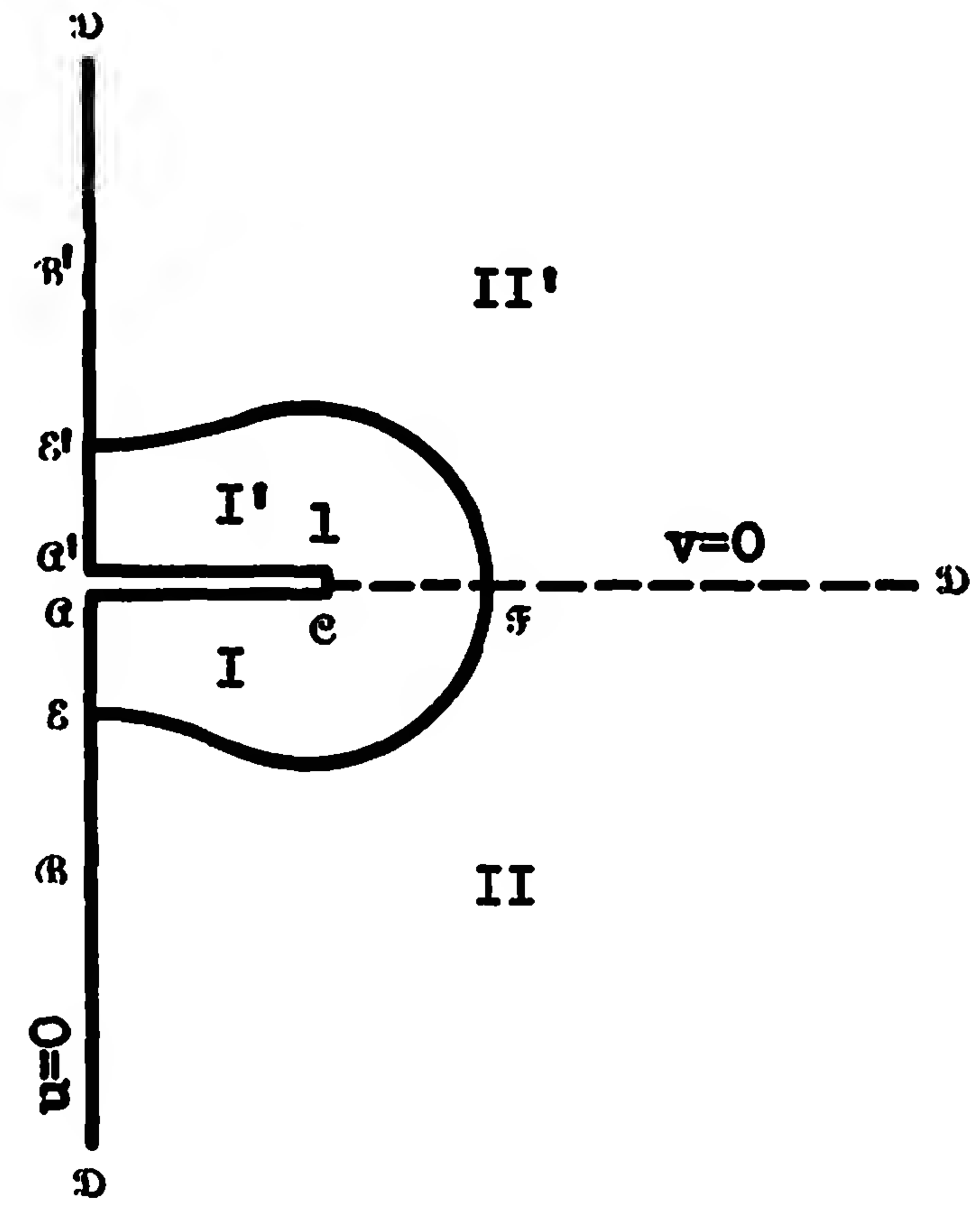
z - plane	w - plane
	
points $z = 0$ (i.e. $\mathfrak{A}$); K ($\mathfrak{B}$); $(1+i)K$ ($\mathfrak{C}$); iK ($\mathfrak{D}$); $\frac{1+i}{2}K$ ($\mathfrak{E}$); $i\frac{K}{2}$ ($\mathfrak{F}$)	points $w = 0$ ($\mathfrak{A}$); 1 ($\mathfrak{B}$); $\sqrt{2}$ ($\mathfrak{C}$); ∞ ($\mathfrak{D}$); $\sqrt[4]{2}\, e^{i\pi/8}$ ($\mathfrak{E}$); $i\sqrt[4]{2}$ ($\mathfrak{F}$).
line-segment $y = \frac{1}{2}K$, $-K \leq x \leq K$	semi-circle $\lvert w \rvert = \sqrt[4]{2}$, $v \geq 0$.
segment $x > 0$, $y > 0$ of $x+y = 1$ (i.e. $\mathfrak{B}\,\mathfrak{E}\,\mathfrak{D}$)	part $u > 0$, $v > 0$ of hyperbola $u^2 - v^2 = 1$.
segment $x < 0$, $y > 0$ of $y-x = 1$ ($\mathfrak{B}'\,\mathfrak{E}'\,\mathfrak{D}$)	part $u < 0$, $v > 0$ of the same hyperbola.
segment $0 < x < K$ of $y = x$ ($\mathfrak{A}\,\mathfrak{E}\,\mathfrak{C}$)	part $u > 0$, $v > 0$ of lemniscate $\lvert w-1 \rvert \lvert w+1 \rvert = 1$
segment $0 > x > -K$ of $y = -x$ ($\mathfrak{A}\,\mathfrak{E}'\,\mathfrak{C}'$)	part $u < 0$, $v > 0$ of the same lemniscate

For the relief of $w = \mathrm{sn}(z, 0.8)$ see Jahnke-Emde, p. 92; for the lines u = constant, v = constant at $w = \mathrm{sn}(z; \sqrt{0.2})$ see Tricomi, p. 108, fig. 22. For the altitude chart of the complete elliptic integral K as a function of $\lambda = k^2$ see Jahnke-Emde, p. 74, for its relief p. 75.

13.3 $\boxed{w = \mathrm{cn}\, z} = \sqrt{1-\mathrm{sn}^2\, z}$; $z = \int_w^1 \frac{dt}{\sqrt{(1-t^2)(k'^2+k^2t^2)}}$; $k'^2+k^2 = 1$.

z - plane	(Cassinian) w - plane





points $z = 0$ (i.e. 𝒞); K (𝒜);

$-K$ (𝒜'); iK' (𝒟); $K+iK'$ (ℬ);

$-K+iK'$ (ℬ'); $iK'/2$ (ℱ);

$K+iK'/2$ (ℰ);

$-K+iK'/2$ (ℰ')

line-segment $y = K'/2$,

$-K \leq x \leq K$

points $w = 1$ (𝒞); 0 (𝒜);

0 (𝒜'); ∞ (𝒟); $-ik'/k$ (ℬ);

ik'/k (ℬ'); $(k+1)^{1/2}k^{-1/2}$ (ℱ);

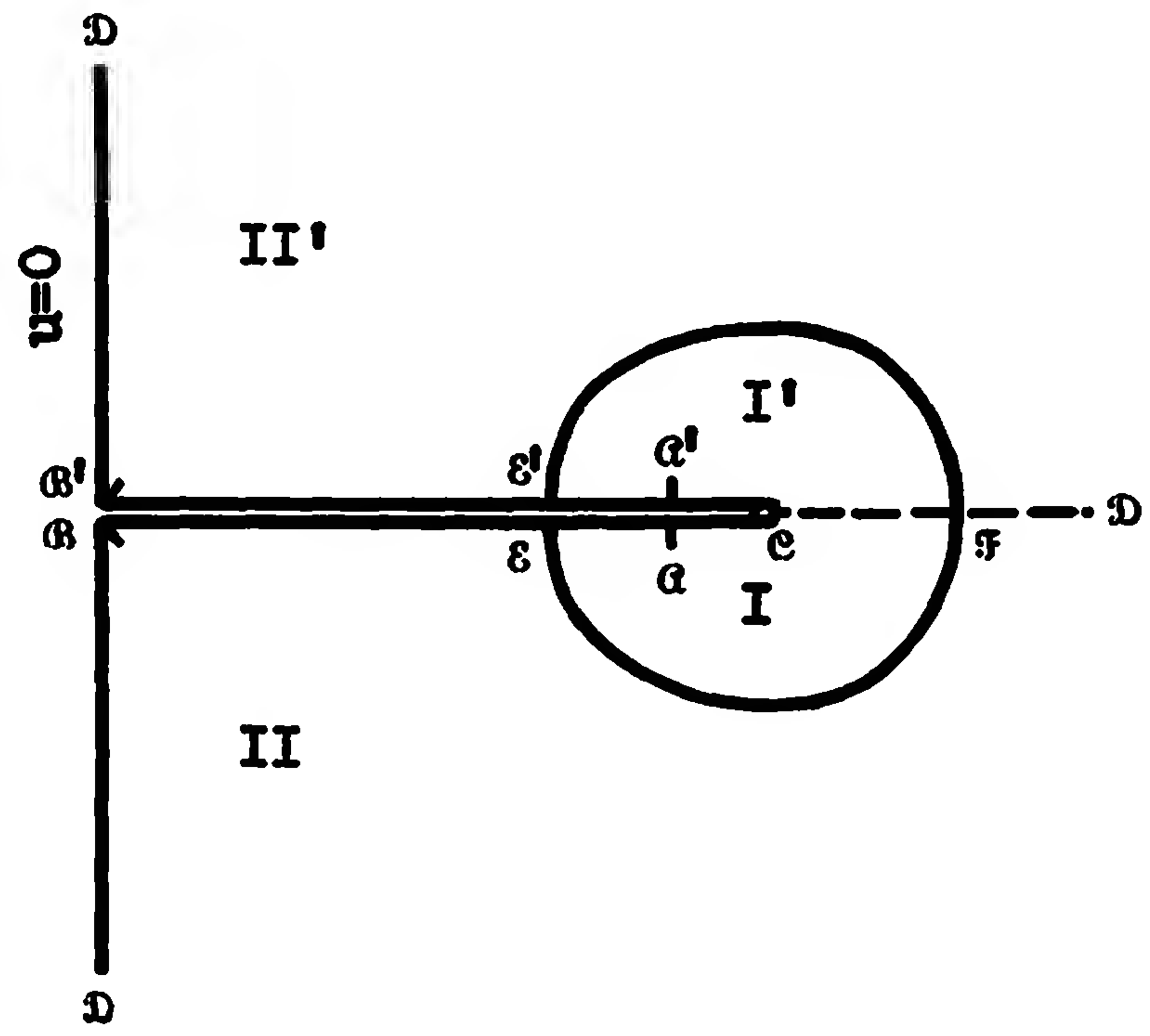
$-i(1-k)^{1/2}k^{-1/2}$ (ℰ);

$i(1-k)^{1/2}k^{-1/2}$ (ℰ').

part $u \geq 0$ of Cassinian

$|w+1||w-1| = k^{-1}$.

$$\boxed{w = \mathrm{dn}\, z} = \sqrt{1-k^2\, \mathrm{sn}^2 z}\,; \quad z = \int_w^1 \frac{dt}{\sqrt{(1-t^2)(t^2-k'^2)}}\,, \quad k'^2+k^2 = 1.$$

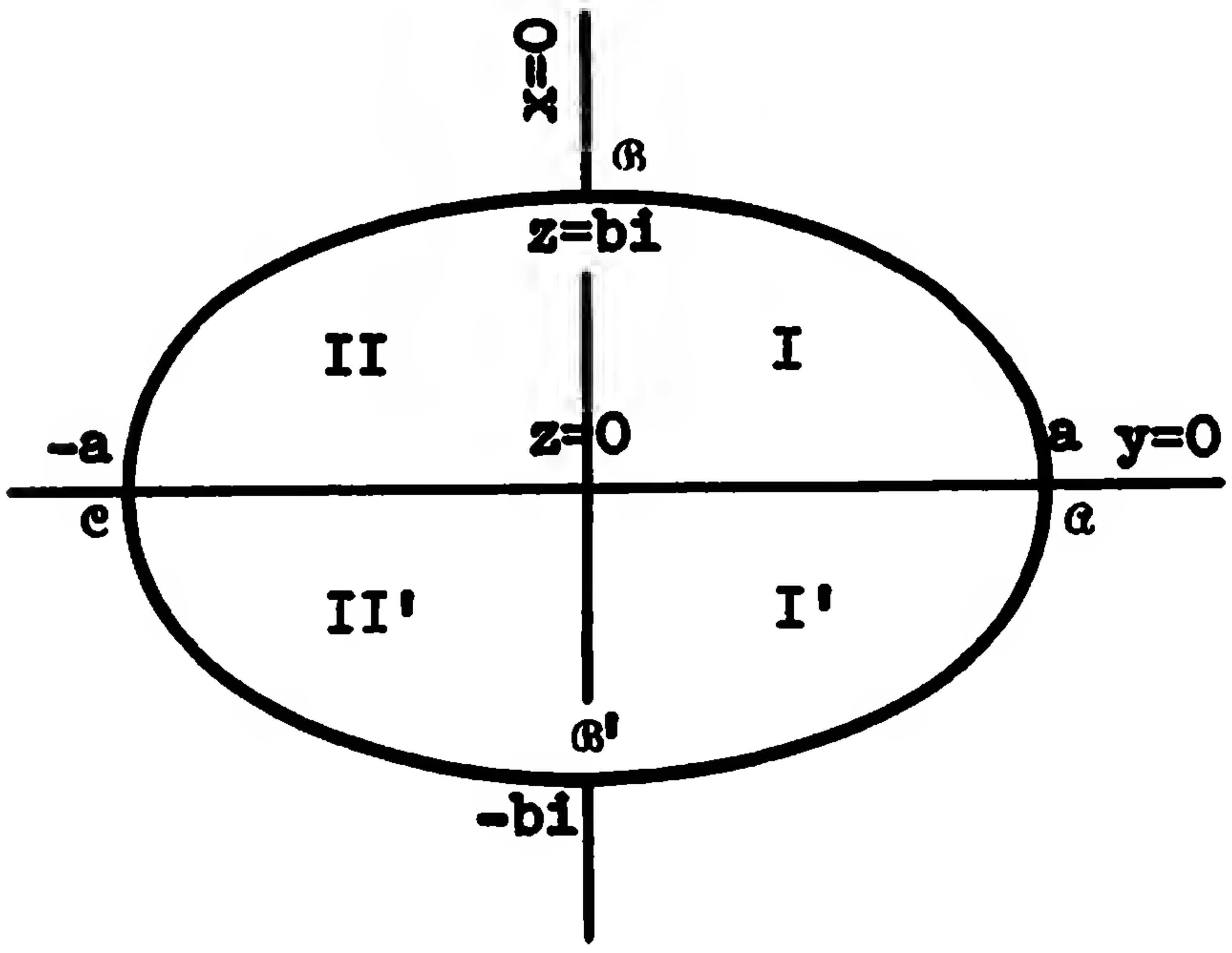
z - plane	w - plane
Figure of rectangle as at $w = \mathrm{cn}\, z$.	
points $z = 0$ (i.e. ℭ); K (𝔄);	points $w = 1$ (i.e. ℭ); k' (𝔄);
$K+iK'/2$ (𝔈); $K+iK'$ (𝔅);	$\sqrt{1-k}$ (𝔈); 0 (𝔅);
iK' (𝔇); $-K+iK'$ (𝔅');	∞ (𝔇); 0 (𝔅');
$-K+iK'/2$ (𝔈'); $-K$ (𝔄');	$\sqrt{1-k}$ (𝔈'); k' (𝔄');
$iK'/2$ (𝔉)	$\sqrt{1+k}$ (𝔉).
line-segment $y = K'/2$, $-K \leq x \leq K$	part $u \geq 0$ of Cassinian $\lvert w+1\rvert\lvert w-1\rvert = k$.

For the reliefs of $w = \mathrm{cn}\,(z, 0.8)$ and $w = \mathrm{dn}\,(z, 0.8)$ see Jahnke-Emde, p. 93.

13.4 <u>Interior of ellipse on interior of circle</u> (cf. §8.3).

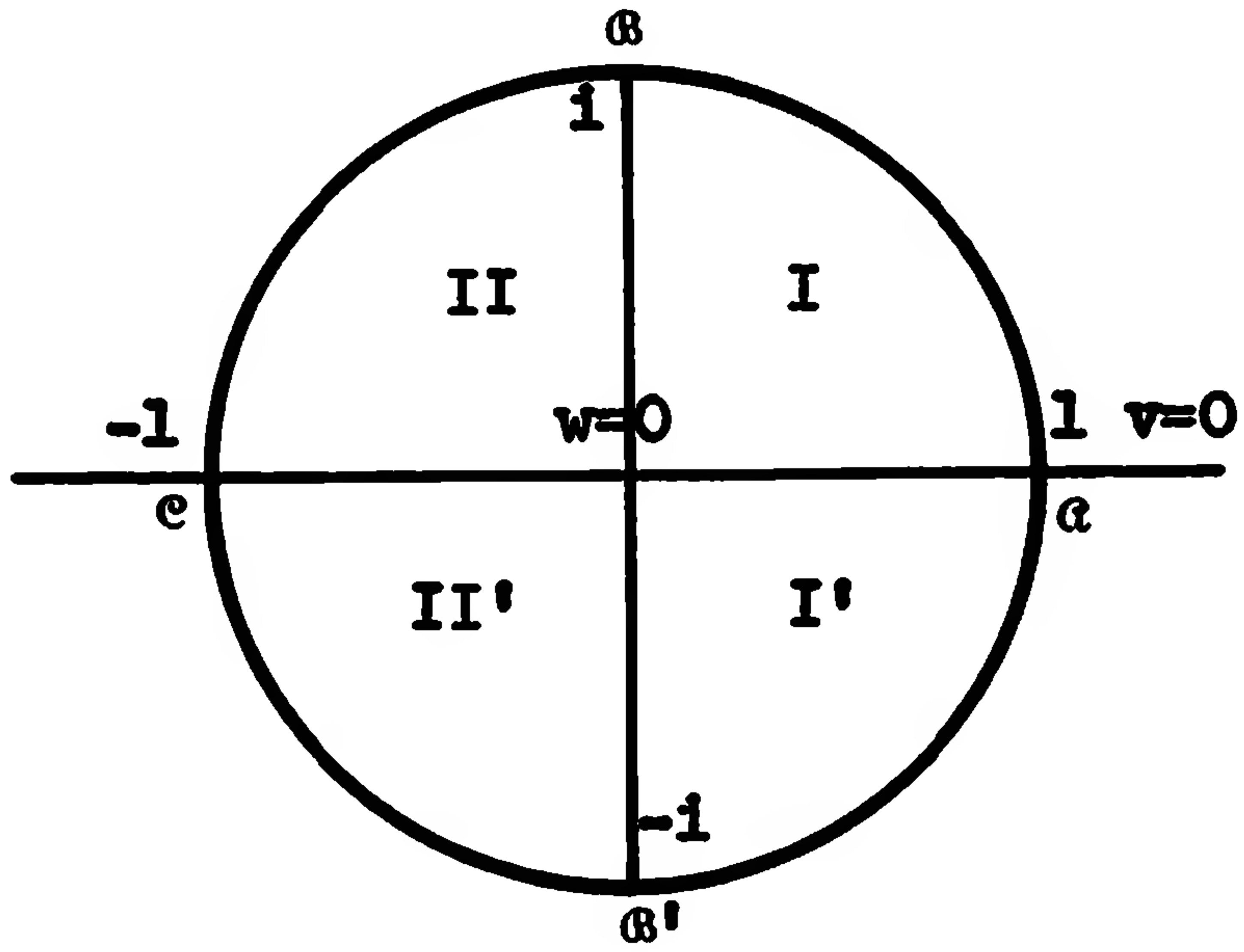
Given: Semi-axes of ellipse a, b $(a > b)$.

z - plane	w - plane



points $z = 0$; $\sqrt{a^2-b^2}$; $-\sqrt{a^2-b^2}$;
a; bi

ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



points $w = 0$; $\sqrt{k}$; $-\sqrt{k}$;
1; i

circle $|w| = 1$.

Transformation required:

$$\boxed{w = \sqrt{k}\ \operatorname{sn}\left(\frac{2K}{\pi}\sin^{-1}\frac{z}{\sqrt{a^2-b^2}}\right)}$$

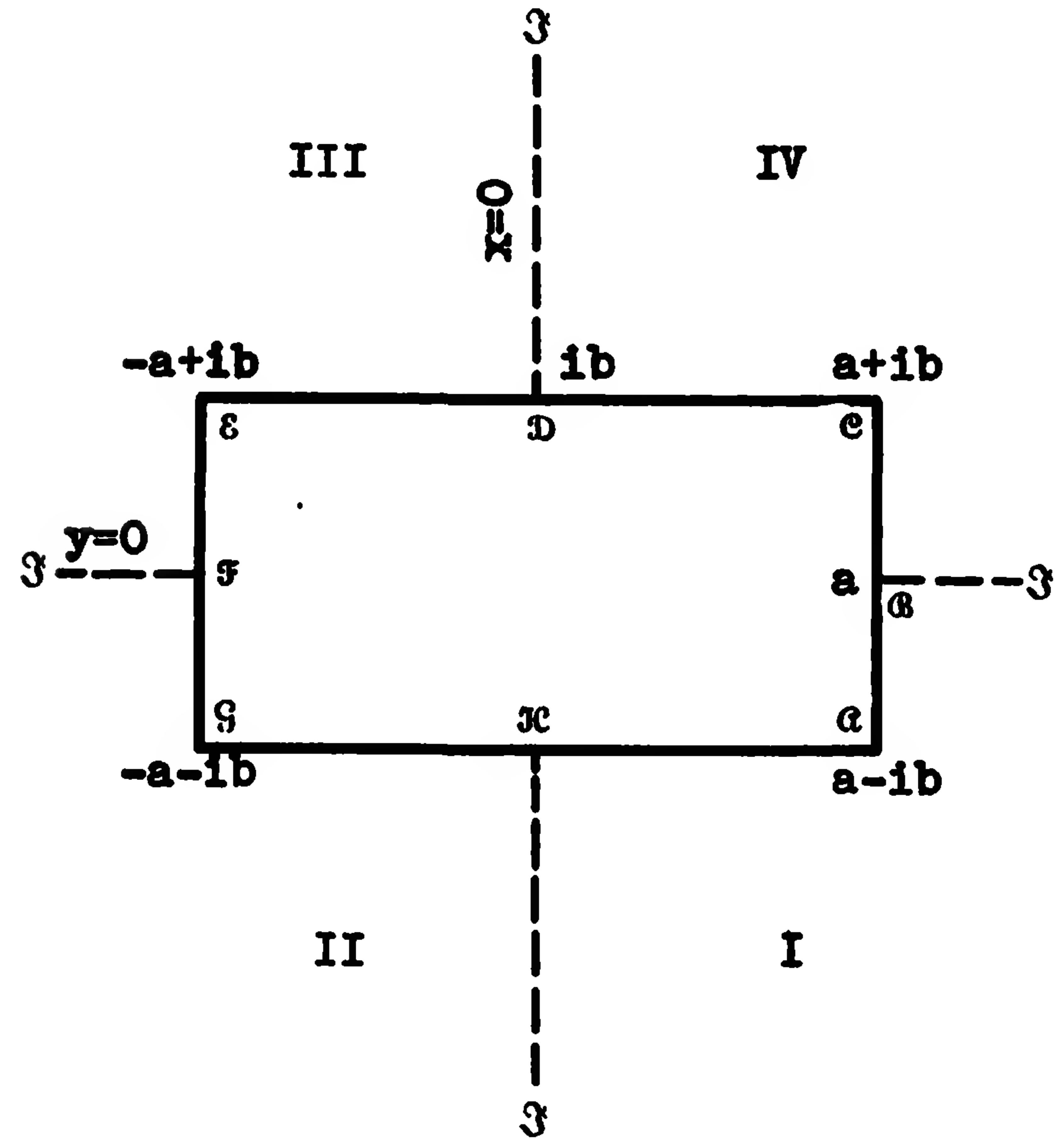
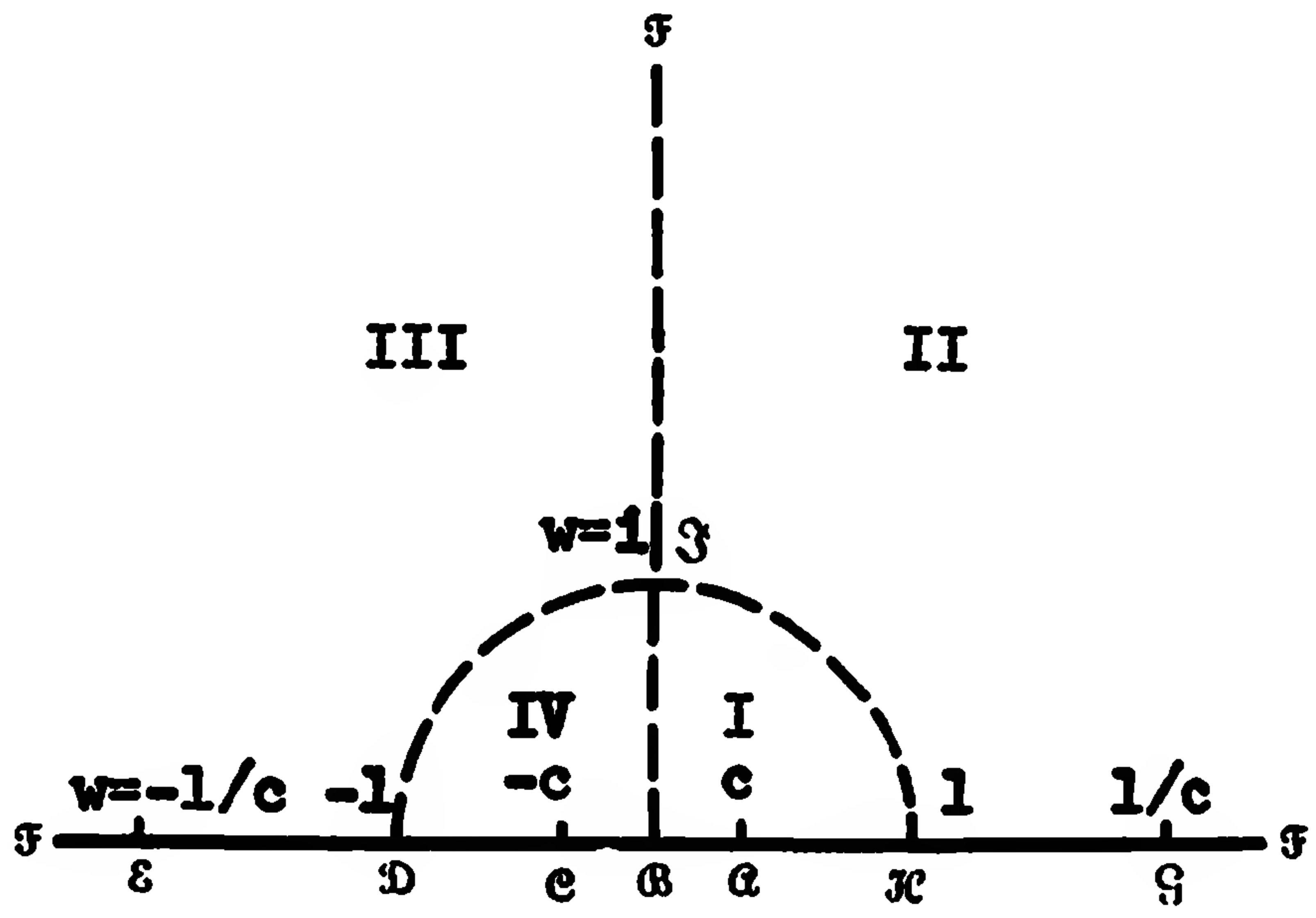
where $\tau = \dfrac{2i}{\pi}\log\dfrac{a+b}{a-b}$, $\quad e^{i\pi\tau} = \left(\dfrac{a-b}{a+b}\right)^2$; $\quad k = \left(\dfrac{\Theta_2(\tau)}{\Theta_3(\tau)}\right)^2$;

$$K = \int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} = F\left(k,\frac{\pi}{2}\right).$$

For the curves in the z-plane and w-plane, corresponding to the lines $\Re\left(\sin^{-1}\dfrac{z}{\sqrt{a^2-b^2}}\right) =$ constant, $\Im\left(\sin^{-1}\dfrac{z}{\sqrt{a^2-b^2}}\right) =$ constant, see H. A. Schwarz.

13.5. Exterior of rectangle on half-plane (cf. §13.10).

Given: a, b ($0 < b \le a$). Vertices of rectangle: $\pm a \pm ib$.

z - plane	w - plane
	

k, which depends on b/a, is found from the equation or graph below (p. 179).

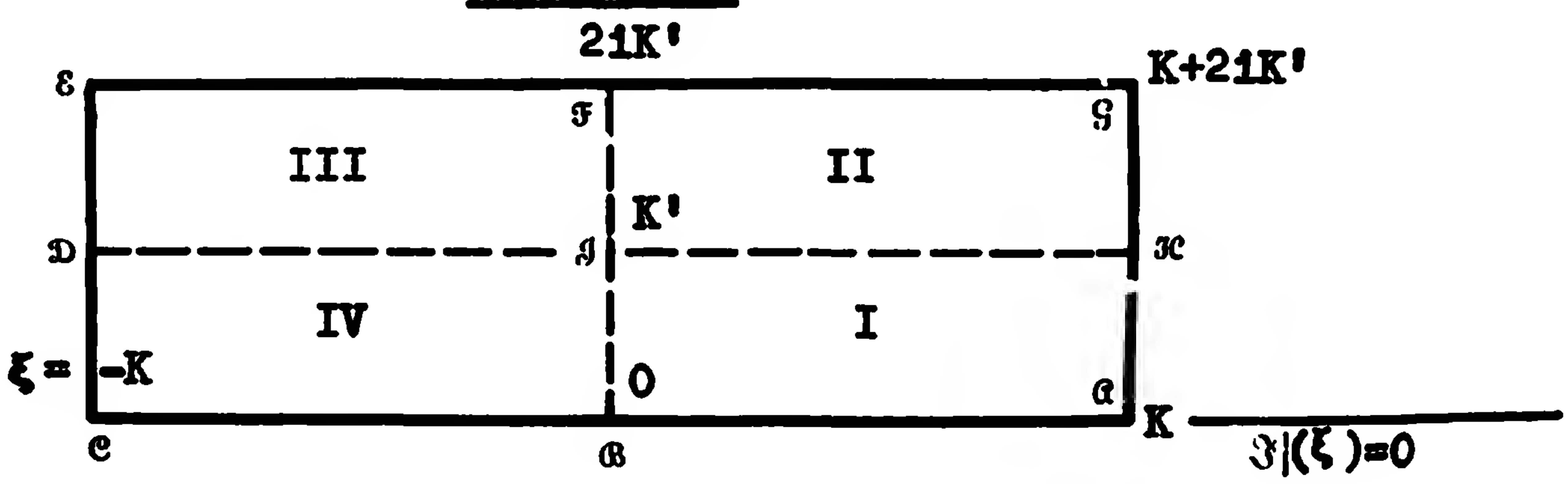
$c = \dfrac{1-\sqrt{1-k^2}}{k}$ $(0 < c < 1)$. If a=b, then $k = 1/\sqrt{2}$, $c = \tan \pi/8 = \sqrt{2}-1$.

Transformation required: $\dfrac{dz}{dw} = C(w^2-c^2)^{1/2}(w^2-c^{-2})^{1/2}(w^2+1)^{-2}$,

$$\boxed{z = \frac{c}{k}\left\{\operatorname{zn}\xi + \left(\frac{E}{K} - k'^2\right)\xi\right\} + a, \quad w = \frac{1-\operatorname{dn}(\xi,k)}{k\,\operatorname{sn}(\xi,k)} = \frac{1-\operatorname{dn}\xi}{k\,\operatorname{sn}\xi}},$$

where ξ is an auxiliary variable, $\operatorname{zn}\xi$ the Jacobian Zeta-function.

ξ - plane

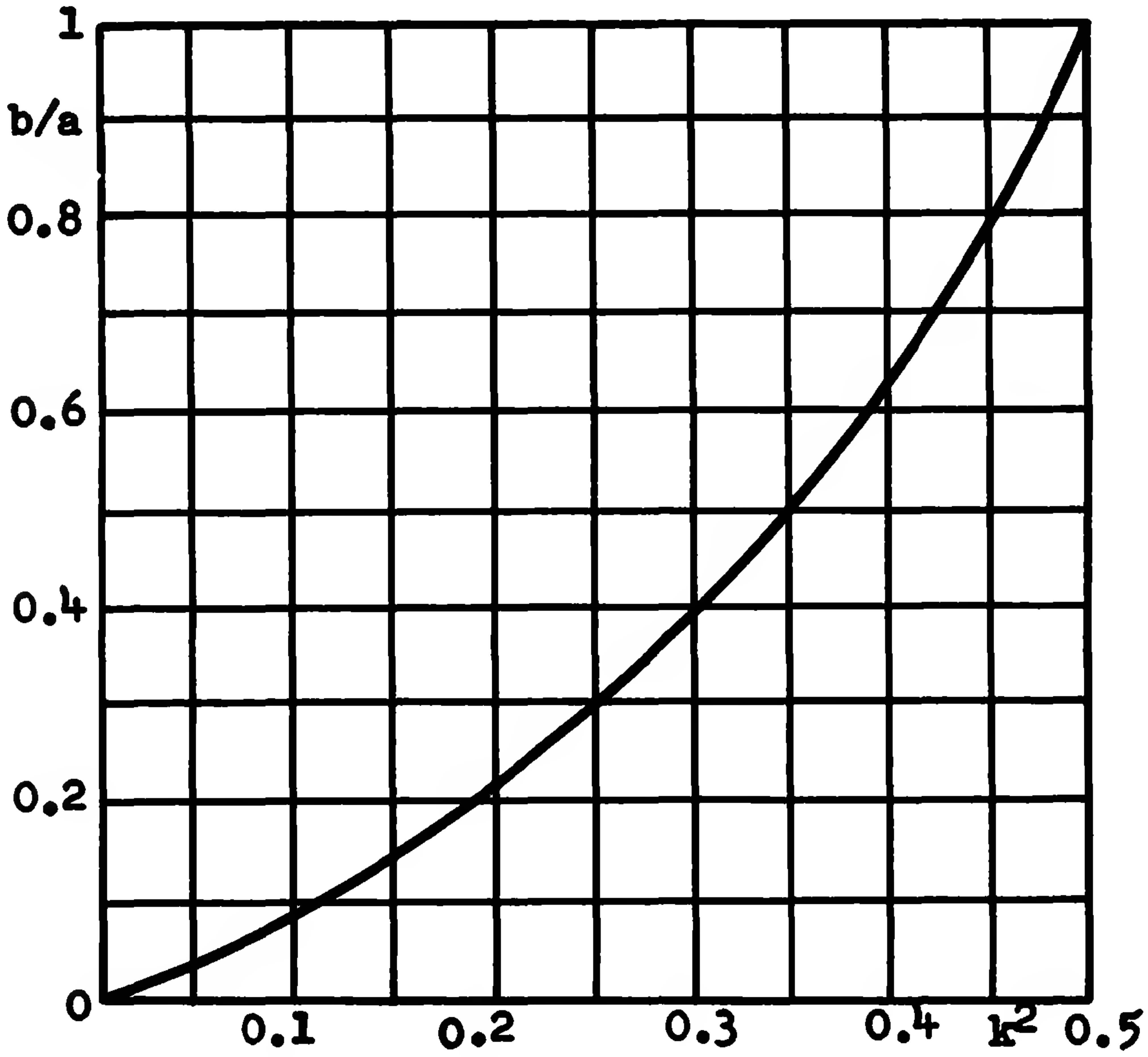


$$E = \int_0^{\pi/2} d\varphi \sqrt{1-k^2\sin^2\varphi}\,, \quad k'^2 = 1-k^2, \quad E' = \int_0^{\pi/2} d\varphi \sqrt{1-k'^2\sin^2\varphi}\,,$$

$$C = -ibk/(E-k'^2K), \quad zn\xi = \frac{\vartheta_0'\left(\frac{\xi}{2K}\middle|\tau\right)}{2K\vartheta_0\left(\frac{\xi}{2K}\middle|\tau\right)}, \quad \tau = \frac{iK'}{K};$$ for K and K',

see p. 171.

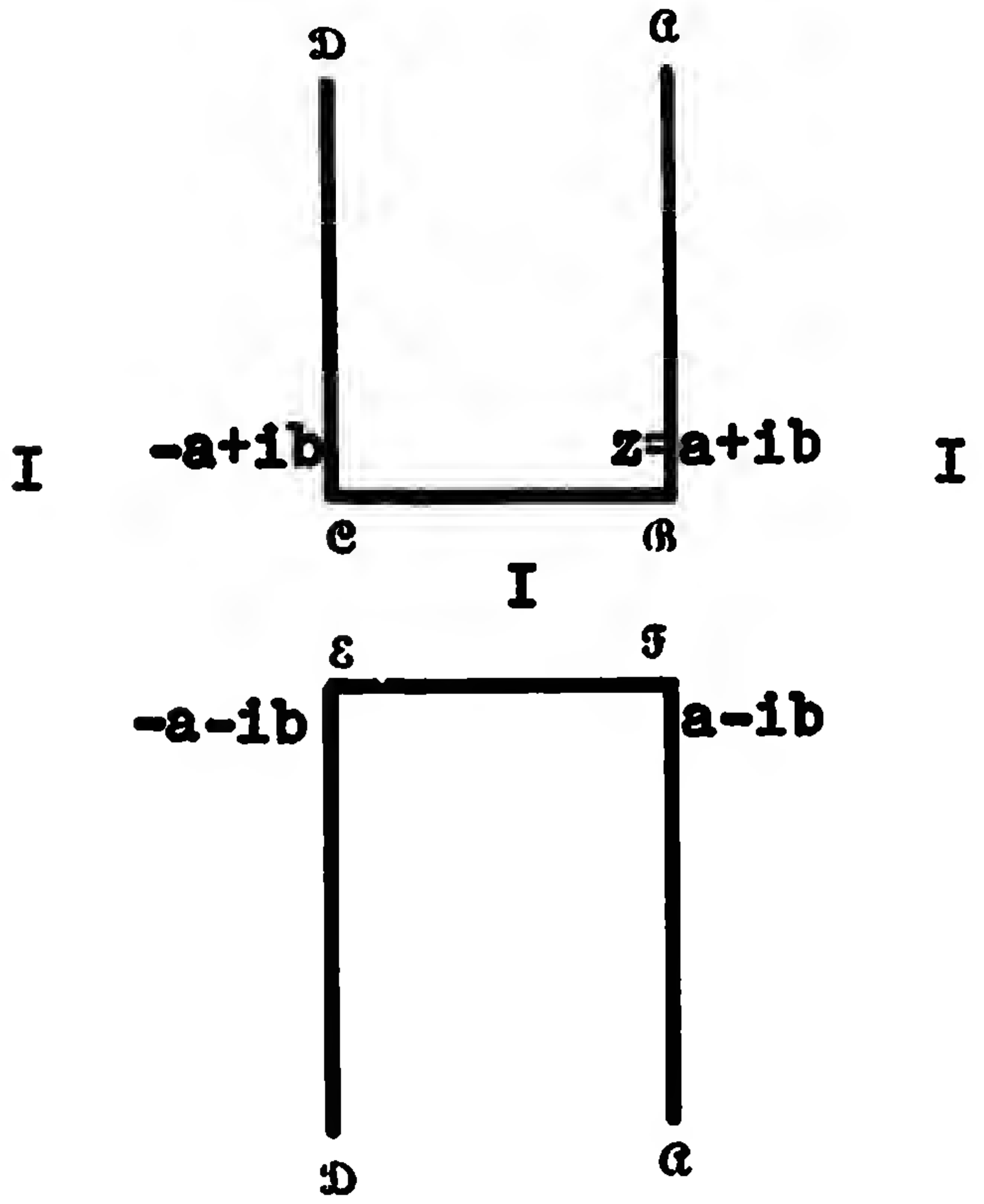
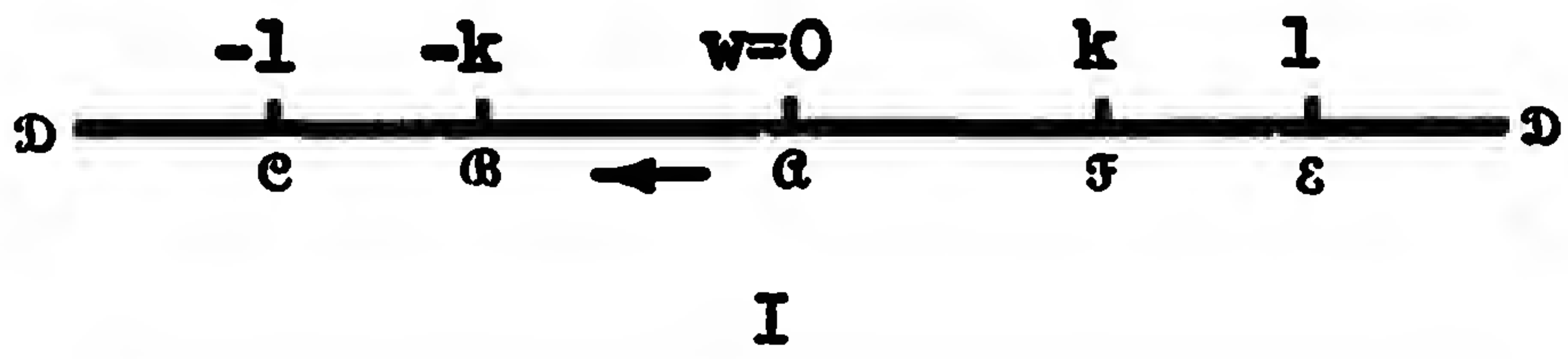
$$\boxed{k \text{ is the root of: } (E-k'^2K)a = (E'-k^2K')b.}$$



For particulars see W. G. Bickley, II. The figure is reprinted by his permission. For the chart of the Jacobian Zeta-function zn ξ see Jahnke-Emde, p. 98. For the altitude chart of the complete elliptic integral E as a function of $\lambda = k^2$ see Jahnke-Emde, p. 76, for its relief p. 77.

13.6 Region exterior to two semi-infinite strips (cf. §13.10).

Given: $a > 0$, $b > 0$.

z - plane	w - plane
	
region exterior to semi-infinite strips	half-plane $v < 0$

k, depending on $\frac{b}{a}$ only, is a root of: $\frac{b}{a} = \frac{K'k'^2-2K'+2E'}{2(Kk'^2-2E)}$.

Transformation required: $\frac{dz}{dw} = A(w^2-k^2)^{1/2}(w^2-1)^{1/2}w^{-2}$,

$$\boxed{z = -A\left\{\xi\left(k'^2 - \frac{2E}{K}\right) - 2\,\mathrm{zn}\,\xi - \frac{\mathrm{cn}\,\xi\,\mathrm{dn}\,\xi}{\mathrm{sn}\,\xi}\right\} - b}$$

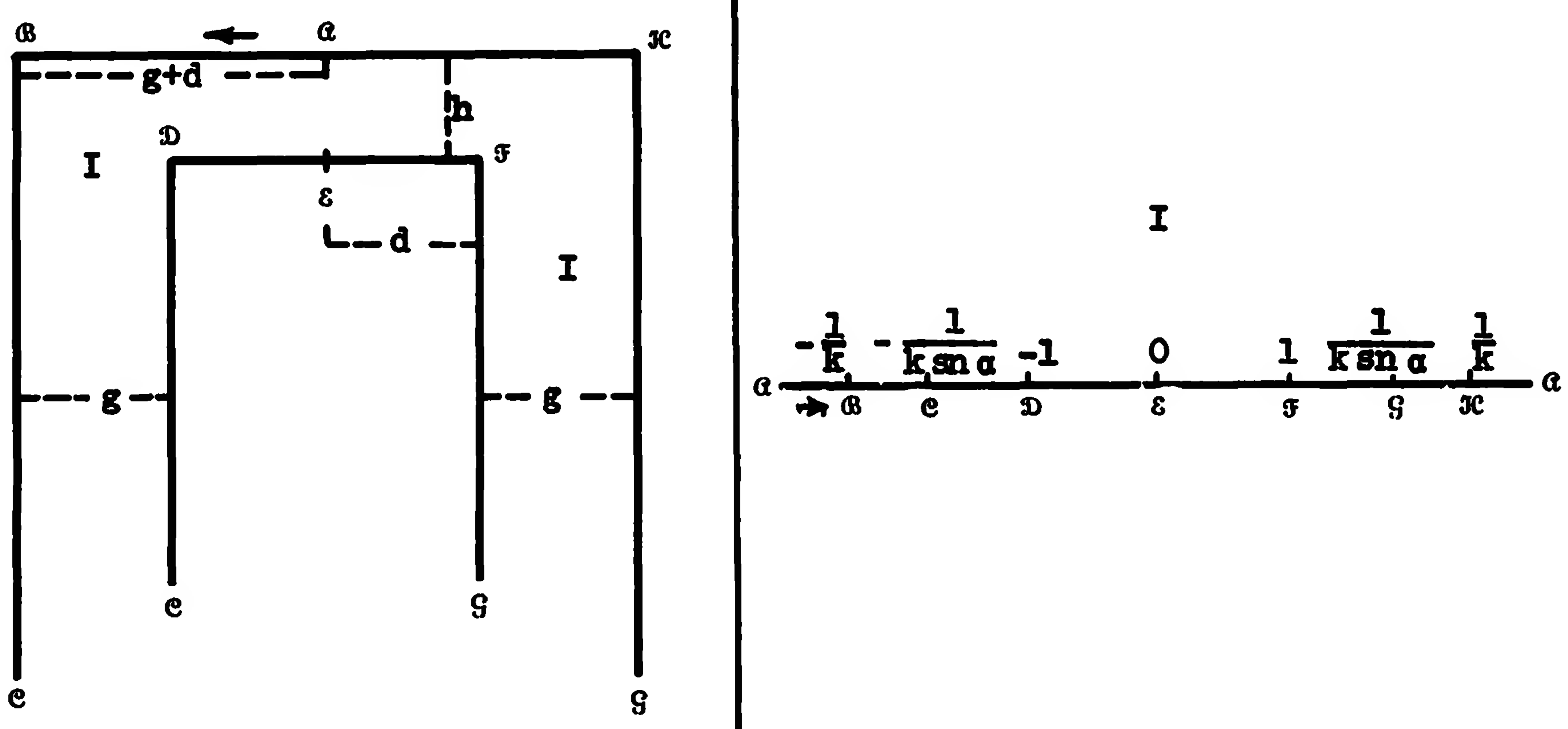
where $w = 1/\mathrm{sn}\,\xi$, $-A = \frac{ai}{2E-k'^2K}$.

For particulars see N. Davy.

Region interior to a semi-infinite strip, but exterior to another one.

Given: k, $0 < k < 1$.

z - plane	w - plane



$$\frac{dz}{dw} = \left(\frac{1-w^2}{1-k^2w^2}\right)^{1/2}(1-k^2w^2\ \mathrm{sn}^2\,\alpha)^{-1}, \qquad \boxed{w = \mathrm{sn}\,\xi,\quad z = \xi - \frac{\mathrm{dn}\,\alpha}{k^2\,\mathrm{sn}\,\alpha\ \mathrm{cn}\,\alpha}\,\Pi(\xi,\alpha)},$$

where $\Pi(\xi,\alpha) = \dfrac{\xi}{2K}\,\dfrac{\theta_0'(\frac{\alpha}{2K}|\tau)}{\theta_0(\frac{\alpha}{2K}|\tau)} + \dfrac{1}{2}\log\dfrac{\theta_0(\frac{\xi-\alpha}{2K}|\tau)}{\theta_0(\frac{\xi+\alpha}{2K}|\tau)}$, $\tau = \dfrac{iK'}{K}$, $\alpha = K - \dfrac{i}{2}K'$;

$d = K\left\{\dfrac{1+k}{2k} - \dfrac{\pi}{4kK}\right\}$, $g = \dfrac{\pi}{2k}$, $h = K'\,\dfrac{1+k}{2k}$.

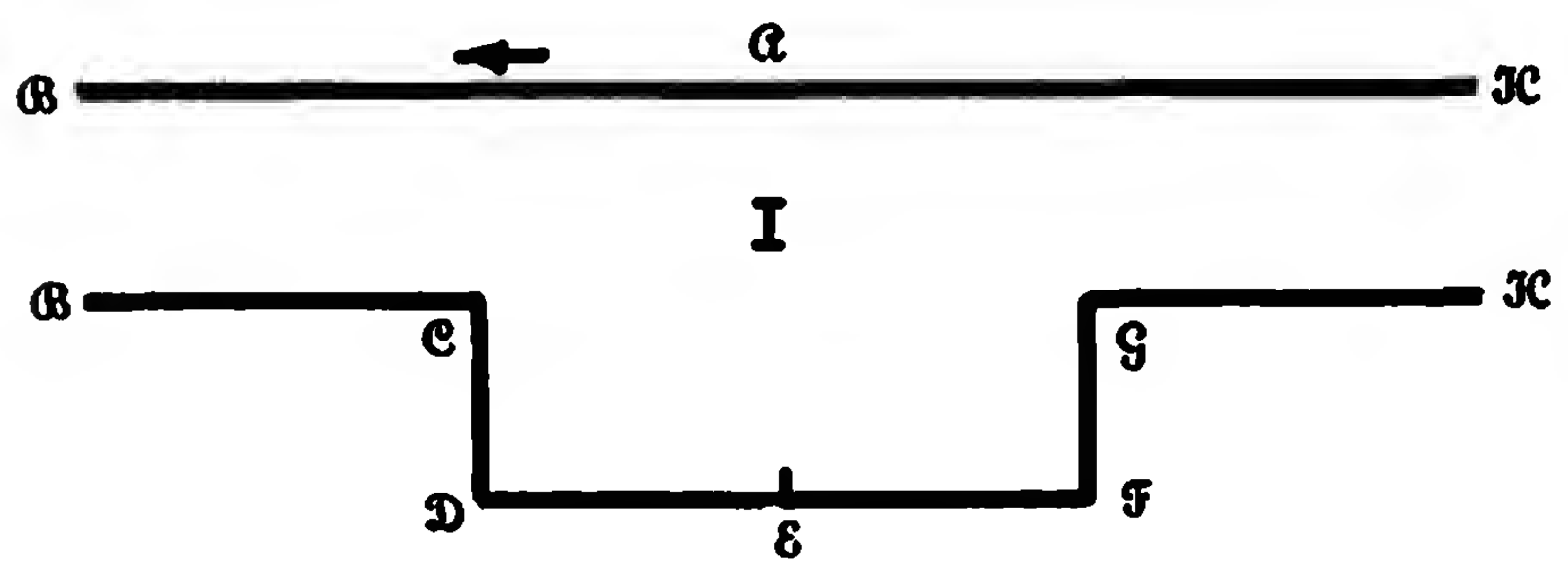
For particulars see J. D. Cockcroft, Appendix 1. The figure is reprinted by his permission.

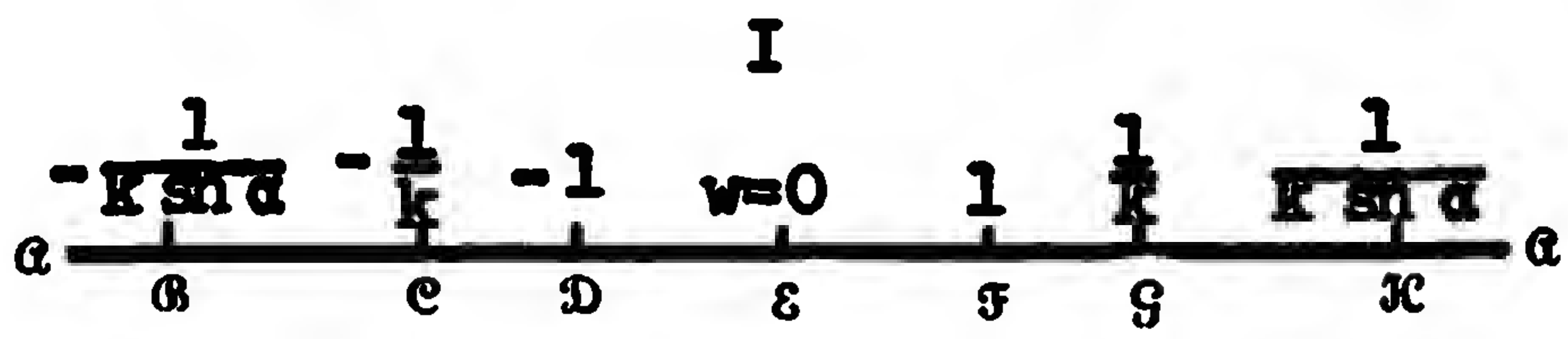
A similar transformation:

$$\frac{dz}{dw} = \left(\frac{1-k^2w^2}{1-w^2}\right)^{1/2}\frac{1}{1-k^2w^2\mathrm{sn}^2\,\alpha},$$

$$\boxed{w = \mathrm{sn}\,\xi,\quad z = \xi - \frac{\mathrm{cn}\,\alpha}{\mathrm{sn}\,\alpha\,\mathrm{dn}\,\alpha}\,\Pi(\xi,\alpha).}$$

z - plane	w - plane





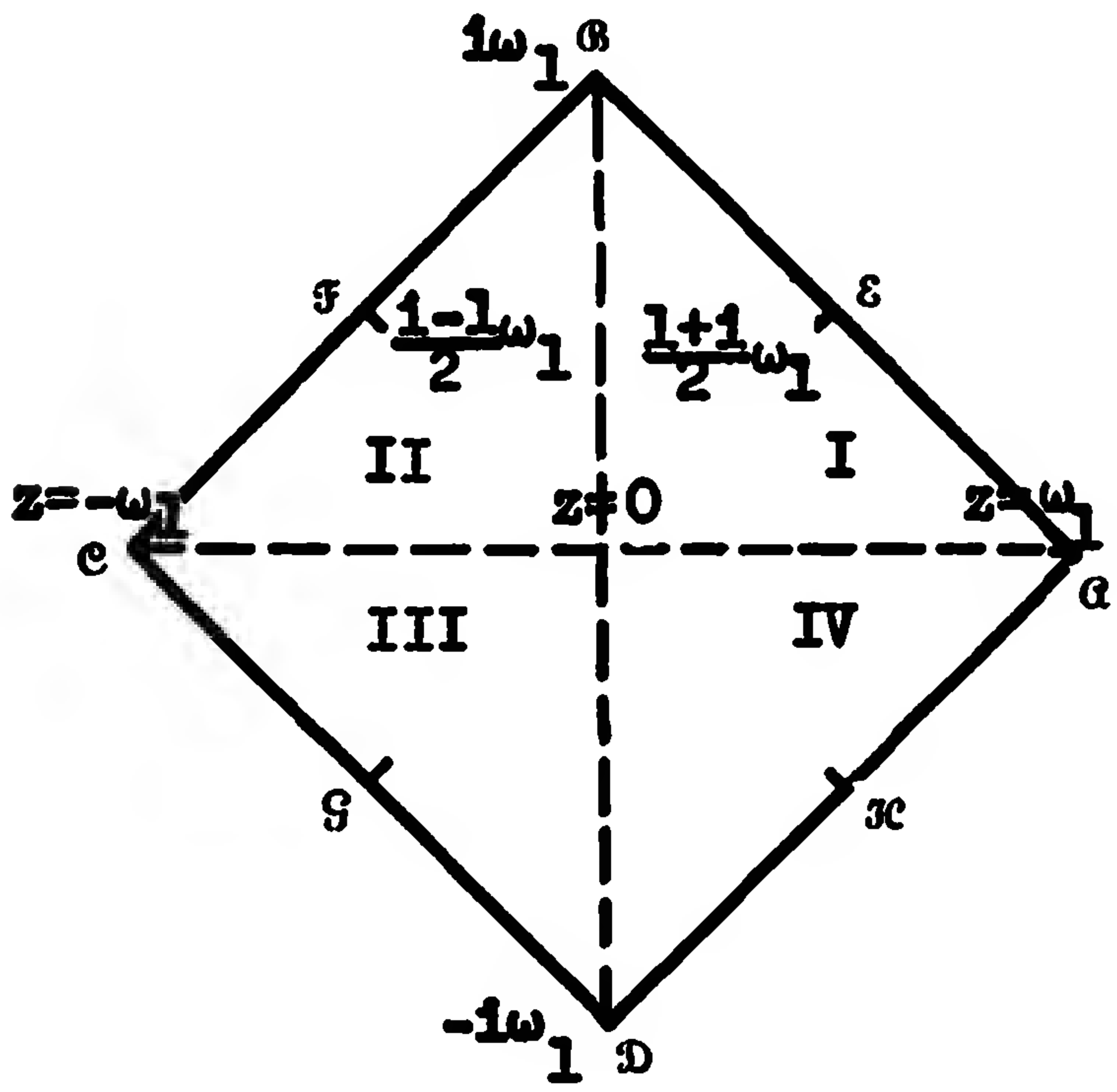
For particulars see F. W. Carter.

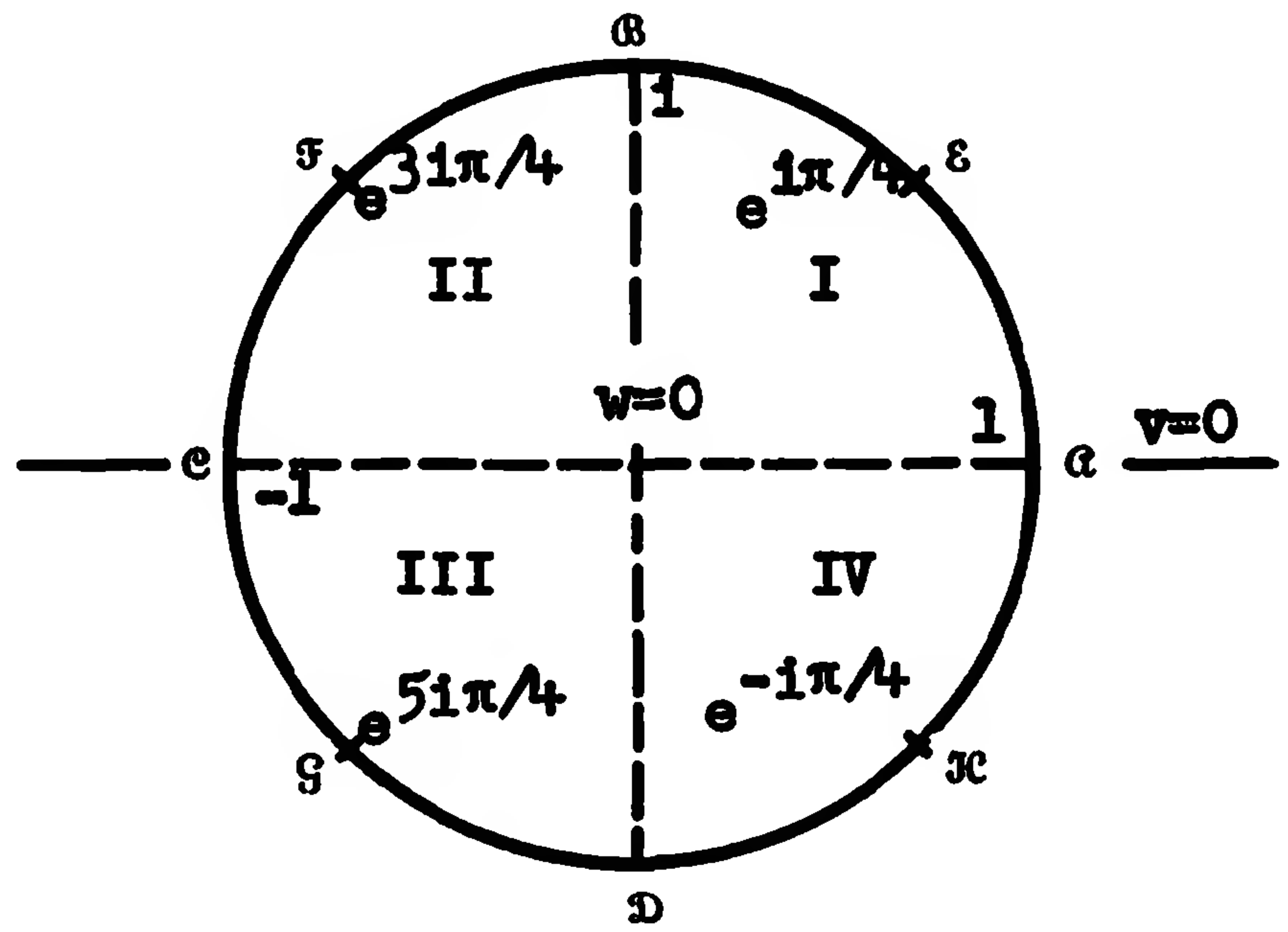
13.7 <u>Square on circle.</u>

$$\boxed{z = \int_0^w \frac{dt}{\sqrt{1-t^4}}} \;;\quad \boxed{w\sqrt{2} = \frac{\mathrm{sn}(z\sqrt{2},k)}{\mathrm{dn}(z\sqrt{2},k)}}\;,\quad \text{where}\quad k = 1/\sqrt{2};$$

$$\omega_1 = \int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{K}{\sqrt{2}},\quad K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = 1.8541.$$

z - plane	w - plane





points $z = 0$; $z = \omega_1 = 1.31$;
$z = i\omega_1 = 1.31i$

points $w = 0$; 1;
i

Another form of the above transformation:

$$\boxed{w^2 = \frac{1}{\wp(z;4,0)}}\ ,\quad \text{i.e.}\quad g_2 = 4,\quad g_3 = 0,\quad e_1 = -e_3 = 1,\quad e_2 = 0;$$

$$\omega_1 \text{ as above},\quad \omega_3 = i\omega_1.$$

Triangle, with angles $\pi/2$, $\pi/4$, $\pi/4$, on half-plane.

$$\boxed{w = \sqrt{2}\ \mathrm{sn}(z\sqrt{2})\ \mathrm{dn}(z\sqrt{2})}\qquad [k,\ \omega_1 \text{ as above}].$$

z - plane	w - plane
points $z = -\omega_1;\ \omega_1;\ i\omega_1$	points $w = -1;\ 1;\ \infty$
Triangle $\mathcal{C}\,\mathcal{A}\,\mathcal{B}$ (see figure on p. 182)	half-plane $v > 0$

Regular polygon with n vertices on unit-circle ($n \geq 3$).

$$\boxed{z = \int_0^w \frac{dt}{(1-t^n)^{2/n}}} = w + \sum_{j=1}^{\infty} \frac{2(2+n)(2+2n)\ \dots\ (2+(j-1)n)}{j!\,n^j(jn+1)}\, w^{jn+1}$$

$$\omega = \int_0^1 dt(1-t^n)^{-2/n}$$

z - plane	w - plane
points $\omega e^{2m\pi i/n}$ $[m = 0,1,\dots,n-1]$;	points $e^{2m\pi i/n}$;
$\dfrac{\omega}{\cos\pi/n}\, e^{(2m+1)\pi i/n}$	$e^{(2m+1)\pi i/n}$
line-segment $\arg z = \dfrac{2m\pi}{n}$, $0 < \lvert z\rvert < \omega$	segment $\arg w = \dfrac{2m\pi}{n}$, $0 < \lvert w\rvert < 1$
line-segment $\arg z = \dfrac{2m+1}{n}\pi$, $0 < \lvert z\rvert < \dfrac{\omega}{\cos\pi/n}$	segment $\arg w = \dfrac{2m+1}{n}\pi$; $0 < \lvert w\rvert < 1$

For the curves in the z-plane corresponding to $|w|$ = constant and arg w = constant when $n = 3$, see E. Kohl.

13.8 Equilateral triangle on half-plane.

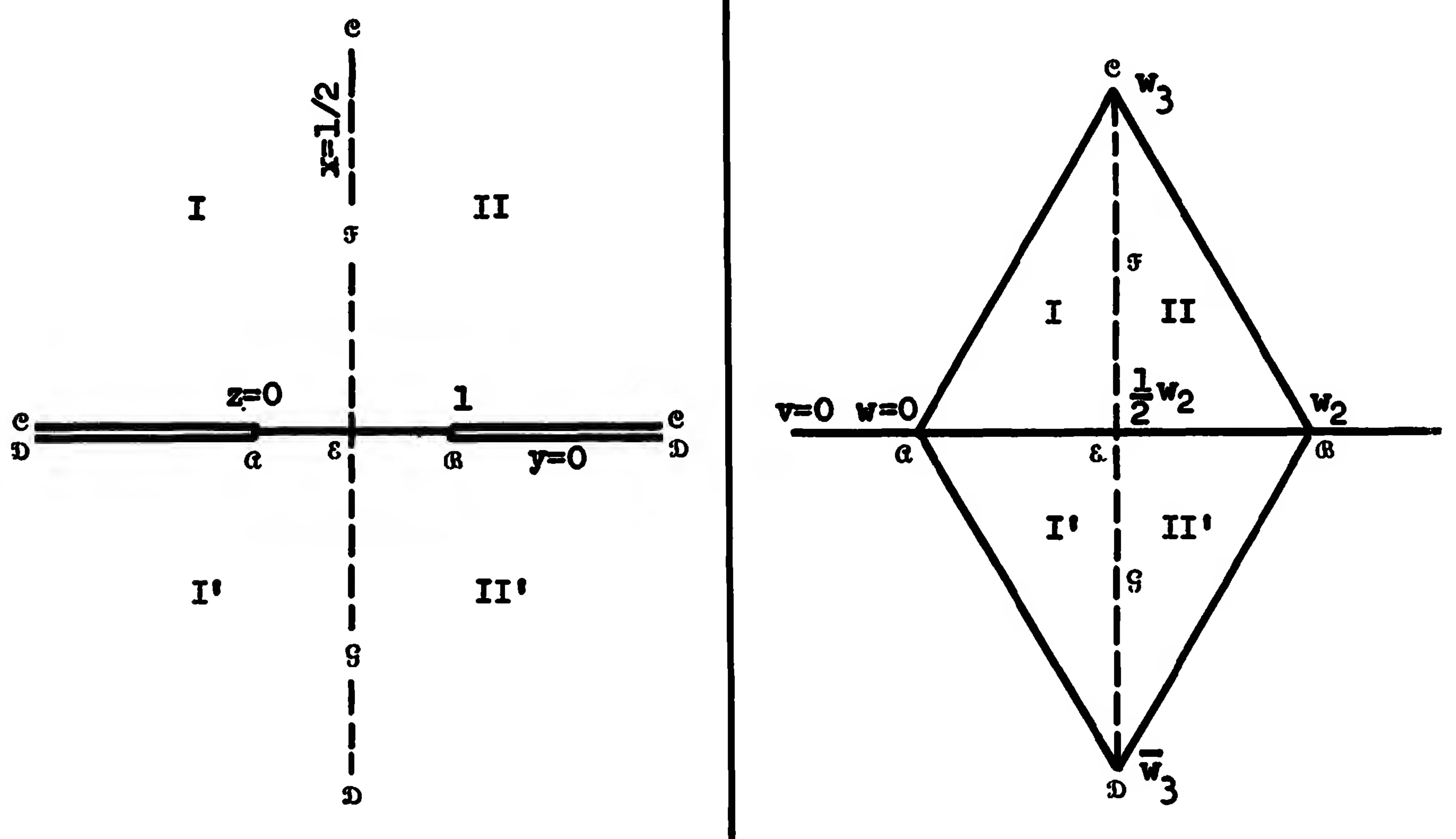
$$\text{(I)}\quad \boxed{w = \int_0^z \frac{dt}{(t-t^2)^{2/3}}} = f(z); \quad \boxed{z - z^2 = \wp^3\left(i\,\frac{w+w_2}{3};\ 0,1\right)}\ , \quad \text{or}$$

$$z = \frac{1}{2} + \frac{A\ \mathrm{sn}(\zeta,k)\,\mathrm{dn}(\zeta,k)}{[1+\mathrm{cn}(\zeta,k)]^2}, \quad \text{where } k = \frac{\sqrt{3}+1}{2\sqrt{2}}, \quad \zeta = \frac{2w - w_2}{\sqrt[4]{27}\ \sqrt[3]{2}},$$

$$A = \sqrt[4]{27}; \quad w_1 = 0, \quad w_2 = \int_0^1 \frac{dt}{(t-t^2)^{2/3}} = 5.298, \quad w_3 = e^{i\pi/3}\, w_2.$$

For the chart of $\wp(\xi;0,1)$ (equianharmonic case, $g_2 = 0$, $g_3 = 1$) see Jahnke-Emde, p. 101.

z - plane	w - plane



points $z = 0$ (i.e. $\mathfrak{A}$); $\frac{1}{2}$ ($\mathfrak{E}$); 1 ($\mathfrak{B}$); ∞ ($\mathfrak{C}$); ∞ ($\mathfrak{D}$)

points $w = 0$; $\frac{1}{2}w_2$; w_2; w_3; $\bar{w}_3$.

z - plane	w - plane
half-line $1 \leq x \leq \infty$, $y = 0$	$\mathcal{B}\mathcal{C}$: $w = e^{i\pi/3}w_2 - e^{2i\pi/3}f(1/z)$ $(1 \leq z \leq \infty)$
half-line $-\infty \leq x \leq 0$, $y = 0$	$\mathcal{C}\mathcal{A}$: $w = e^{i\pi/3}w_2 - e^{i\pi/3}f(1/(1-z))$ $(-\infty \leq z \leq 0)$

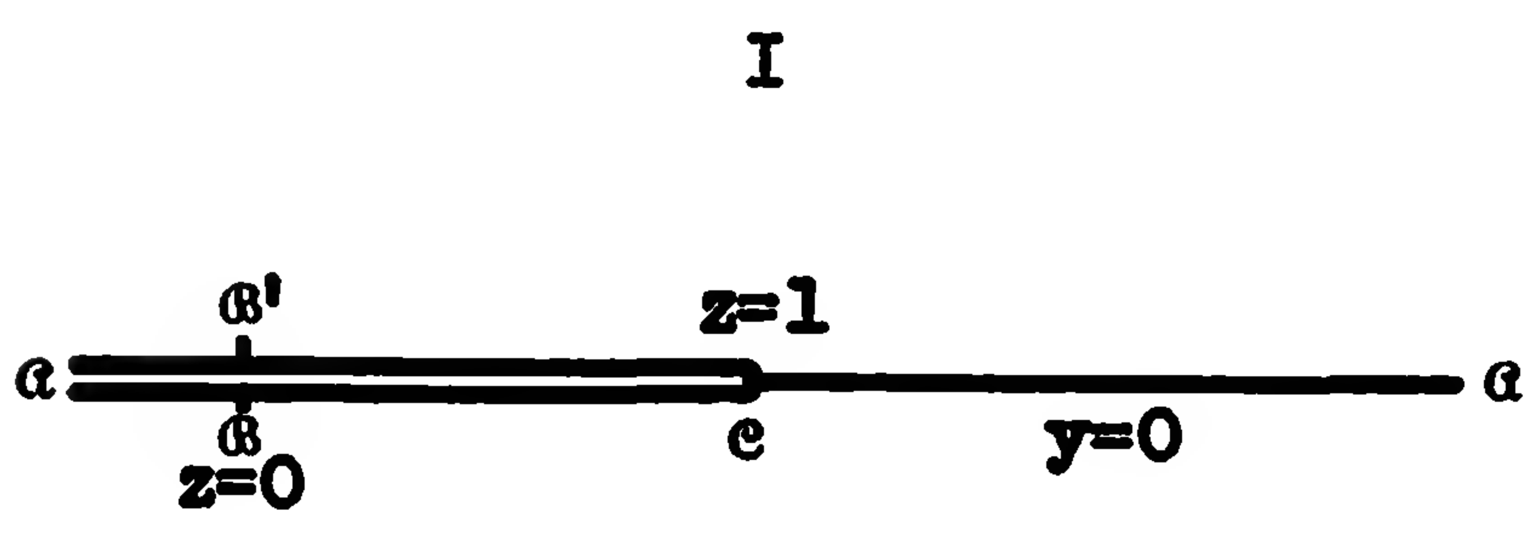
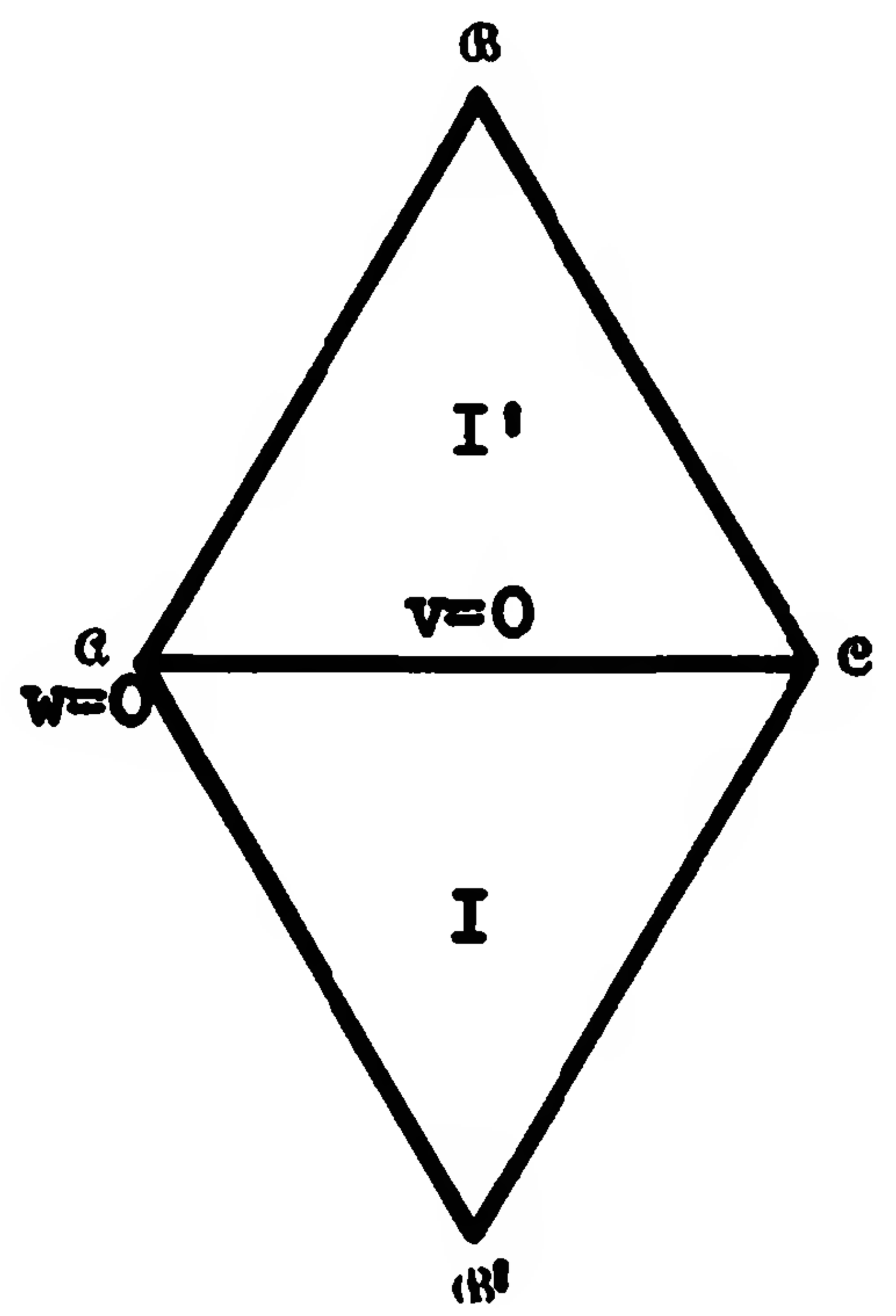
When z real and $0 \leq z \leq 1$: $f(z) - \dfrac{w_2}{2} = \mp \dfrac{\sqrt[4]{27}}{\sqrt[3]{4}} \displaystyle\int_0^{\gamma} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$

for $\begin{matrix} 0 \leq z \leq \frac{1}{2} \\ \frac{1}{2} \leq z \leq 1 \end{matrix}$, where $k = \sin\dfrac{5\pi}{12} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$;

$$\tan\frac{\gamma}{2} = 3^{-1/4}\left\{1-(4z-4z^2)^{1/3}\right\}^{1/2}, \quad 0 \leq \gamma \leq 1.842.$$

Or (II) $\boxed{\dfrac{w\sqrt[4]{27}}{2} = \displaystyle\int_z^{\infty} \frac{dt}{(t^3-t)^{2/3}}}$; $\boxed{z = c\,\dfrac{(1+\mathrm{cn}\,w)^2}{\mathrm{sn}\,w\;\mathrm{dn}\,w}}$

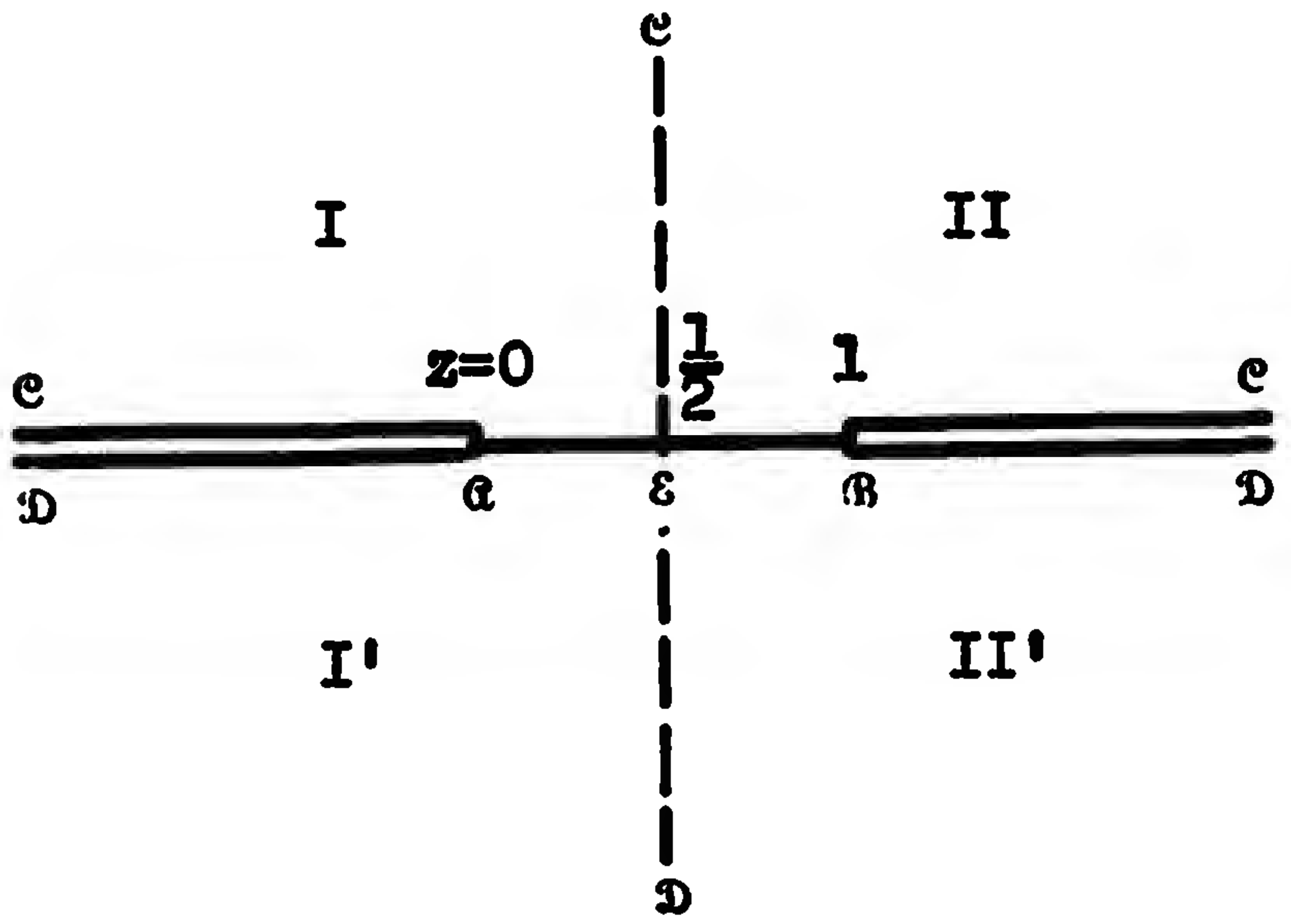
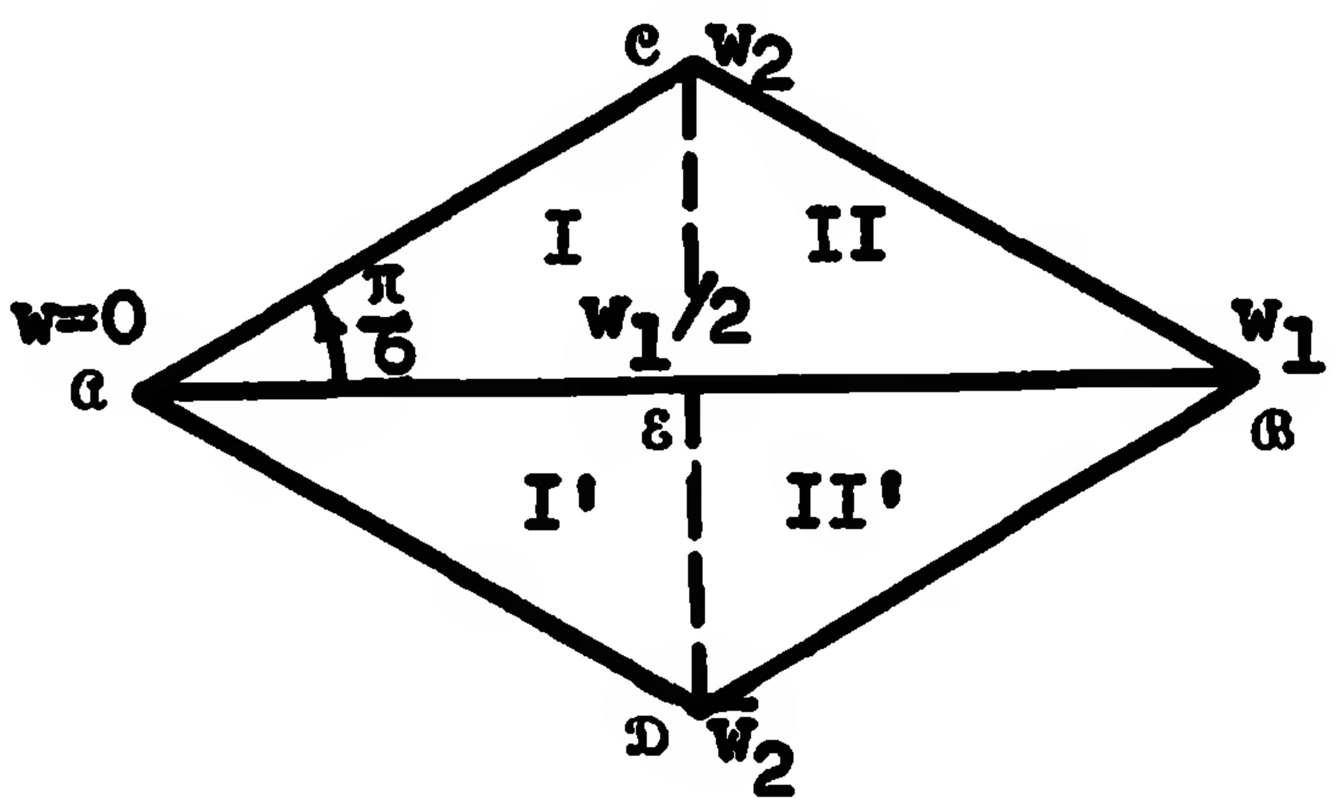
with $k = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$, $c = \dfrac{\sqrt[4]{3}}{6}$.

z - plane	w - plane
	

13.9 Triangles with angles $2\pi/3,\ \pi/6,\ \pi/6$, and $\pi/2,\ \pi/3,\ \pi/6$.

(I) $$\boxed{w = \int_0^z \frac{dt}{(t-t^2)^{5/6}}}\ ; \qquad z-z^2 = \frac{1}{16}\,\wp^{-3}\left(\frac{w}{3\sqrt[3]{2}};\ 0,\ 1\right).$$

$$w_1 = \int_0^1 \frac{dt}{(t-t^2)^{5/6}}; \quad w_2 = \frac{e^{i\pi/6}w_1}{\sqrt{3}}.$$

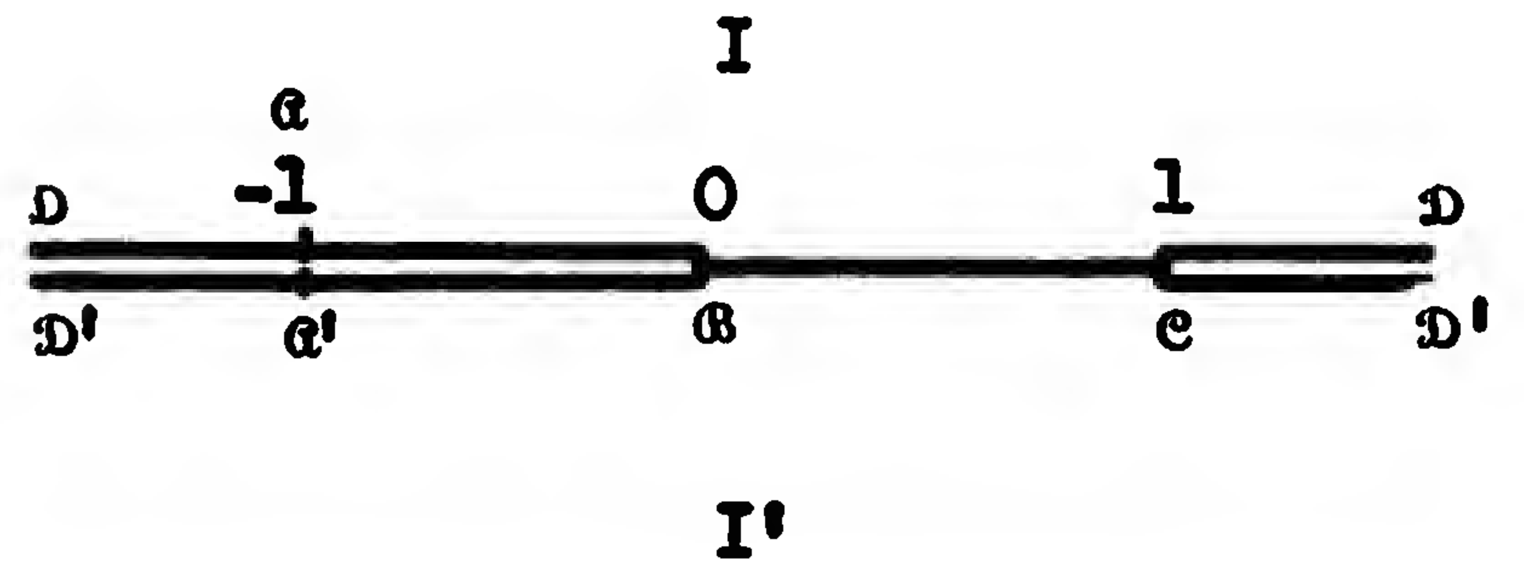
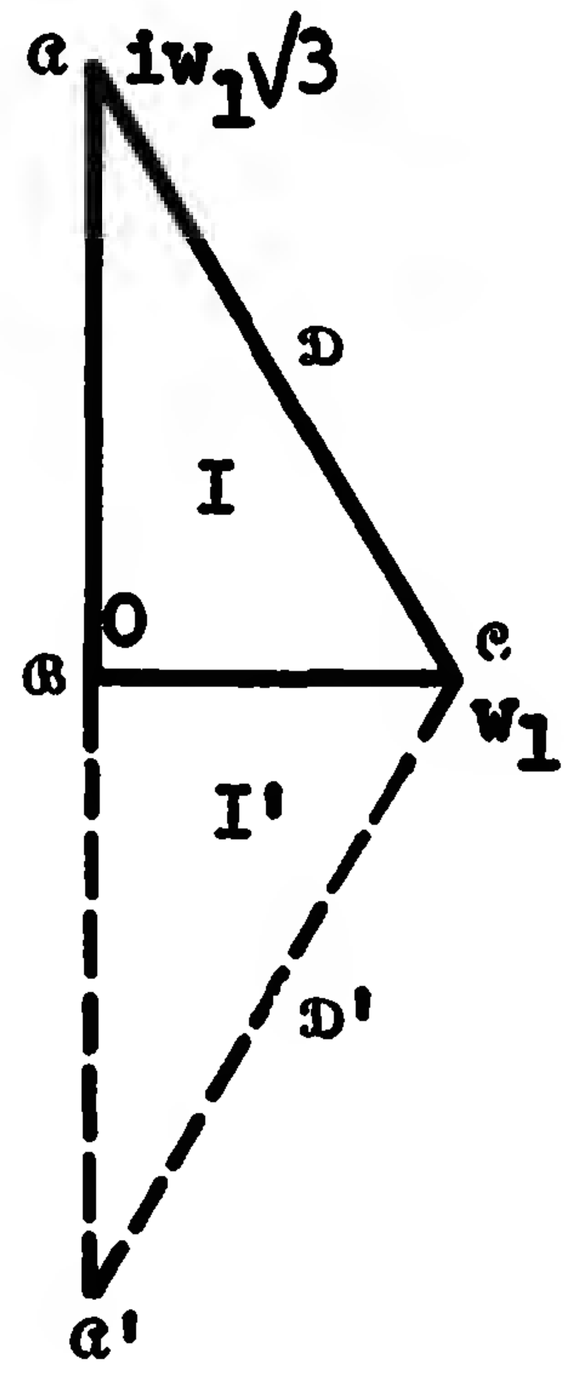
<table>
<tr><th>z - plane</th><th>w - plane</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Quarter of plane</td><td>interior of a triangle with angles $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$</td></tr>
<tr><td>half-plane $y > 0$</td><td>interior of $\mathcal{A}\,\mathcal{B}\,\mathcal{C}$; angles $\frac{\pi}{6}, \frac{\pi}{6}, \frac{2\pi}{3}$.</td></tr>
</table>

Or (II) $$\boxed{w\sqrt[4]{108} = \int_0^z \frac{dt}{t^{1/2}(1-t)^{2/3}(1+t)^{5/6}}}\ ;$$

$$\boxed{\frac{1-z}{1+z} = \left(1-\sqrt{3}\,\frac{\mathrm{sn}^2\,w}{\mathrm{cn}^2\,w}\right)^3}\ ,$$

where $k = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

See figures on next page

z - plane	w - plane
	

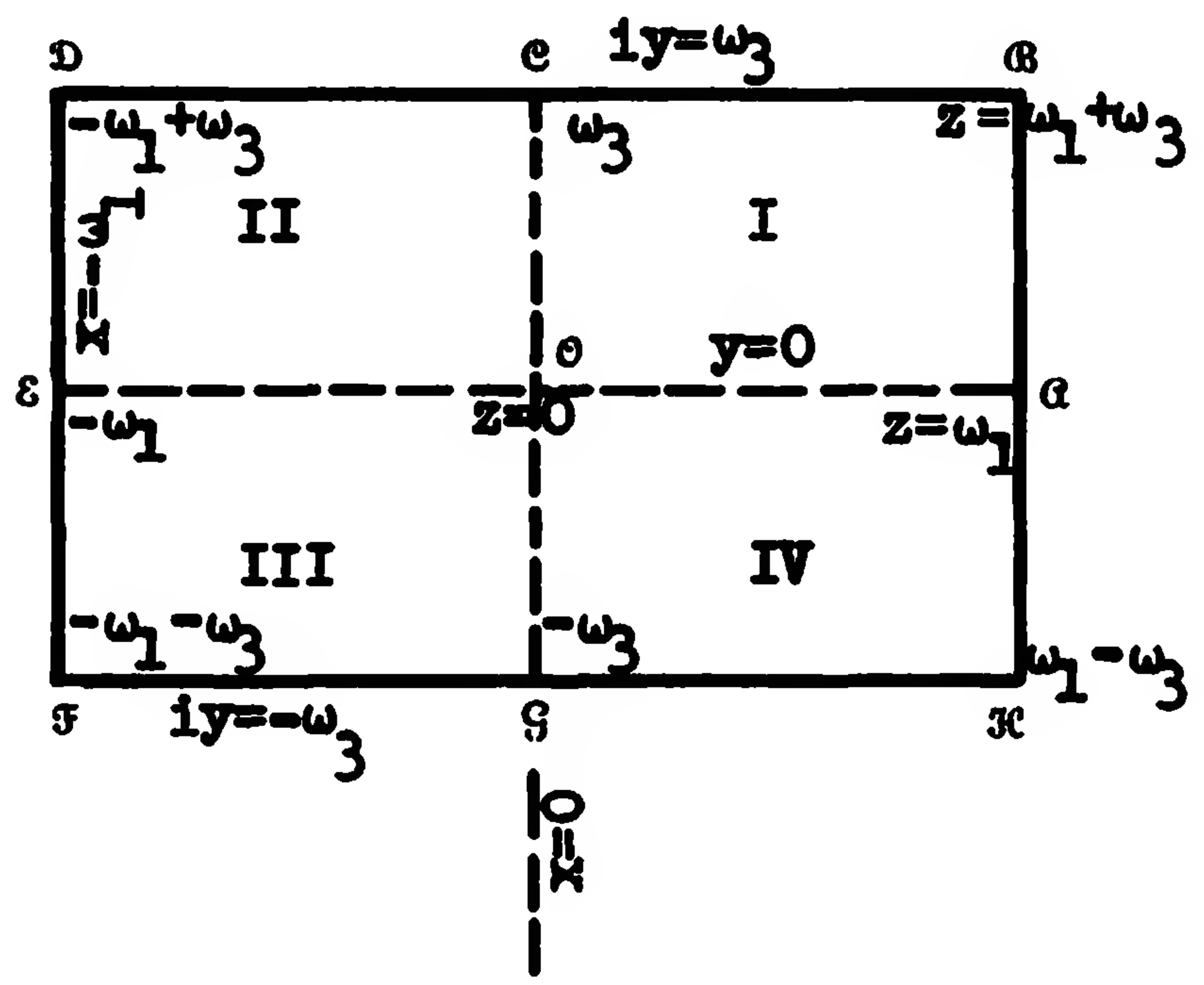
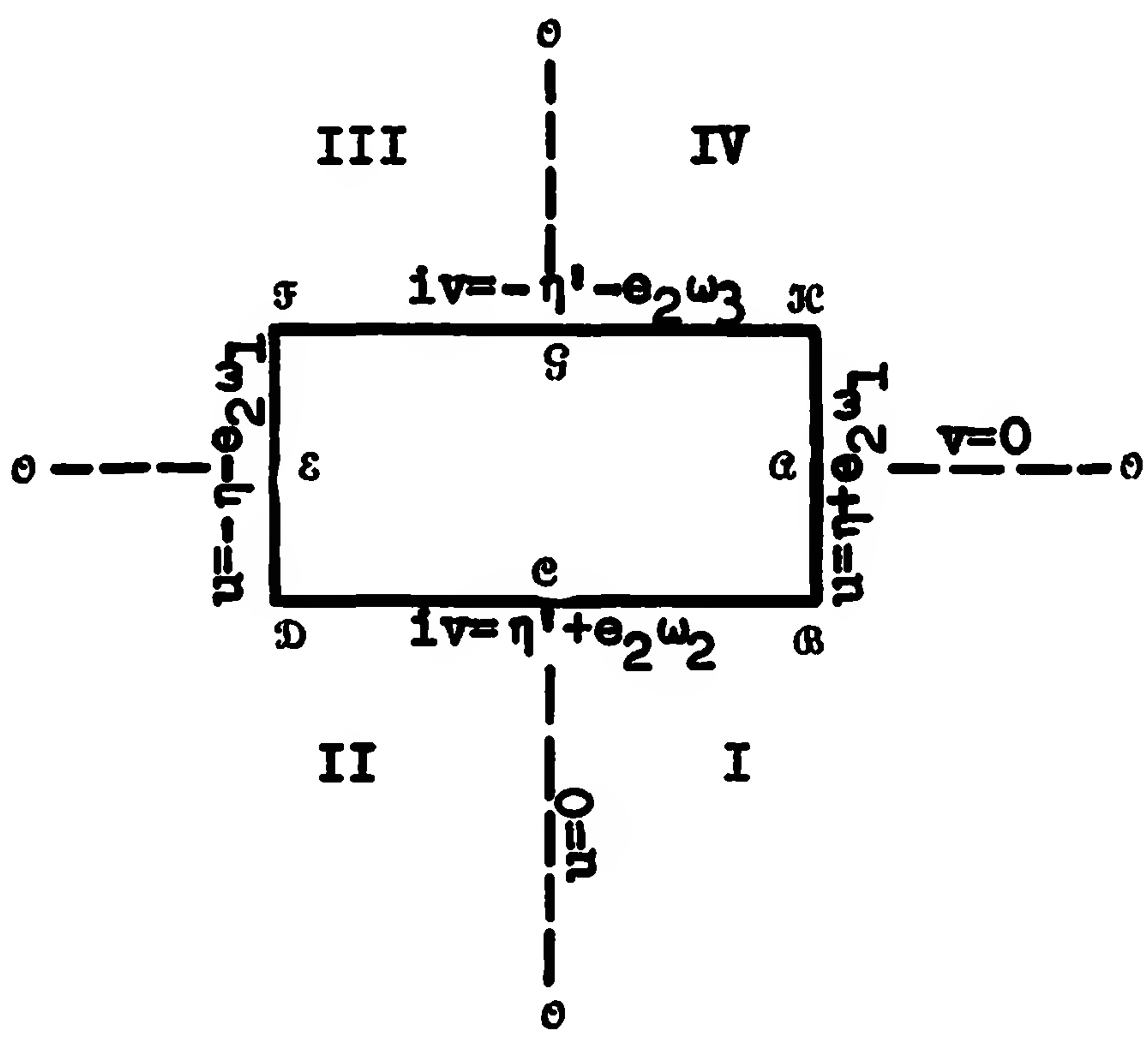
13.10 Interior of rectangle on exterior of another rectangle (cf. §13.5).

$$\boxed{w = \zeta(z) + e_2 z}\ , \text{ or } \boxed{\frac{dw}{ds} = -\frac{1}{2}(s-e_1)^{-1/2}(s-e_2)^{1/2}(s-e_3)^{-1/2};\ s = \wp z},$$

where $\omega_1 > 0$, $\frac{\omega_3}{i} > 0$; $e_1 > e_2 > e_3$ (conf. §13.1).

$$\zeta(z) = \frac{1}{z} - \int_0^z (\wp t - \frac{1}{t^2})dt, \quad \zeta'(z) = -\wp z.$$

$$\tau = \omega_3/\omega_1; \quad \zeta(\omega_1) = \eta, \quad \zeta(\omega_3) = \eta'; \quad \eta\omega_3 - \eta'\omega_1 = i\pi/2.$$

z - plane	w - plane
	

z - plane	w - plane
$z = \omega_1$ (i.e. $\mathfrak{A}$); $-\omega_1$ ($\mathfrak{E}$);	$w = \eta + e_2\omega_1$ ($\mathfrak{A}$); $-\eta - e_2\omega_1$ ($\mathfrak{E}$);
ω_3 ($\mathfrak{C}$); $-\omega_3$ ($\mathfrak{G}$);	$\eta' + e_2\omega_3$ ($\mathfrak{C}$); $-\eta' - e_2\omega_3$ ($\mathfrak{G}$);
$\omega_1 + \omega_3$ ($\mathfrak{B}$);	$\eta + \eta' + e_2(\omega_1 + \omega_3)$ ($\mathfrak{B}$);
$\omega_3 - \omega_1$ ($\mathfrak{D}$);	$\eta' - \eta + e_2(\omega_3 - \omega_1)$ ($\mathfrak{D}$);
$-\omega_1 - \omega_3$ ($\mathfrak{F}$);	$-\eta - \eta' - e_2(\omega_1 + \omega_3)$ ($\mathfrak{F}$);
$\omega_1 - \omega_3$ ($\mathfrak{K}$); 0 ($\mathfrak{O}$).	$\eta - \eta' + e_2(\omega_1 - \omega_3)$ ($\mathfrak{K}$); ∞ ($\mathfrak{O}$).

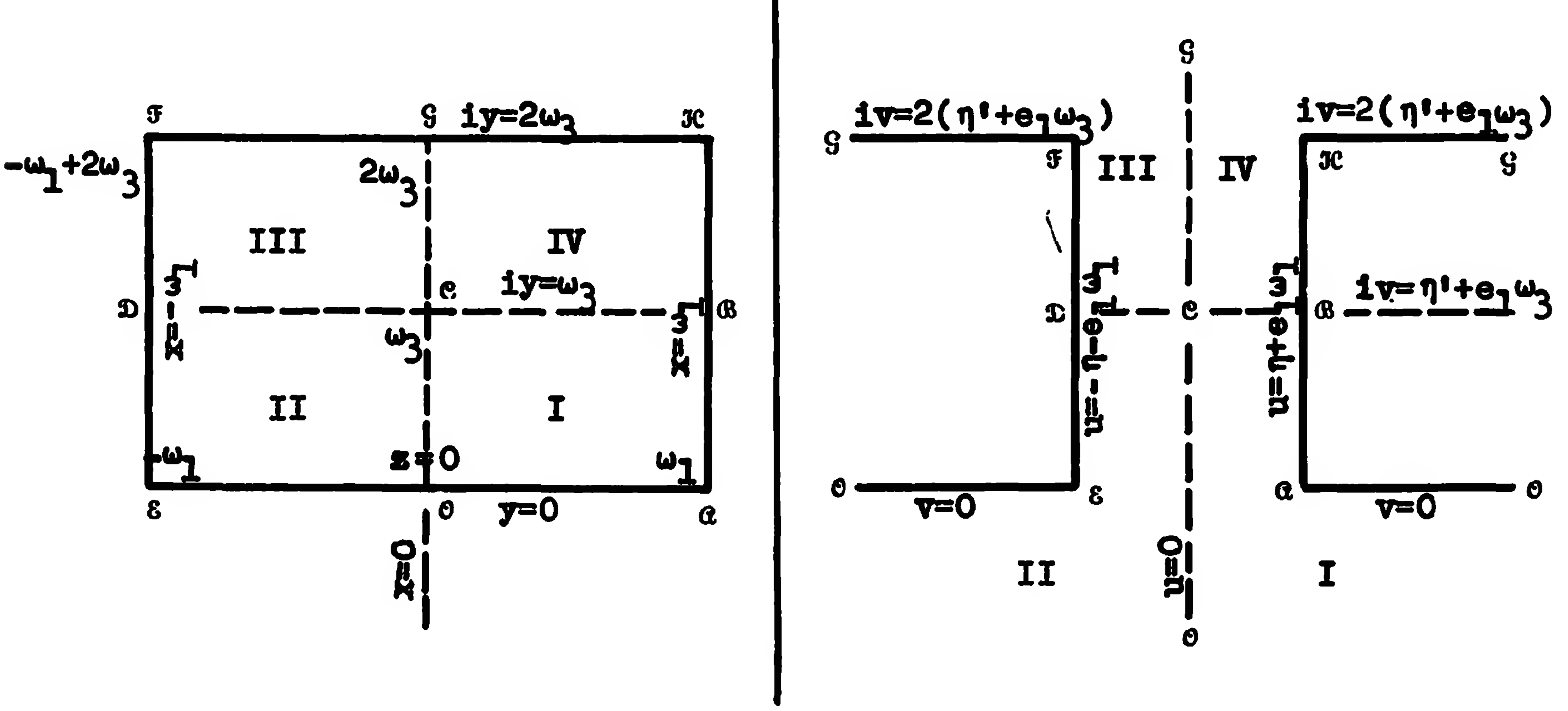
If $\omega_3 = i\omega_1$, then $e_2 = e_1 + e_3 = 0$, $w = \zeta(z)$; rectangle in z-plane is a square.

<u>Interior of rectangle on region exterior to two semi-infinite strips</u> (cf. §13.6).

$\boxed{w = \zeta(z) + e_1 z}$, or $\boxed{\frac{dw}{ds} = -\frac{1}{2}(s-e_1)^{1/2}(s-e_2)^{-1/2}(s-e_3)^{-1/2};\ s = \wp(z)}$

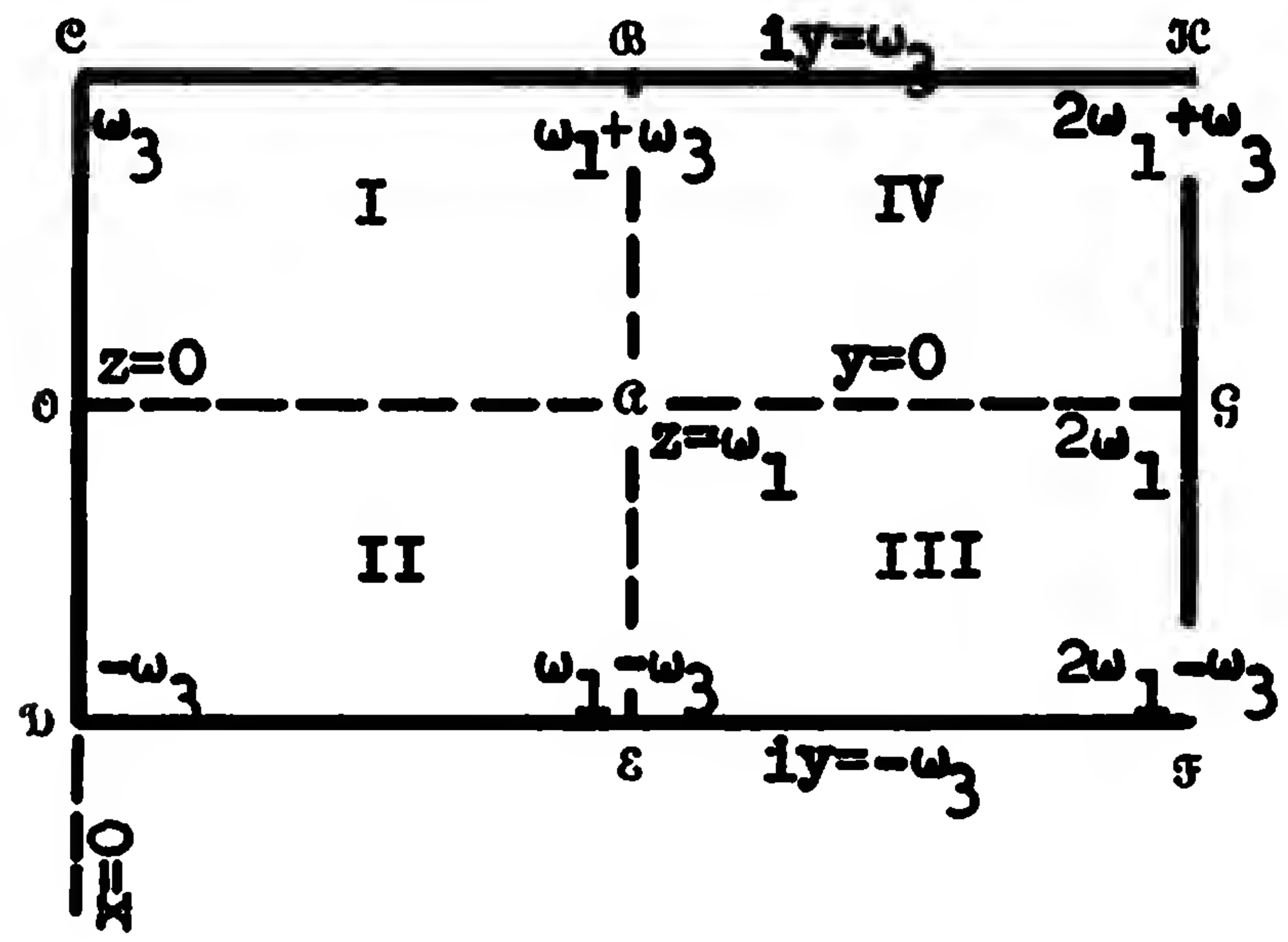
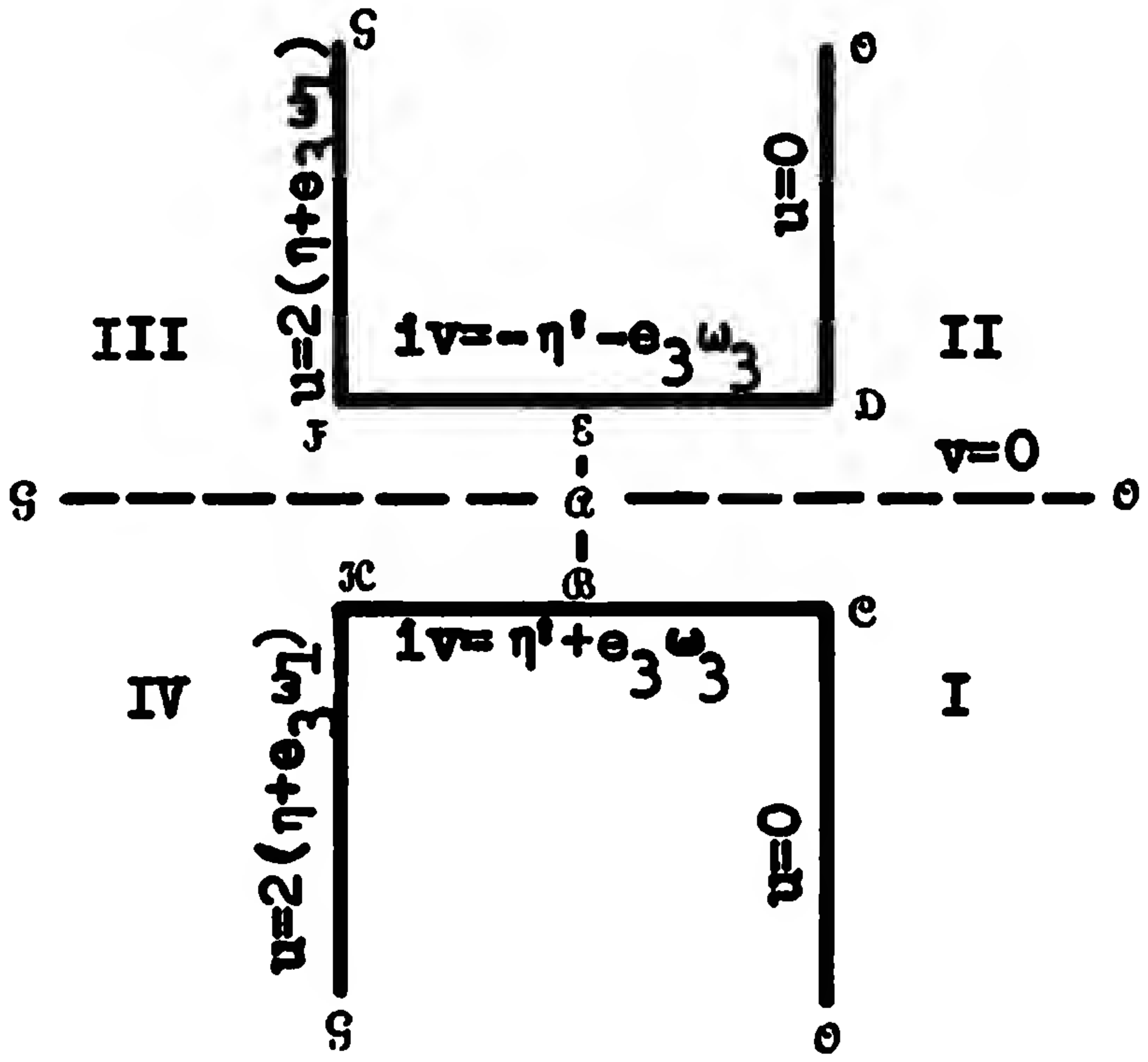
ω_1, ω_3; e_1, e_2, e_3; η, η' as above.

z - plane	w - plane



z - plane	w - plane
$z = \omega_1$ (i.e. $\mathcal{A}$); $\omega_1+\omega_3$ ($\mathcal{B}$);	$w = \eta+e_1\omega_1$ ($\mathcal{A}$); $\eta+\eta'+e_1(\omega_1+\omega_3)$ ($\mathcal{B}$);
$\omega_1+2\omega_3$ ($\mathcal{K}$);	$\eta+2\eta'+e_1(\omega_1+2\omega_3)$ ($\mathcal{K}$);
ω_3 ($\mathcal{C}$); $2\omega_3$ ($\mathcal{G}$);	$\eta'+e_1\omega_3$ ($\mathcal{C}$); ∞ ($\mathcal{G}$);
0 ($\mathcal{O}$); $-\omega_1$ ($\mathcal{E}$);	∞ ($\mathcal{O}$); $-\eta-e_1\omega_1$ ($\mathcal{E}$);
$-\omega_1+\omega_3$ ($\mathcal{D}$);	$-\eta+\eta'+e_1(\omega_3-\omega_1)$ ($\mathcal{D}$);
$-\omega_1+2\omega_3$ ($\mathcal{F}$).	$-\eta+2\eta'+e_1(2\omega_3-\omega_1)$ ($\mathcal{F}$).

$$\boxed{w = \zeta(z)+e_3 z}$$

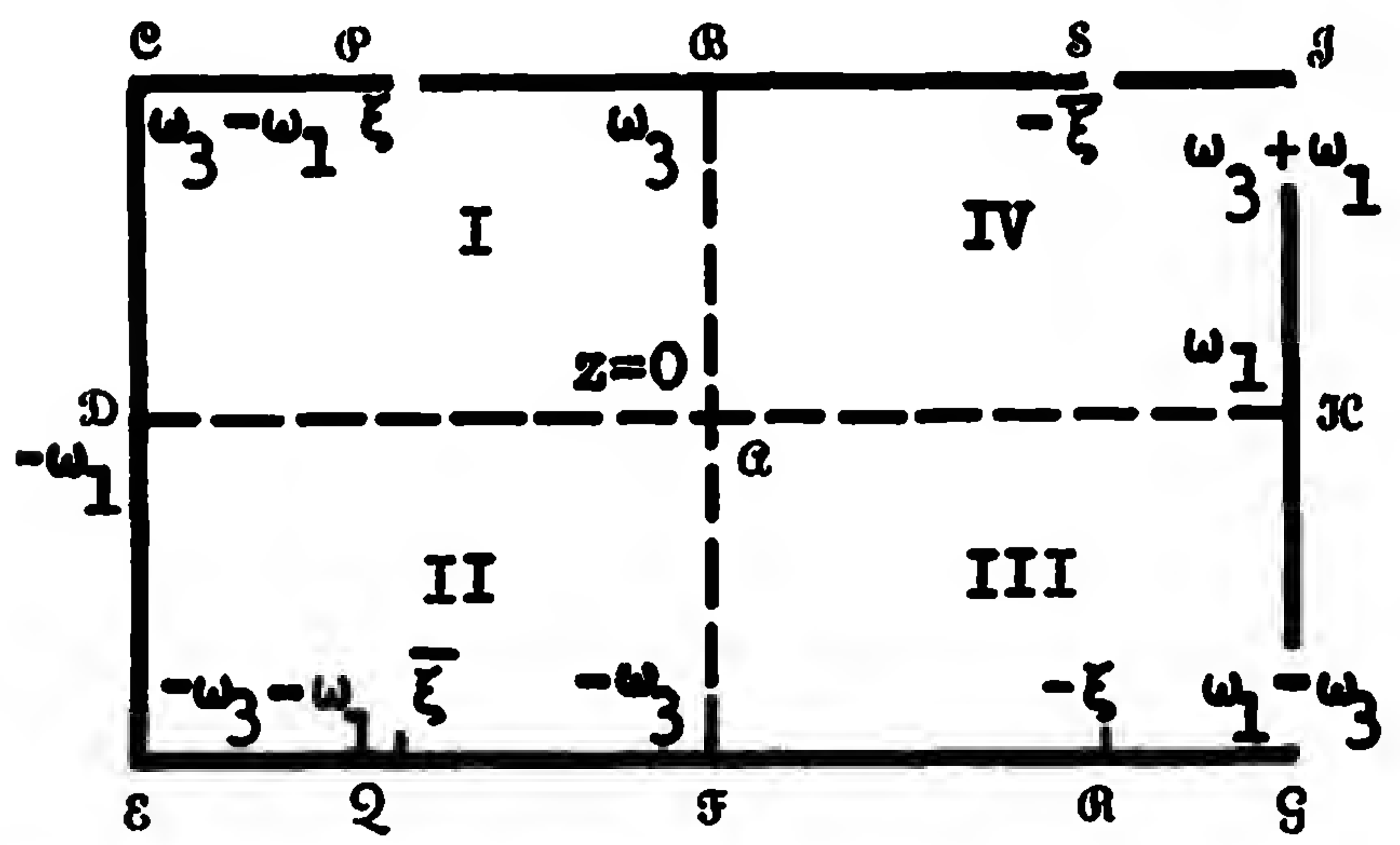
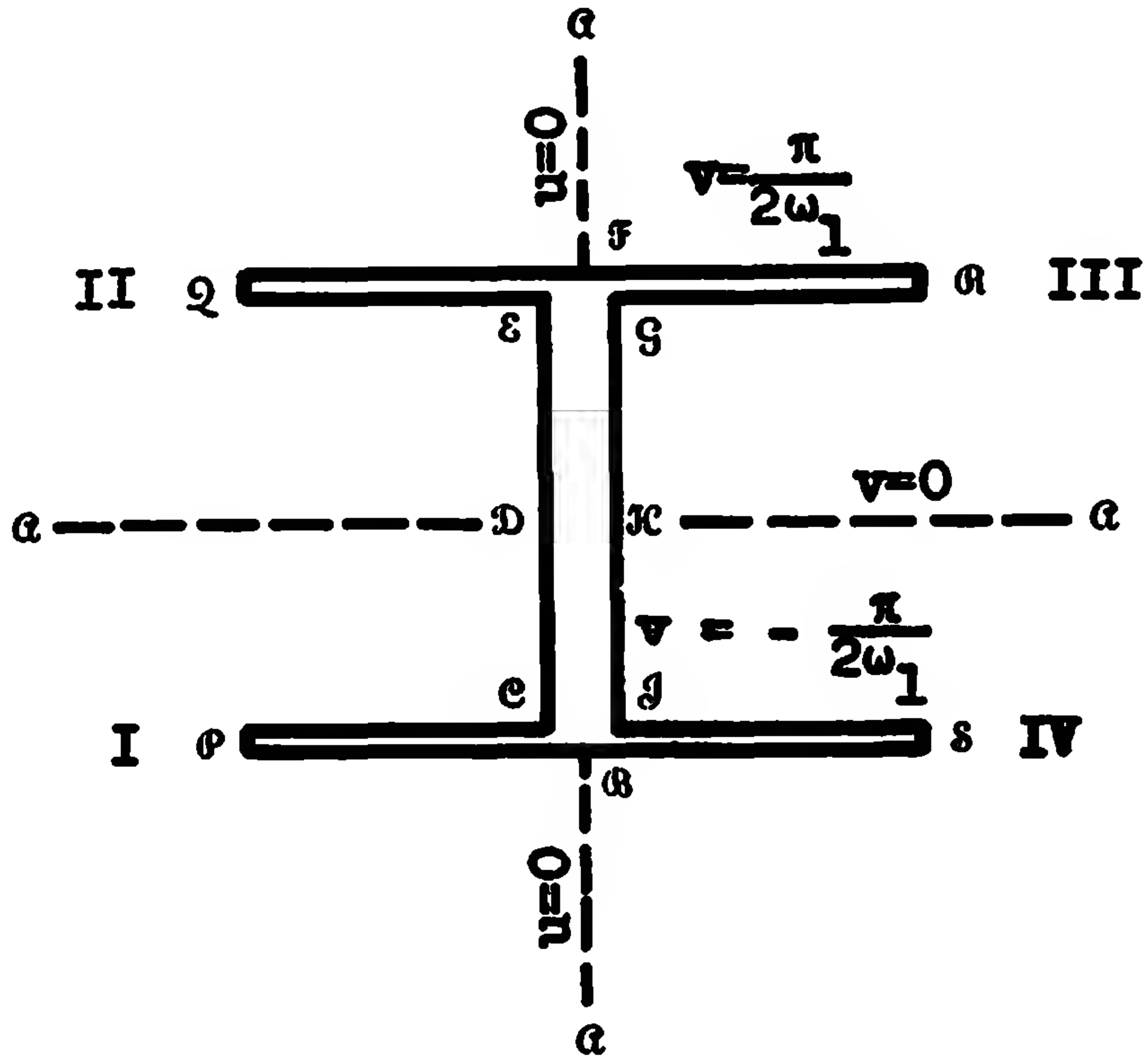
z - plane	w - plane
	
$z = \omega_1-\omega_3$ (i.e. $\mathcal{E}$); ω_1 ($\mathcal{A}$);	$w = \eta-\eta'+e_3(\omega_1-\omega_3)$ ($\mathcal{E}$); $\eta+e_3\omega_1$ ($\mathcal{A}$);
$\omega_1+\omega_3$ ($\mathcal{B}$);	$\eta+\eta'+e_3(\omega_1+\omega_3)$ ($\mathcal{B}$);
ω_3 ($\mathcal{C}$); 0 ($\mathcal{O}$);	$\eta'+e_3\omega_3$ ($\mathcal{C}$); ∞ ($\mathcal{O}$);
$-\omega_3$ ($\mathcal{D}$); $2\omega_1-\omega_3$ ($\mathcal{F}$);	$-\eta'-e_3\omega_3$ ($\mathcal{D}$); $2\eta-\eta'+e_3(2\omega_1-\omega_3)$ ($\mathcal{F}$);
$2\omega_1$ ($\mathcal{G}$); $2\omega_1+\omega_3$ ($\mathcal{K}$).	∞ ($\mathcal{G}$); $2\eta+\eta'+e_3(2\omega_1+\omega_3)$ ($\mathcal{K}$).

Rectangle on cut plane.

$$\boxed{w = \zeta(z) - \frac{\eta}{\omega_1} z} = \frac{1}{2\omega_1}\, \vartheta_1'\!\left(\frac{z}{2\omega_1}\Big|\tau\right) \Big/ \vartheta_1\!\left(\frac{z}{2\omega_1}\Big|\tau\right).$$

$\omega_1 > 0$, $\omega_3/i > 0$, $e_1, e_2, e_3, \eta, \eta'$ as above; $\tau = \omega_3/\omega_1$.

ξ is a root of $\wp\xi = -\eta/\omega_1$; $\Im(\xi) = \omega_3/i$, $0 > \Re(\xi) > -\omega_1$.

z - plane	w - plane
	
$z = \omega_3-\omega_1$ (i.e. $\mathfrak{C}$); $\omega_1+\omega_3$ ($\mathfrak{J}$); ω_3 ($\mathfrak{B}$);	$w = -\frac{\pi i}{2\omega_1}$ (i.e. $\mathfrak{C}$; $\mathfrak{J}$; $\mathfrak{B}$)
$z = -\omega_3-\omega_1$ (i.e. $\mathfrak{E}$); $\omega_1-\omega_3$ ($\mathfrak{G}$); $-\omega_3$ ($\mathfrak{F}$);	$w = \frac{\pi i}{2\omega_1}$ ($\mathfrak{E}$; $\mathfrak{G}$; $\mathfrak{F}$)
$z = -\omega_1$ (i.e. $\mathfrak{D}$); ω_1 ($\mathfrak{K}$); 0 ($\mathfrak{A}$)	$w = 0$ ($\mathfrak{D}$); 0 ($\mathfrak{K}$); ∞ ($\mathfrak{A}$).
$z = \xi$ (i.e. $\mathfrak{P}$); $\bar{\xi}$ ($\mathfrak{Q}$); $-\bar{\xi}$ ($\mathfrak{S}$); $-\xi$ ($\mathfrak{R}$)	$w_p = \zeta(\xi) - \frac{\eta\xi}{\omega_1}$; $\bar{w}_p$ ($\mathfrak{Q}$); $-\bar{w}_p$ ($\mathfrak{S}$); $-w_p$ ($\mathfrak{R}$).
line segments $y = 0$, $-\omega_1 < x < 0$ or $0 < x < \omega_1$, respectively	half-lines $v = 0$, $0 > u > -\infty$ or $\infty > u > 0$.

The transformation is a combination of

$$\frac{dw}{ds} = -\frac{s + \eta/\omega_1}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}} \quad \text{and} \quad s = \wp z.$$

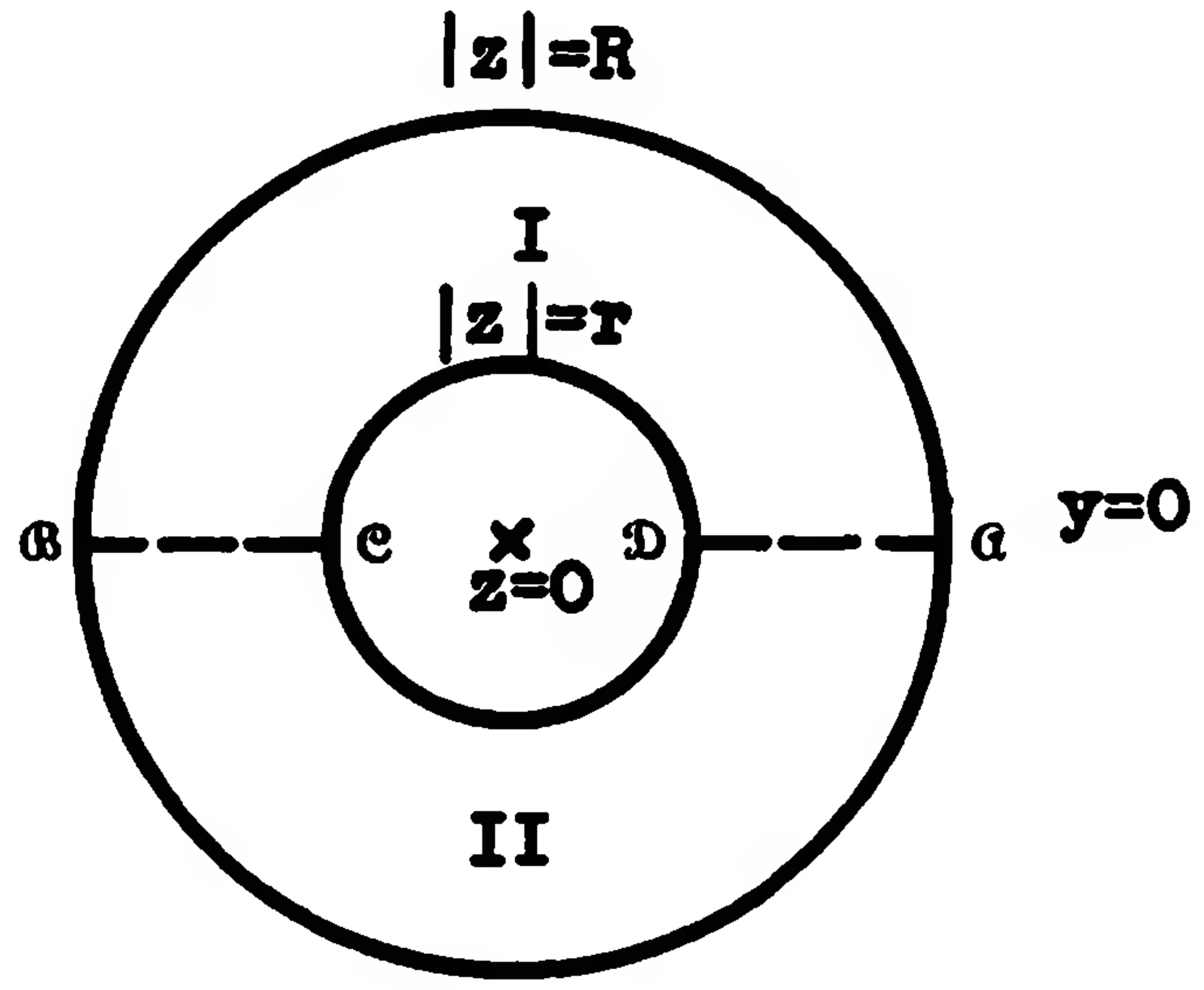
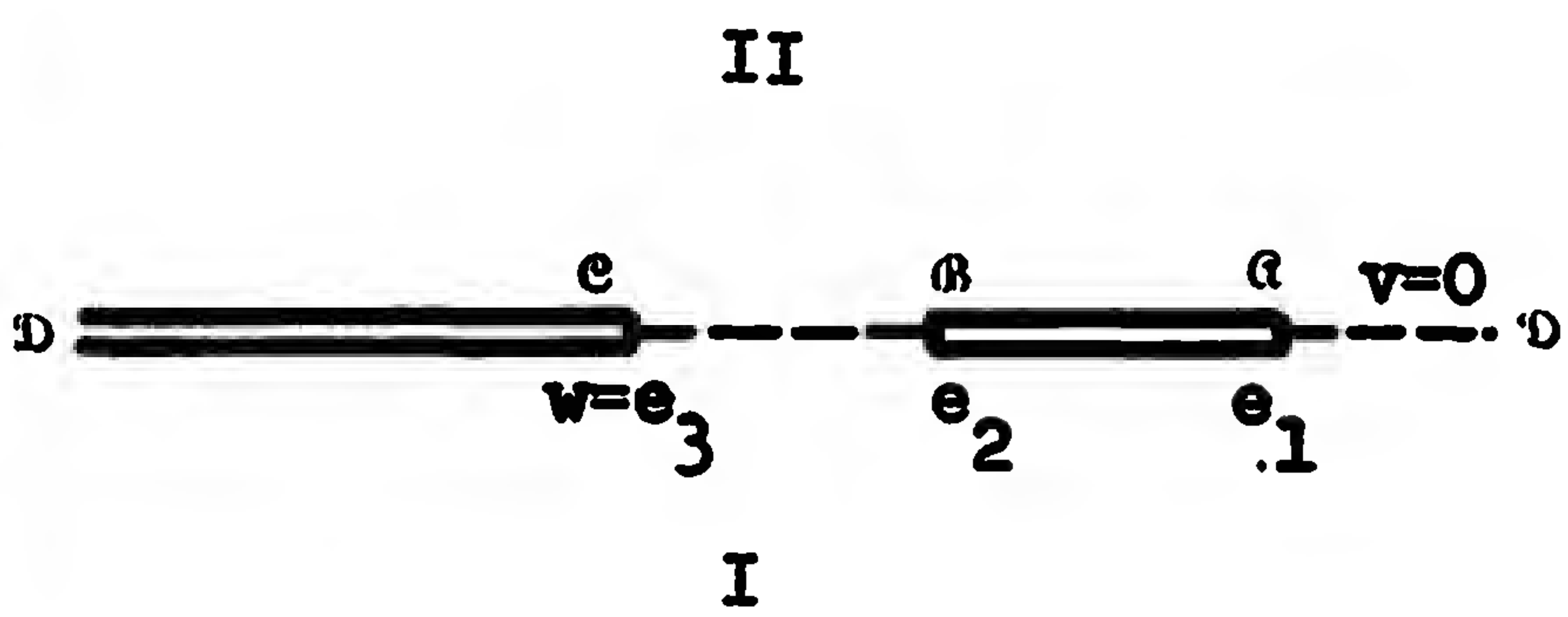
Some doubly connected regions

on plane with two slits at least one of which is finite.

13.11 Ring on plane with two slits in line.

$\boxed{w = \wp(\log \frac{z}{r})}$; $\omega_1 = \log \frac{R}{r}$, $\omega_3 = i\pi$.

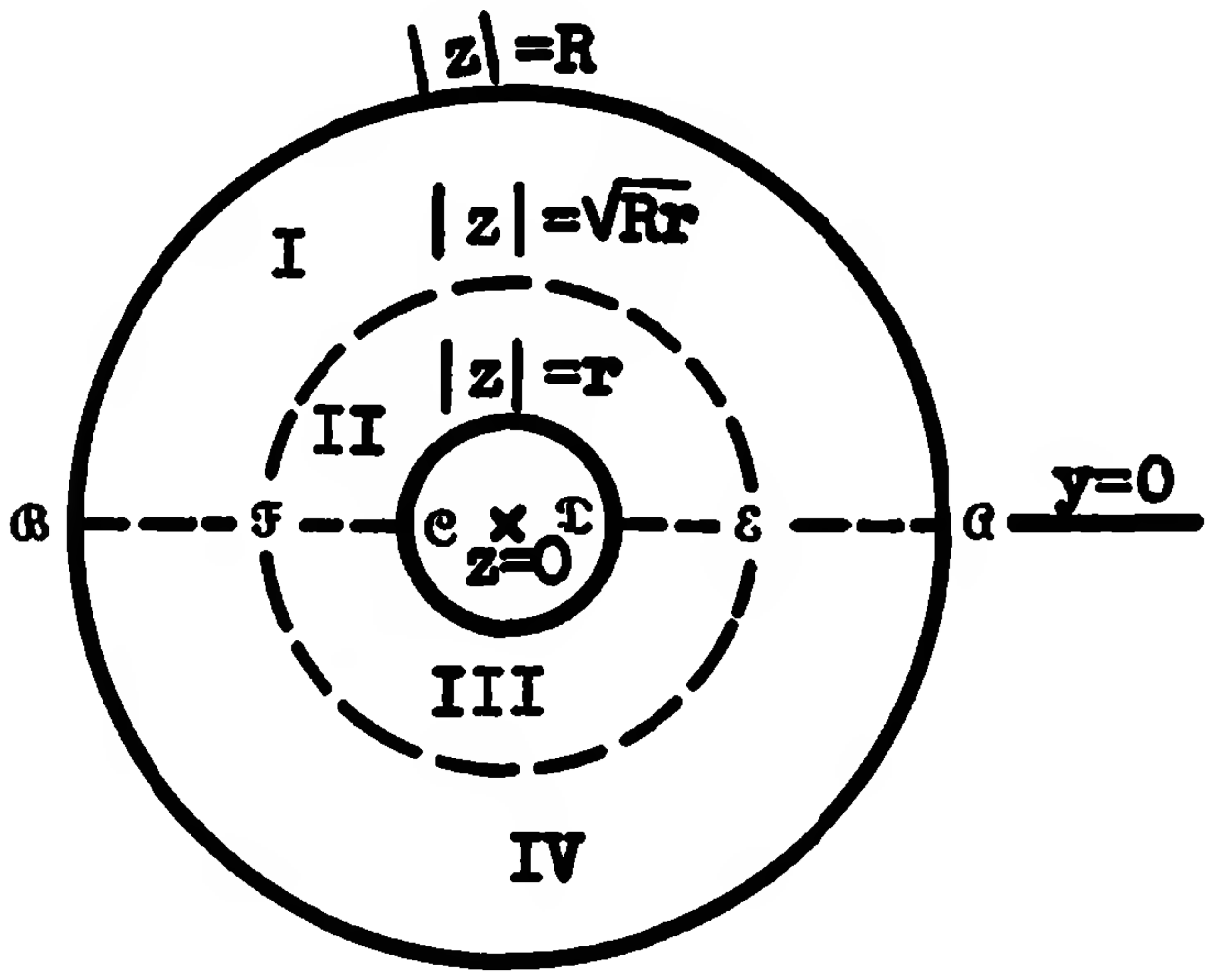
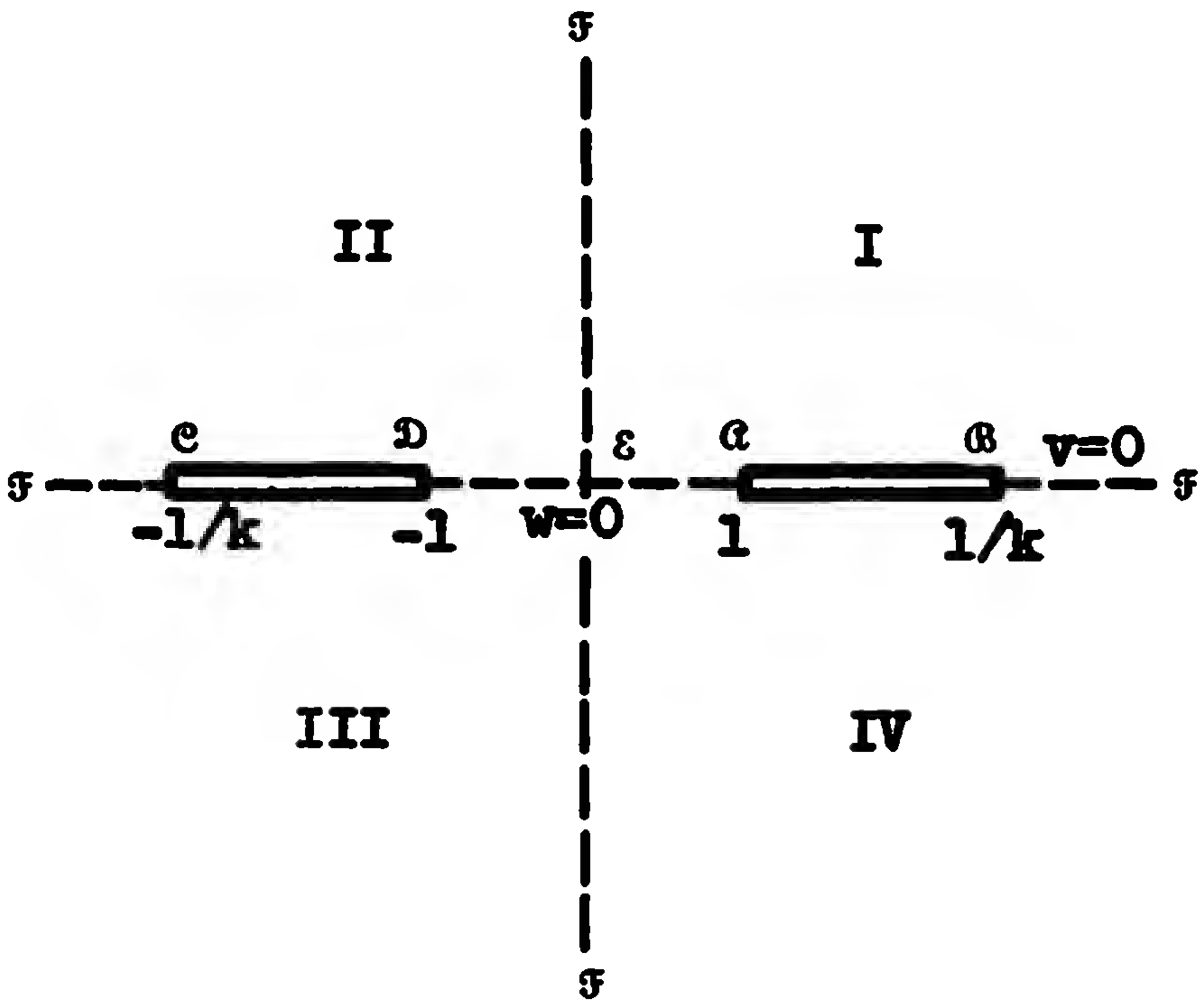
Given: $r > 0$, $R > r$.

z - plane	w - plane
	
$z = R$ (i.e. a); r (D); $-R$ (ß); $-r$ (C)	$w = e_1$ (a); ∞ (D); e_2 (ß); e_3 (C)
circle $\lvert z\rvert = R$	segment $e_2 \le u \le e_1$ of $v = 0$, counted twice.
circle $\lvert z\rvert = r$	half-line $-\infty \le u \le e_3$, $v = 0$, counted twice.

$$\boxed{w = \operatorname{sn}\left(\frac{2K}{\log R/r}\log\frac{z}{\sqrt{Rr}}\right)}$$

Given: r, R; $R > r > 0$. $\tau = \dfrac{2i\pi}{\log R/r}$, $k = \left(\dfrac{\theta_2(\tau)}{\theta_3(\tau)}\right)^2$; K, K' as in §13.1;

here $\dfrac{K'}{K} = \dfrac{2\pi}{\log R/r}$.

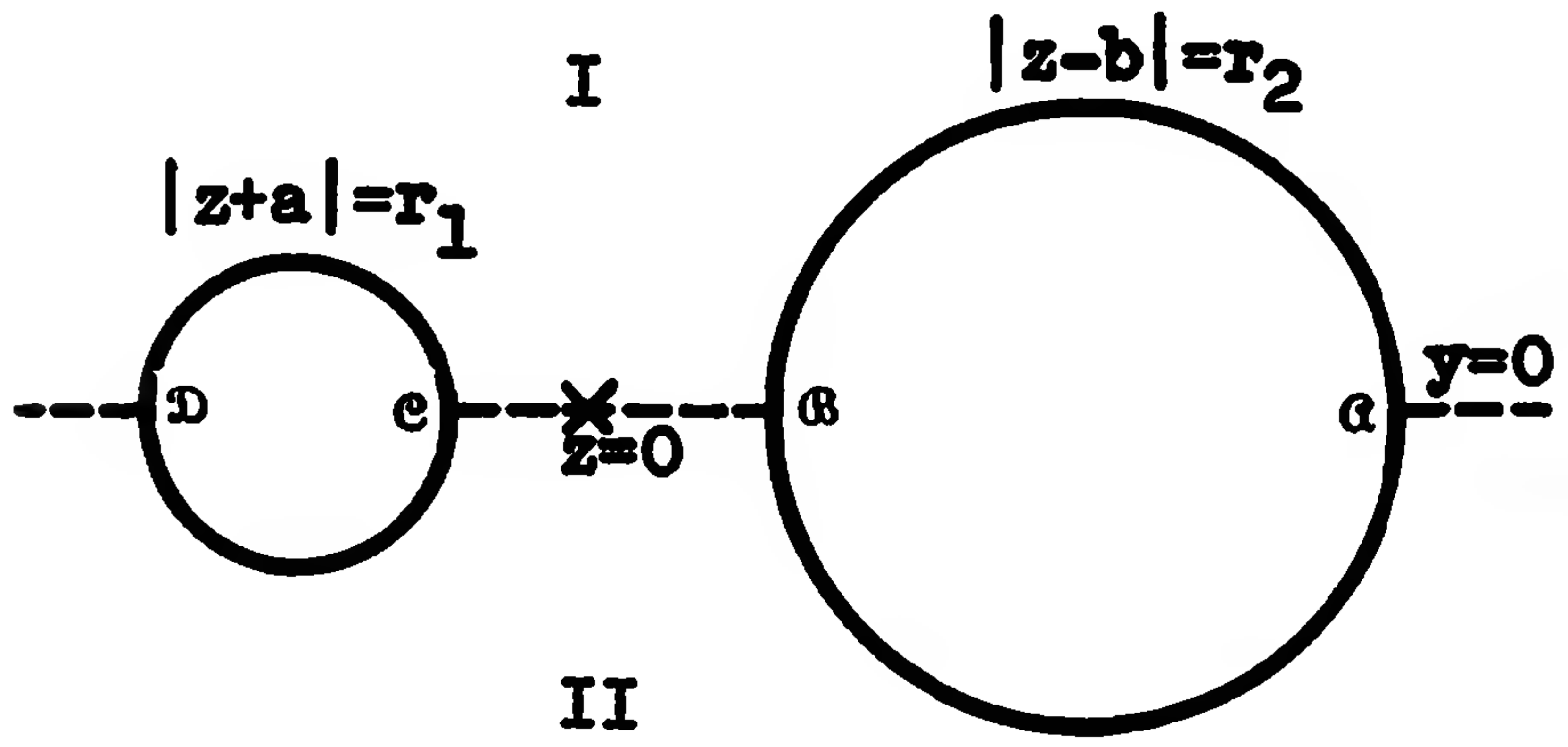
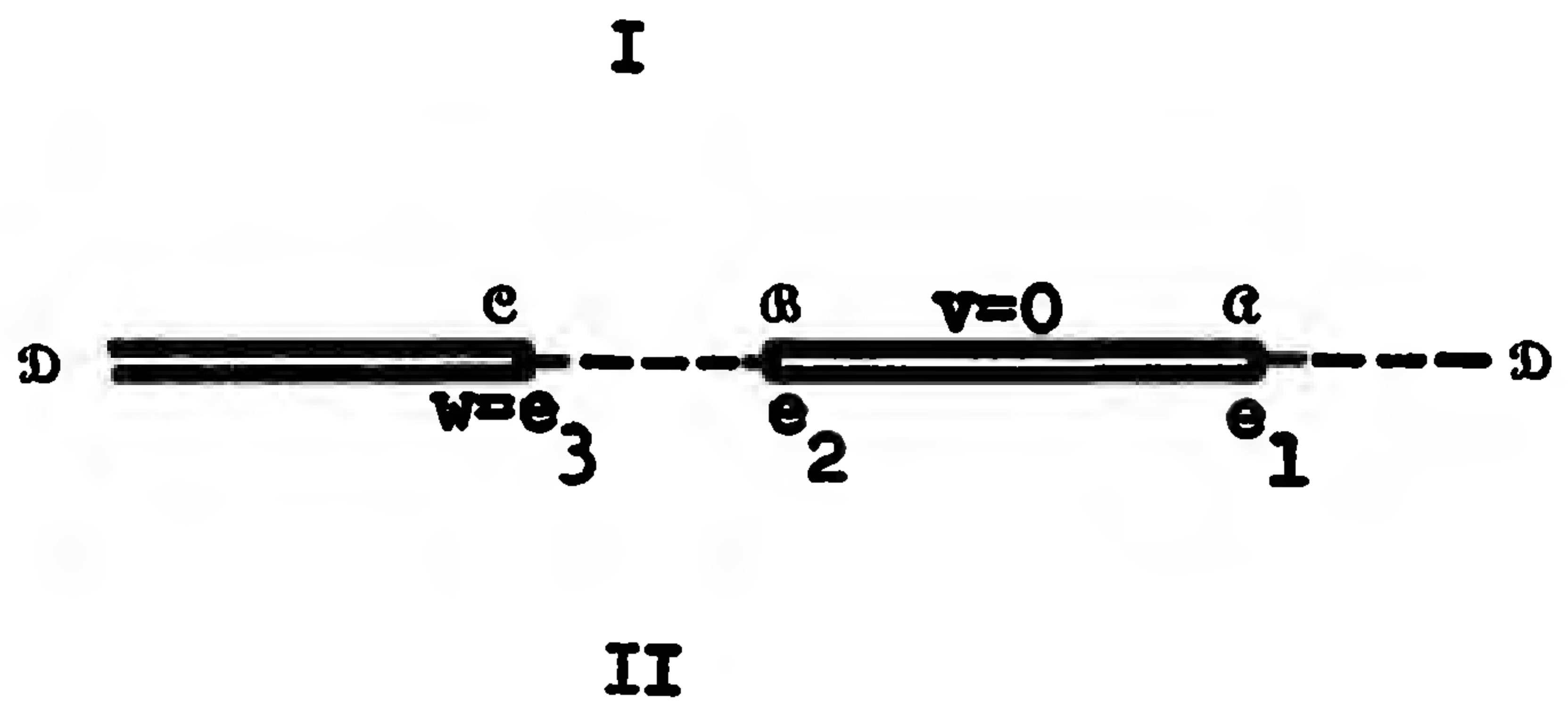
z - plane	w - plane
	
$z = R$ (i.e. $\mathfrak{A}$); $\sqrt{Rr}$ ($\mathfrak{E}$); r ($\mathfrak{D}$); $-r$ ($\mathfrak{C}$); $-\sqrt{Rr}$ ($\mathfrak{F}$); $-R$ ($\mathfrak{B}$).	$w = 1$ ($\mathfrak{A}$); 0 ($\mathfrak{E}$); -1 ($\mathfrak{D}$); $-1/k$ ($\mathfrak{C}$); ∞ ($\mathfrak{F}$); $1/k$ ($\mathfrak{B}$).
circle $\|z\| = R$	segment $1 \leq u \leq 1/k$ of $v = 0$, counted twice.
circle $\|z\| = r$	segment $-1/k \leq u \leq -1$ of $v = 0$, counted twice.
circle $\|z\| = \sqrt{Rr}$	line $u = 0$.

Plane with two circular holes on plane with two **slits in line**.

$$\boxed{w = \wp\left(\log \frac{z+c}{z-c} + \frac{1}{2}\log\frac{a+c}{a-c}\right)} \quad ;$$

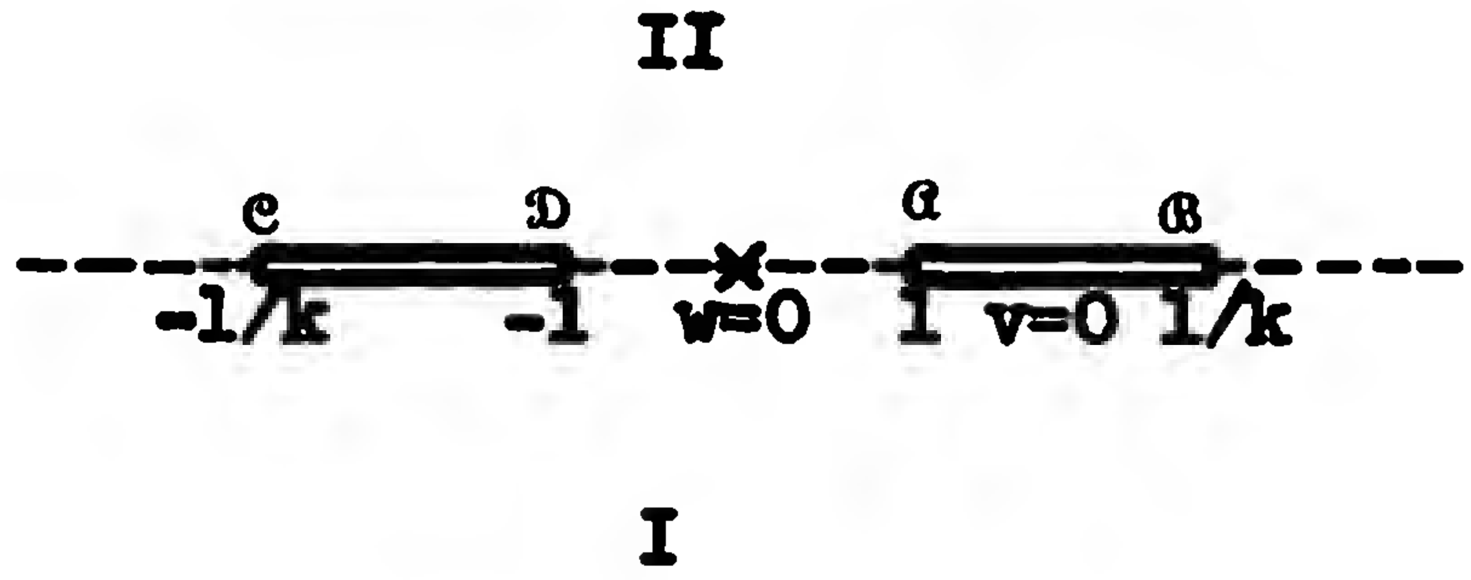
$$\omega_1 = \frac{1}{2}\log\frac{(b+c)(a+c)}{(b-c)(a-c)}, \quad \omega_3 = i\pi.$$

Given: a, b, r_1, r_2, all positive; but $a^2-r_1^2 = b^2-r_2^2$; $c = \sqrt{a^2-r_1^2} > 0$.

z - plane	w - plane
	
$z = b+r_2$ (i.e. $\mathcal{A}$); $b-r_2$ ($\mathcal{B}$); $-a+r_1$ ($\mathcal{C}$); $-a-r_1$ ($\mathcal{D}$)	$w = e_1$; e_2; e_3; ∞
circle $\lvert z-b\rvert = r_2$	segment $e_2 \le u \le e_1$ of $v = 0$, counted twice
circle $\lvert z+a\rvert = r_1$	half-line $v = 0$, $e_3 \ge u \ge -\infty$, counted twice.

$$\boxed{w = \mathrm{sn}\left\{\frac{K}{\lambda}\log\frac{z+c}{z-c} + \rho\right\}} ;$$

$$\lambda = \sqrt[4]{\frac{b+c}{b-c}\cdot\frac{a+c}{a-c}}, \quad \tau = \frac{i\pi}{\lambda}, \quad k = \left(\frac{\theta_2(\tau)}{\theta_3(\tau)}\right)^2 ; \quad \rho = K\log\left(\frac{a+c}{a-c}\cdot\frac{b-c}{b+c}\right)\Big/(4\lambda).$$

z - plane	w - plane
Figure as above	
$z = b+r_2$ (i.e. $\mathcal{A}$); $b-r_2$ ($\mathcal{B}$); $-a+r_1$ ($\mathcal{C}$); $-a-r_1$ ($\mathcal{D}$)	$w = 1;\ 1/k;$ $-1/k;\ -1.$
circle $\lvert z-b\rvert = r_2$	segment $1 \leq u \leq 1/k$ of $v = 0$, counted twice
circle $\lvert z+a\rvert = r_1$	segment $-1 \geq u \geq -1/k$ of $v = 0$, counted twice.

13.12 <u>Ring on plane with two parallel slits.</u>

$$\boxed{w = -\zeta\left(i \log \frac{z}{\sqrt{Rr}}\right) + \frac{\eta i}{\pi} \log \frac{z}{\sqrt{Rr}}}$$

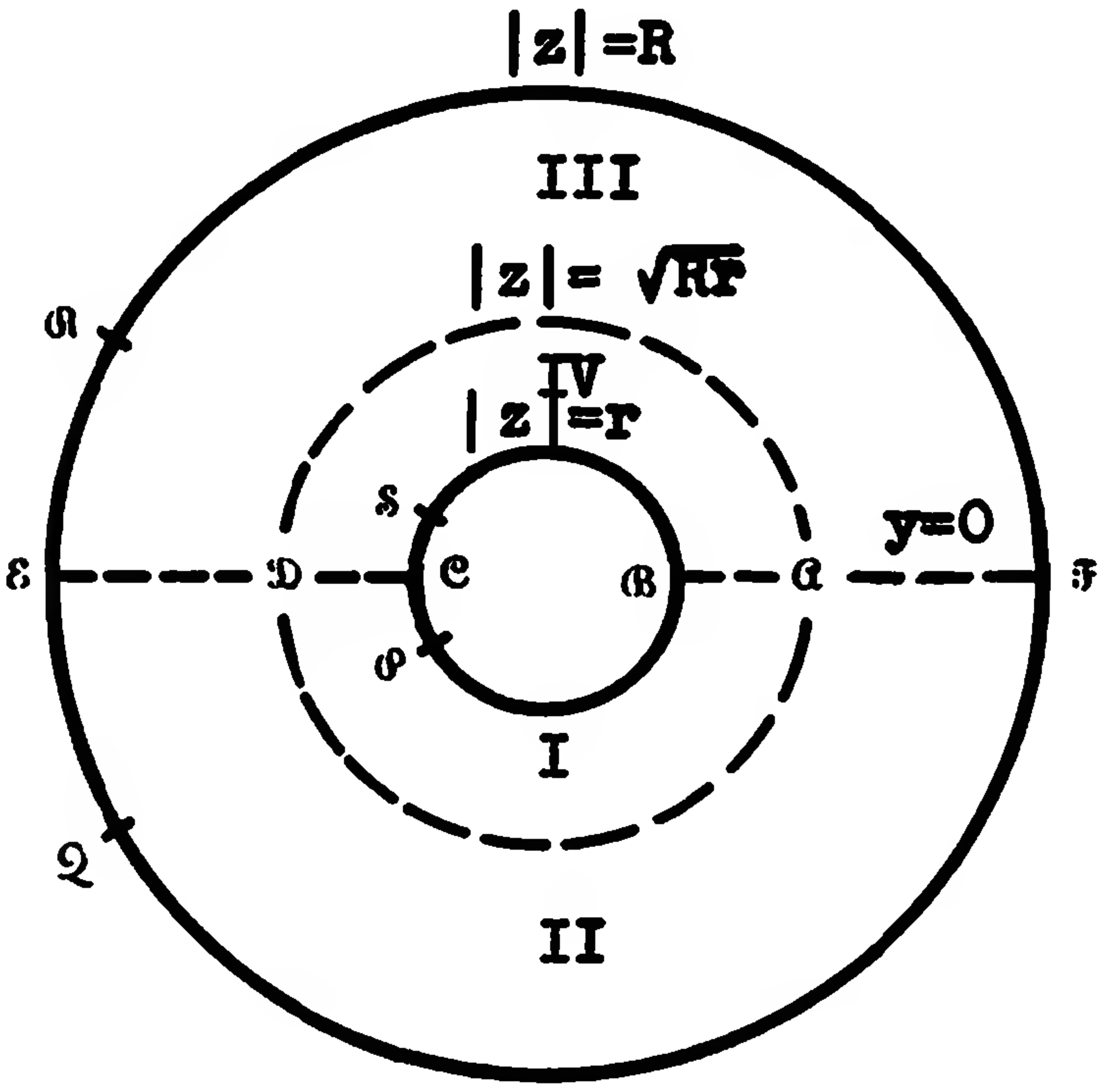
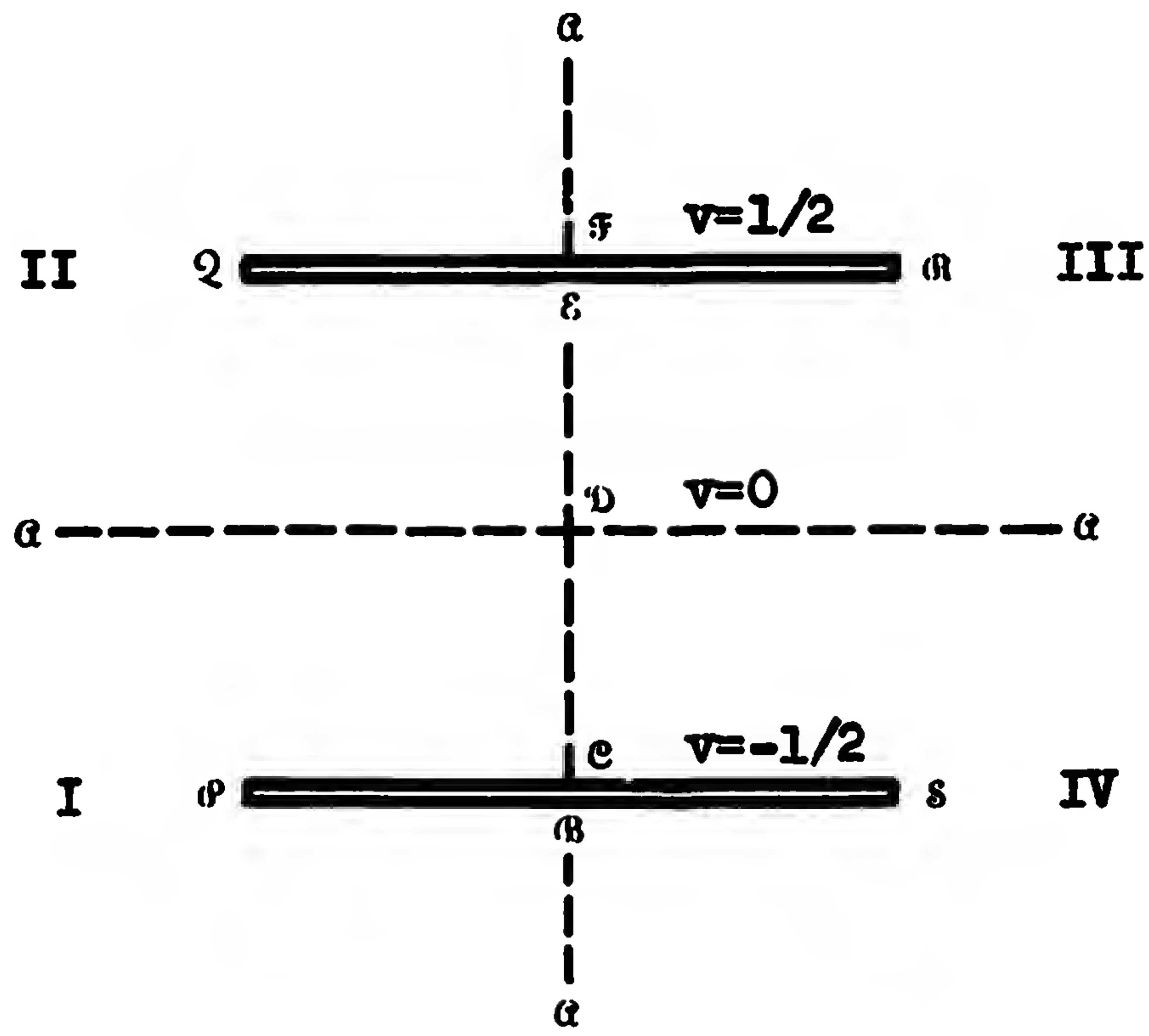
(cf. §10.3 and §13.10; pp. 92, 190)

Given: $r, R;\quad r > 0,\quad R > r.$

$\omega_1 = \pi,\quad \omega_3 = \frac{1}{2}\log\frac{R}{r}$. λ is the root of $\wp(\lambda+\omega_3) = -\frac{\eta}{\pi}$,

$-\pi < \lambda < 0;\quad \zeta(\omega_1) = \eta.\quad \xi = \lambda + \omega_3,\quad \bar{\xi} = \lambda - \omega_3.$

z - plane	w - plane
See figures on next page	
$z = R$ (i.e. $\mathcal{F}$); $\sqrt{Rr}$ ($\mathcal{A}$); r ($\mathcal{B}$);	$w = \frac{1}{2}$ ($\mathcal{F}$); ∞ ($\mathcal{A}$); $-\frac{1}{2}$ ($\mathcal{B}$);
$-r$ ($\mathcal{C}$); $-\sqrt{Rr}$ ($\mathcal{D}$); $-R$ ($\mathcal{E}$);	$-\frac{1}{2}$ ($\mathcal{C}$); 0 ($\mathcal{D}$); $\frac{1}{2}$ ($\mathcal{E}$);
$re^{i\lambda}$ ($\mathcal{P}$); $Re^{i\lambda}$ ($\mathcal{Q}$);	$\zeta(\xi) - \frac{\eta\xi}{\pi}$ ($\mathcal{P}$); $\zeta(\bar{\xi}) - \frac{\eta\bar{\xi}}{\pi}$ ($\mathcal{Q}$);
$re^{-i\lambda}$ ($\mathcal{S}$); $Re^{-i\lambda}$ ($\mathcal{R}$)	$-\zeta(\bar{\xi}) + \frac{\eta\bar{\xi}}{\pi}$ ($\mathcal{S}$); $-\zeta(\xi) + \frac{\eta\xi}{\pi}$ ($\mathcal{R}$).

z - plane	w - plane
	
circles $\|z\| = R$; $\|z\| = r$	slits $\mathcal{Q}\mathcal{R}$; $\mathcal{P}\,\mathcal{S}$
circle $\|z\| = \sqrt{Rr}$	line $v = 0$
The positions of $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$ in the diagram are not supposed to be exact.	

<u>Plane, with two circular holes, on plane with two parallel slits.</u>

$$\boxed{w = \zeta(\chi) + \frac{\eta\chi}{\pi}}, \quad \text{where } \chi = i \log\left\{\frac{z+c}{z-c}\left(\frac{a+c}{a-c}\cdot\frac{b-c}{b+c}\right)^{1/4}\right\};$$

$$\omega_1 = \pi, \quad \omega_3 = i \log\left(\frac{a+c}{a-c}\cdot\frac{b+c}{b-c}\right)^{1/4}; \quad \zeta(\omega_1) = \eta, \quad \zeta(\omega_3) = \eta'.$$

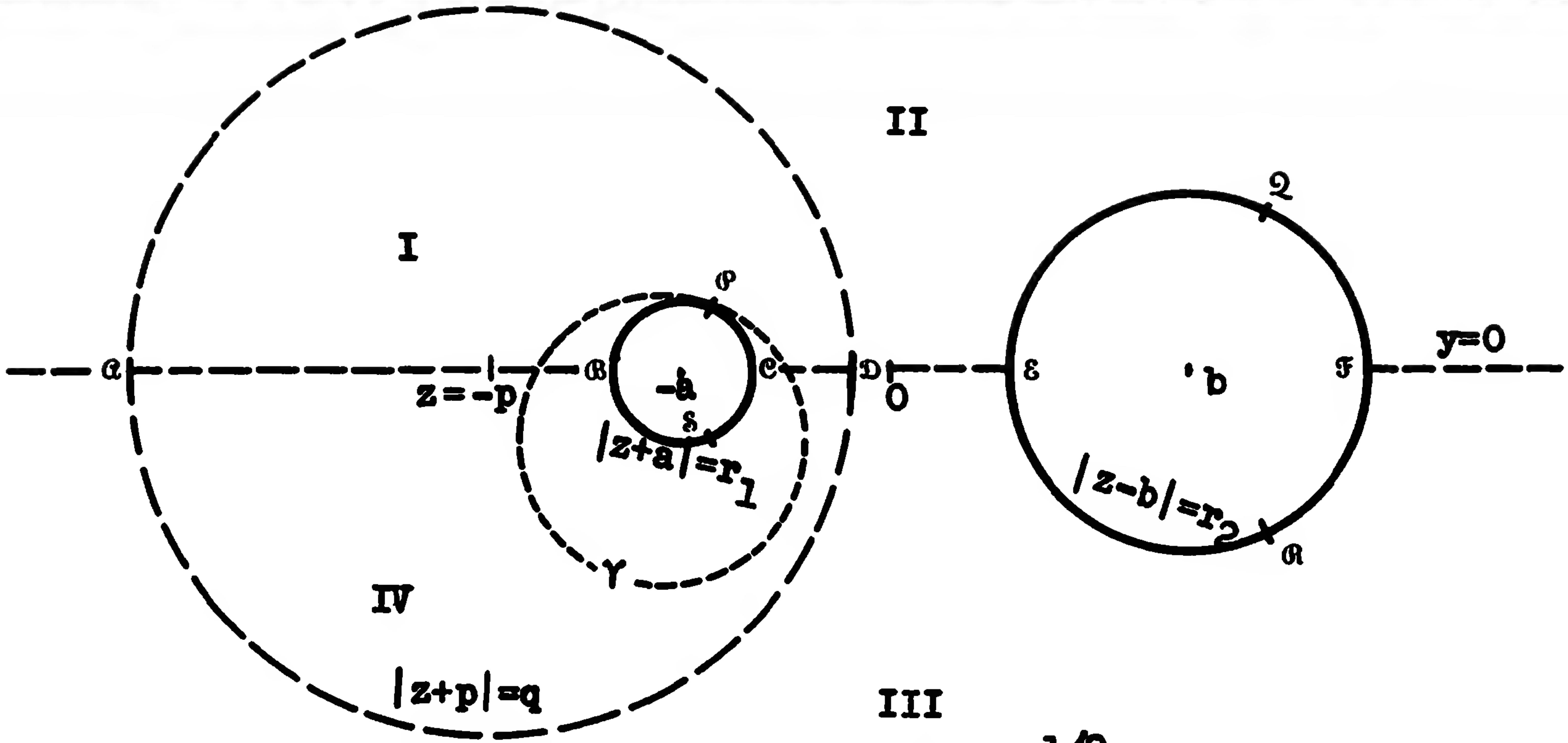
Given: a, b, r_1, r_2, all positive, but $a^2-r_1^2 = b^2-r_2^2$; $c = \sqrt{a^2-r_1^2} > 0$.

λ is the root of $\wp(\lambda+\omega_3) = -\eta/\pi$; $-\pi < \lambda < 0$.

Figure in w - plane as before.

z - plane
See figure on next page

z - plane



$$p = c\,\frac{\mu+1}{\mu-1}\,, \quad q = \frac{2c\sqrt{\mu}}{|\mu-1|}, \text{ where } \mu = \left(\frac{a-c}{a+c}\,\frac{b+c}{b-c}\right)^{1/2}.$$

$$\mathfrak{P}: \quad z = -a+r_1e^{i\varphi}\,, \quad \text{where } \tan\varphi = \frac{c\sin\lambda}{a\cos\lambda - r_1}; \quad \mathfrak{S}: \quad z = -a+r_1e^{-i\varphi}$$

$$\mathfrak{Q}: \quad z = b+r_2e^{i\psi}\,, \quad \text{where } \tan\psi = \frac{c\sin\lambda}{r_2 - b\cos\lambda}; \quad \mathfrak{R}: \quad z = b+r_2e^{-i\psi}\,.$$

The positions of these points in the diagram are not exact.

z - plane	w - plane
circles $\lvert z+a\rvert = r_1$; $\lvert z-b\rvert = r_2$	slit $\mathfrak{P}\,\mathfrak{S}$; slit $\mathfrak{Q}\,\mathfrak{R}$
circle $\lvert z+p\rvert = q$	line $v = 0$
segment $-a+r_1 \le x \le b-r_2$ of $y=0$	segment $-\frac{1}{2} \le v \le \frac{1}{2}$ of $u=0$ (i.e. $\mathfrak{C}\,\mathfrak{E}$)
segment $-p-q < x \le -a-r_1$ of $y=0$	half-line $u=0$, $-\infty < v \le -\frac{1}{2}$(i.e. $\mathfrak{A}\,\mathfrak{B}$)
half-lines $y=0$, $x \ge b+r_2$, and $y=0$, $-\infty \le x \le -p-q$, together	half-line $u=0$, $\frac{1}{2} < v < \infty$ (i.e. $\mathfrak{F}\,\mathfrak{A}$)
circle γ, touching $\lvert z+a\rvert = r_1$ at $\mathfrak{P}$, exterior to $\lvert z+a\rvert = r_1$ and to $\lvert z-b\rvert = r_2$	aerofoil, surrounding slit $\mathfrak{P}\,\mathfrak{S}$, cusp at $\mathfrak{P}$ (not in figure)

If $a=b$, then $r_1=r_2$, and $|z+p|=q$ is replaced by $x=0$.

For a similar transformation, with details, see Tricomi. pp. 239-246, and C. Ferrari.

14. OTHER FUNCTIONS.

14.1 The modular function $w = \lambda(\tau)$.

Curvilinear triangles with angles

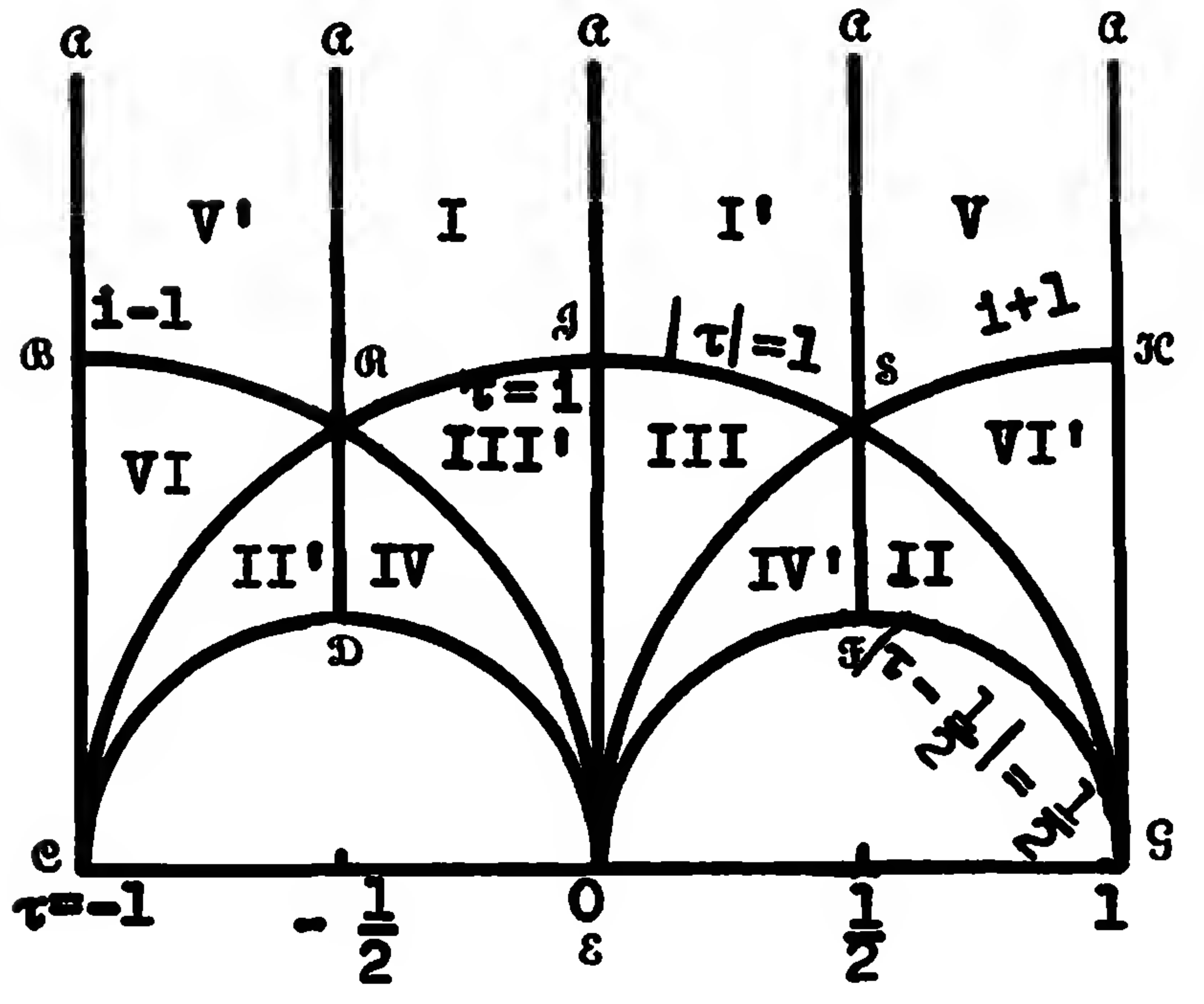
$0, 0, 0;\quad \pi/3, \pi/3, 0;\quad \pi/2, \pi/3, 0;\quad \pi/2, 0, 0;\quad 2\pi/3, 0, 0.$

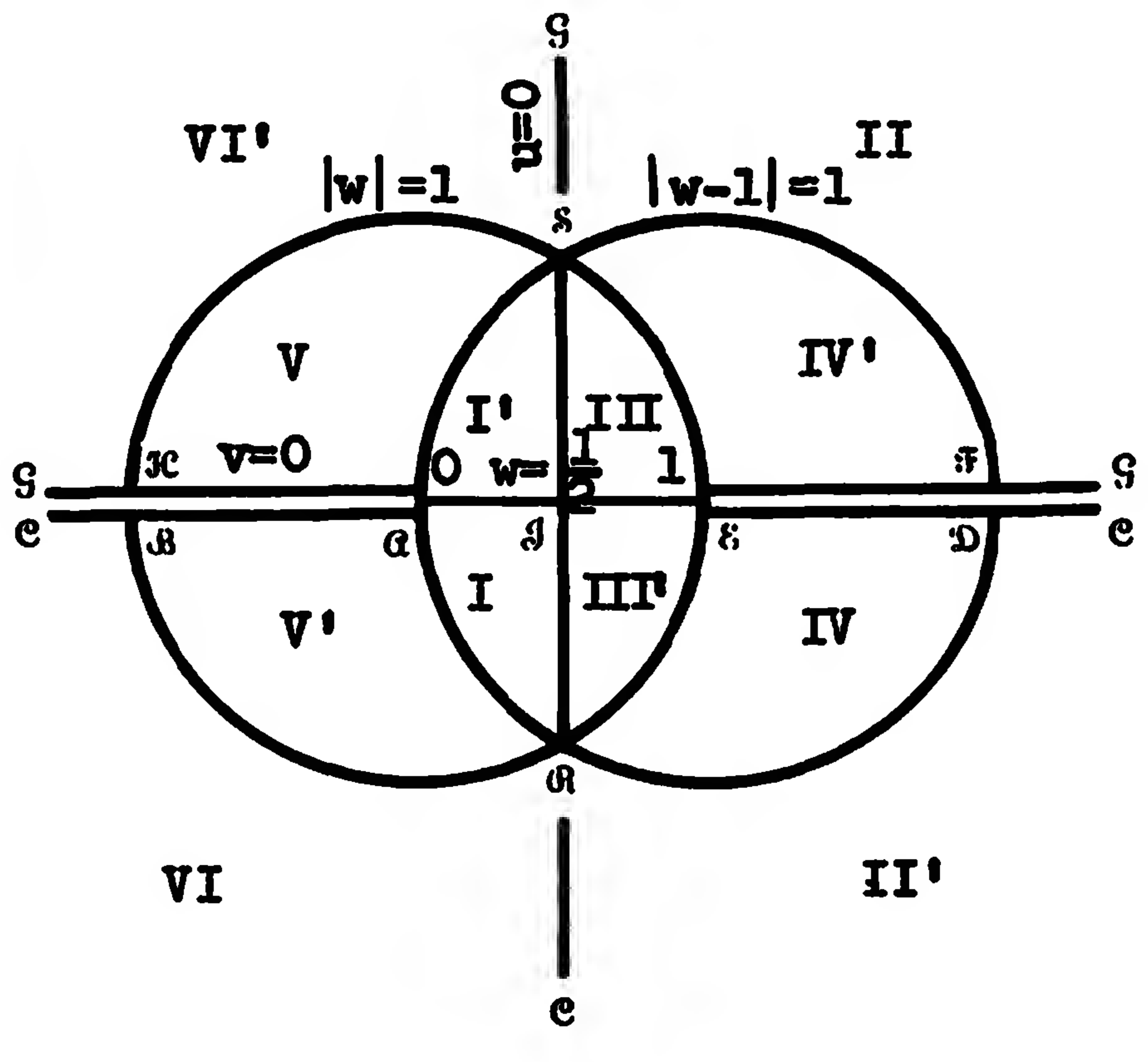
$$\boxed{w = \lambda(\tau) = \frac{\theta_2^4(\tau)}{\theta_3^4(\tau)}} = k^2, \quad \mathfrak{I}(\tau) > 0; \quad \tau = x+iy.$$

$\alpha, \beta, \gamma, \delta$ integers; α and δ odd, γ and β even; $\alpha\delta - \gamma\beta = 1.$

Critical points: $\tau = \infty$; any point on $y = 0$.

τ - plane	w - plane
Points $\frac{\gamma+\delta\tau}{\alpha+\beta\tau}$; $\frac{\tau}{\tau+1}$; $\frac{1}{\tau}$; $\tau+1$; $\frac{1}{\tau+1}$; $\frac{\tau+1}{\tau}$;	$w = \lambda(\tau)$; $\frac{1}{\lambda(\tau)}$; $1-\lambda(\tau)$; $\frac{\lambda(\tau)}{\lambda(\tau)-1}$; $\frac{1}{1-\lambda(\tau)}$; $\frac{\lambda(\tau)-1}{\lambda(\tau)}$
$\tau = i$ (i.e. $\mathfrak{J}$); $i-1$ ($\mathfrak{B}$);	$w = \frac{1}{2}$ ($\mathfrak{J}$); -1 ($\mathfrak{B}$);
$-\frac{1}{2}+\frac{i}{2}\sqrt{3}$ ($\mathfrak{R}$); $\frac{1}{2}+\frac{i}{2}\sqrt{3}$ ($\mathfrak{S}$);	$\frac{1}{2}-\frac{i}{2}\sqrt{3}$ ($\mathfrak{R}$); $\frac{1}{2}+\frac{i}{2}\sqrt{3}$ ($\mathfrak{S}$);
$\pm\frac{1}{2}+\frac{i}{2}$ ($\mathfrak{F}$, $\mathfrak{D}$).	2 ($\mathfrak{F}$, $\mathfrak{D}$).





τ - plane	w - plane
Area of curvilinear quadrilateral, bounded by half-lines $x = \pm 1$ $(y \geq 0)$ and by semi-circles $\lvert\tau \pm \frac{1}{2}\rvert = \frac{1}{2}$ $(y \geq 0)$	whole plane, with the two slits
Area of curvilinear triangle $\mathfrak{a}\,\mathfrak{e}\,\mathfrak{g}\,\mathfrak{G}\,\mathfrak{a}$, with angles 0, 0, 0	cut half-plane $u < 0$ (i.e. VI+V'+I+I'+V+VI')
Area of curvilinear triangle $\mathfrak{a}\,\mathfrak{R}\,\mathfrak{g}\,\mathfrak{s}\,\mathfrak{a}$, angles $\pi/3$, $\pi/3$, 0	region I+I' (see §7.1)
Area of curvilinear triangle $\mathfrak{a}\,\mathfrak{g}\,\mathfrak{s}\,\mathfrak{a}$, angles $\pi/2$, $\pi/3$, 0	region I' (see §7.3)
Area of curvilinear triangle $\mathfrak{a}\,\mathfrak{g}\,\mathfrak{s}\,\mathfrak{G}\,\mathfrak{a}$, angles $\pi/2$, 0, 0	quadrant $u < 0$, $v > 0$ (i.e. I'+V+VI')
Area of curvilinear triangle $\mathfrak{a}\,\mathfrak{g}\,\varepsilon\,\mathfrak{s}\,\mathfrak{a}$, angles 0, $2\pi/3$, 0	region I'+III (see §7.3)

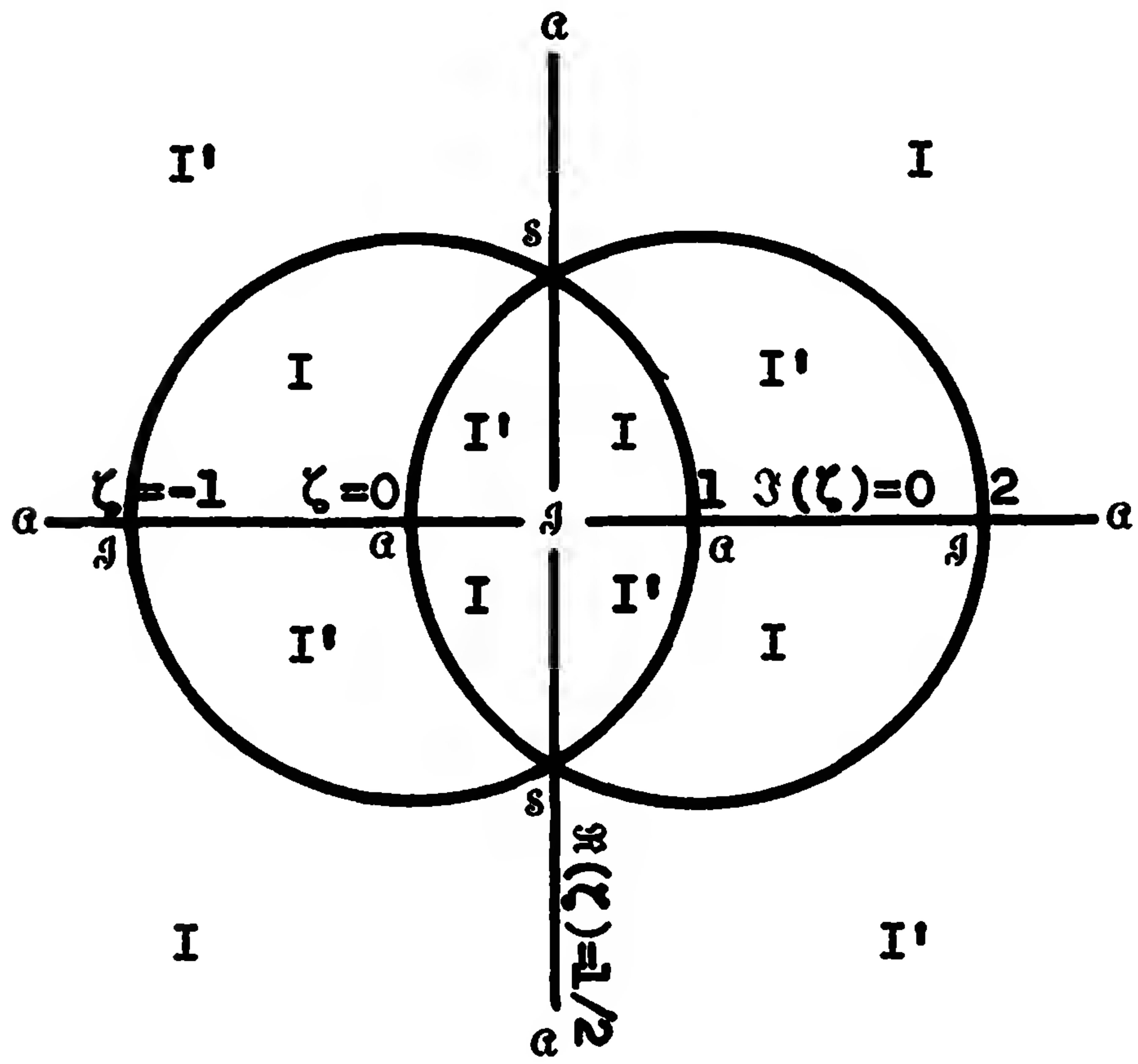
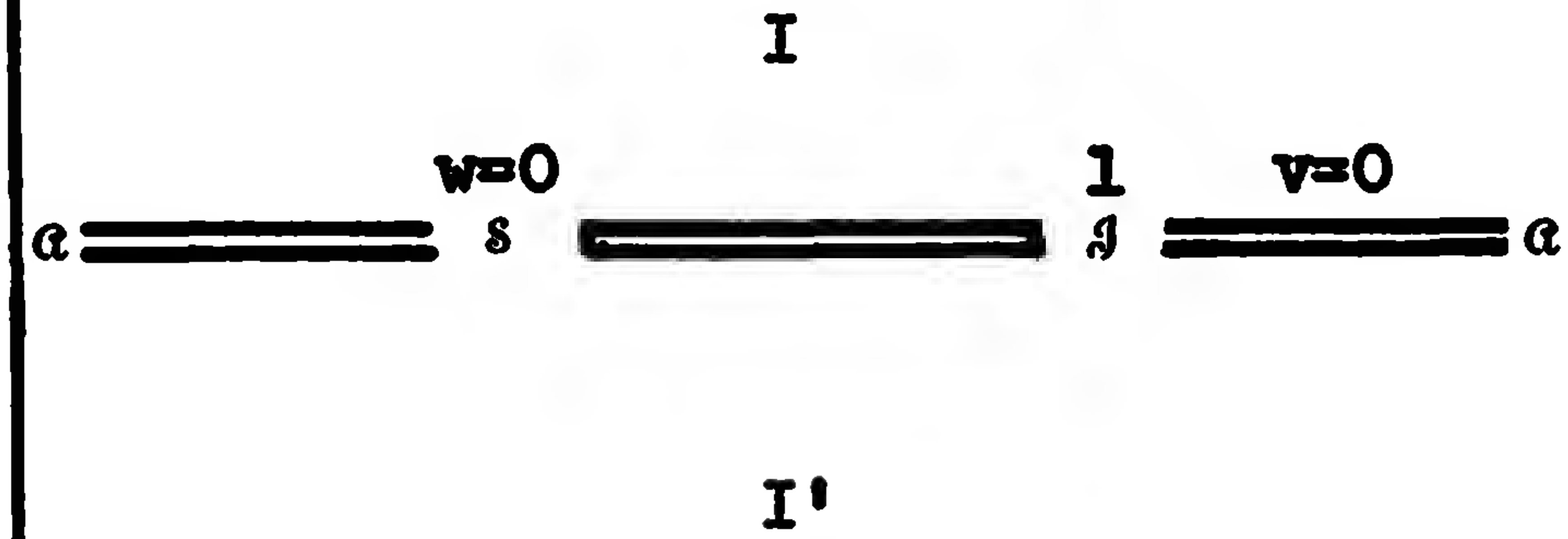
By $\tau' = \frac{\gamma+\delta\tau}{\alpha+\beta\tau}$ the area of the above quadrilateral is mapped on that of another curvilinear quadrilateral, also with angles 0, 0, 0, 0. By $w = \lambda(\tau')$ each of them is mapped on the whole plane, with the two slits as above. All the quadrilaterals together cover the half-plane $y > 0$.

14.2
$$\boxed{w = J(\tau) = \frac{1}{54}\cdot\frac{\Theta_0^8(\tau)+\Theta_2^8(\tau)+\Theta_3^8(\tau)}{\left\{\pi\Theta_0(\tau)\Theta_2(\tau)\Theta_3(\tau)\right\}^8} = \frac{4}{27}\,\frac{\left\{1-\lambda(\tau)+\lambda^2(\tau)\right\}^3}{\left\{\lambda^2(\tau)-\lambda(\tau)\right\}^2}}$$

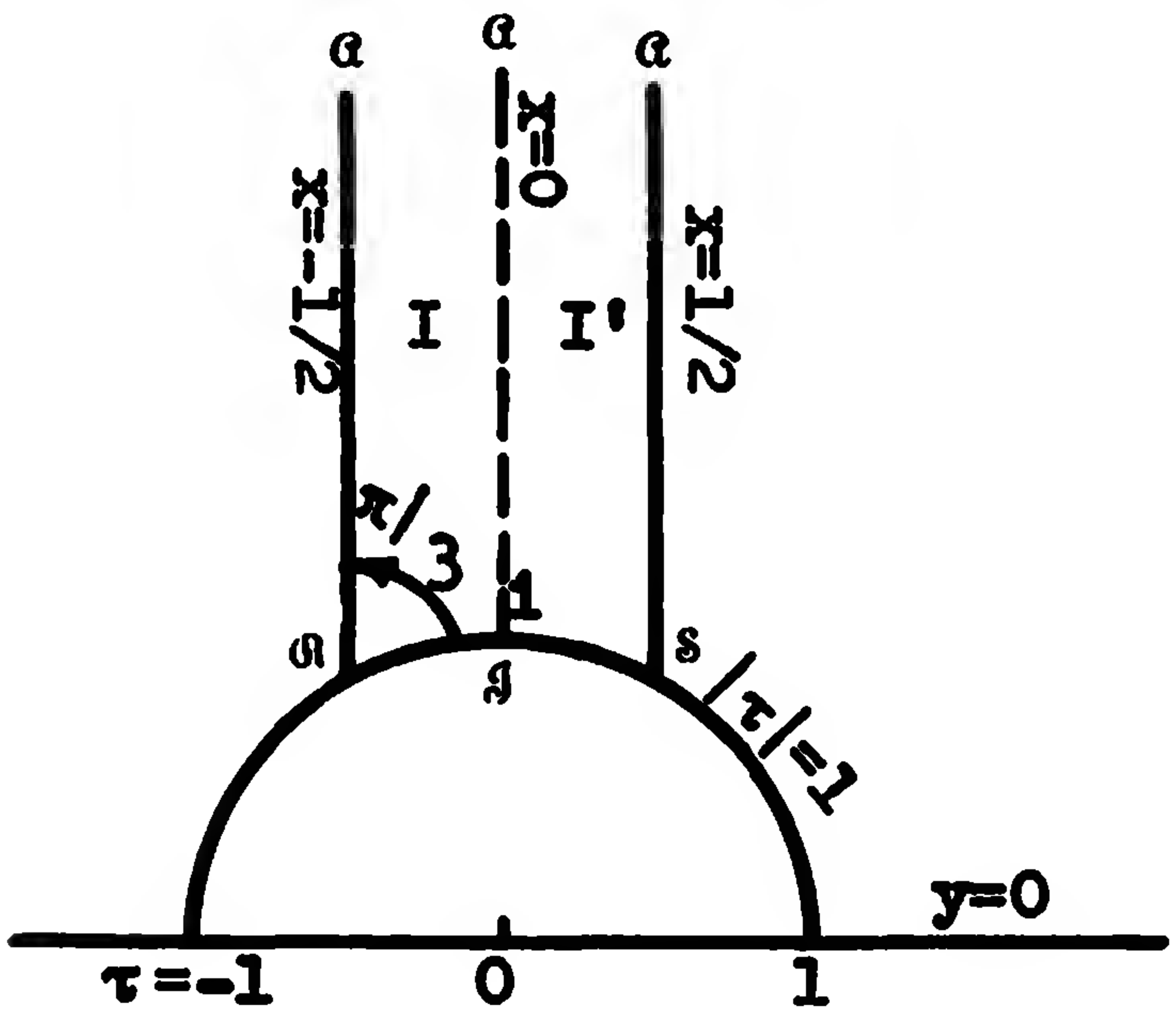
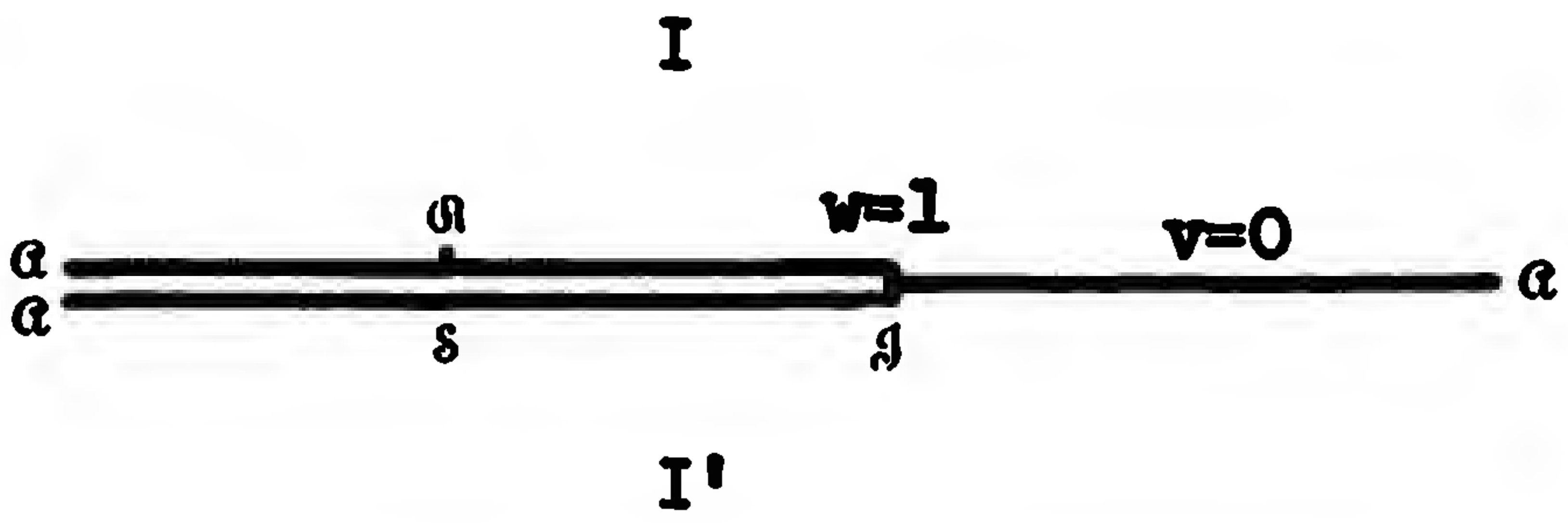
$$= \frac{27g_2^3}{27g_2^3-g_3^2}$$ (cf. §13.1), $y > 0$ $(\tau = x+iy)$. Also $\pi\Theta_0(\tau)\Theta_2(\tau)\Theta_3(\tau) = \Theta_1'$.

Critical points: $\tau = \infty$; any point on the axis $y = 0$.

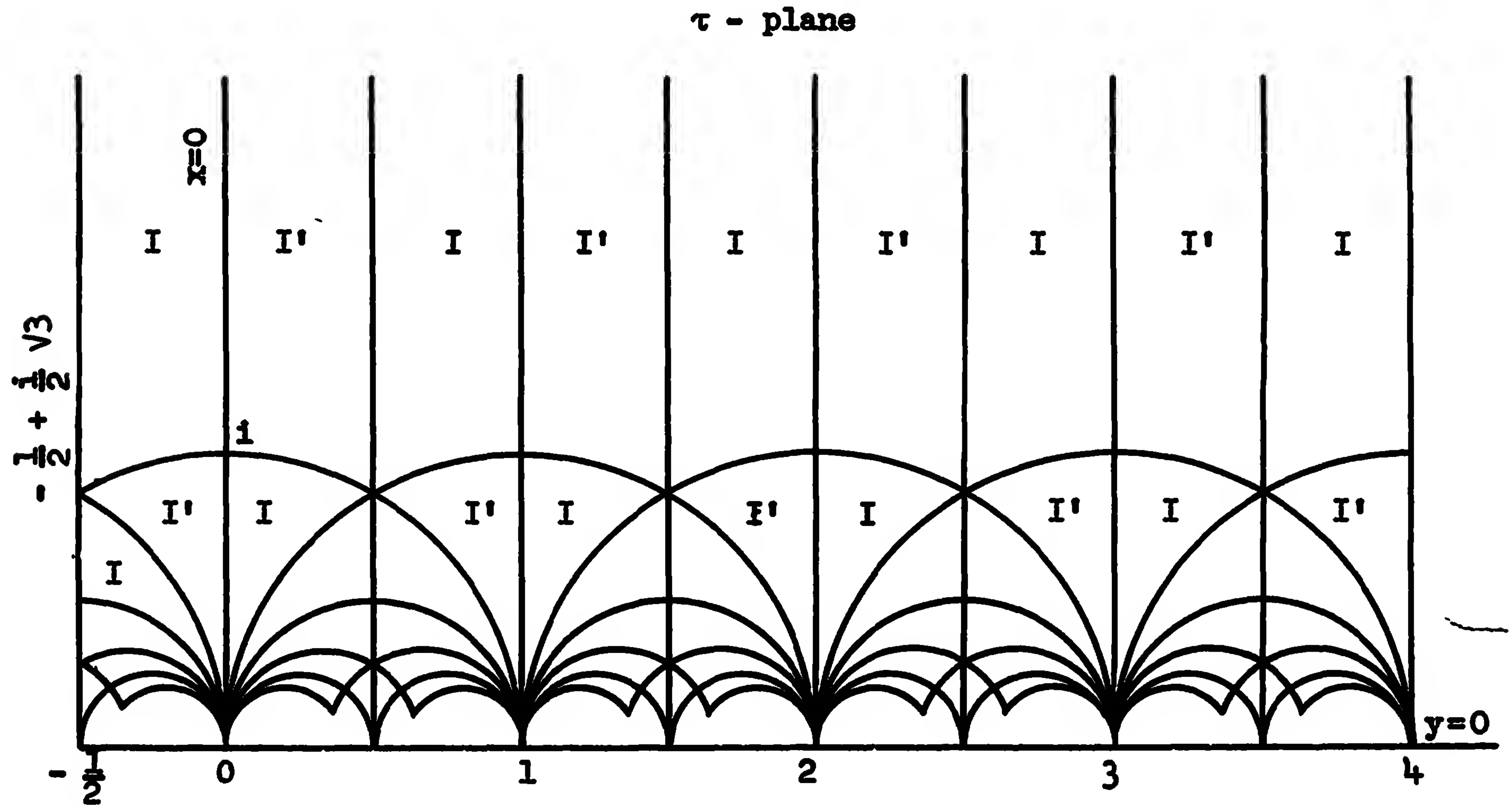
Combination of $\zeta = \lambda(\tau)$ and $w = \frac{4}{27}\frac{(1-\zeta+\zeta^2)^3}{(\zeta^2-\zeta)^2}$, cf. §7.3, pp. 54, 55.

ζ - plane	w - plane
	
Each of the regions I	half-plane $v > 0$
Each of the regions I'	half-plane $v < 0$

Here α, β, γ, δ are any integers such that $\alpha\delta - \gamma\beta = 1$.

τ - plane	plane of $w = J(\tau)$
	
points $\tau = i$ (i.e. $\mathfrak{J}$); $\frac{1}{2}+\frac{1}{2}\sqrt{3}$ ($\mathfrak{s}$); $-\frac{1}{2}+\frac{1}{2}\sqrt{3}$ ($\mathfrak{R}$)	points $w = 1$ ($\mathfrak{J}$); 0 ($\mathfrak{s}$); 0 ($\mathfrak{R}$)
point $\tau' = \dfrac{\gamma+\delta\tau}{\alpha+\beta\tau}$	point $w = J(\tau) = J(\tau')$

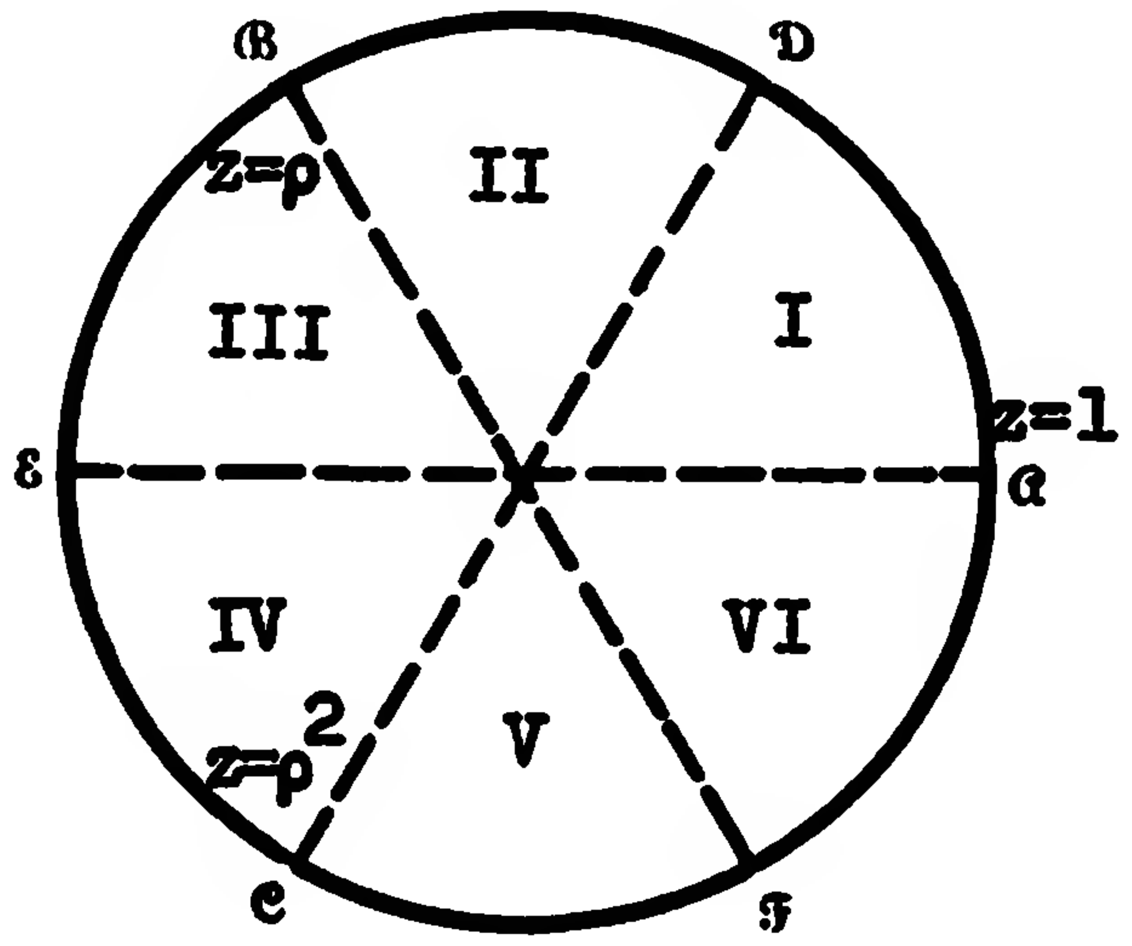
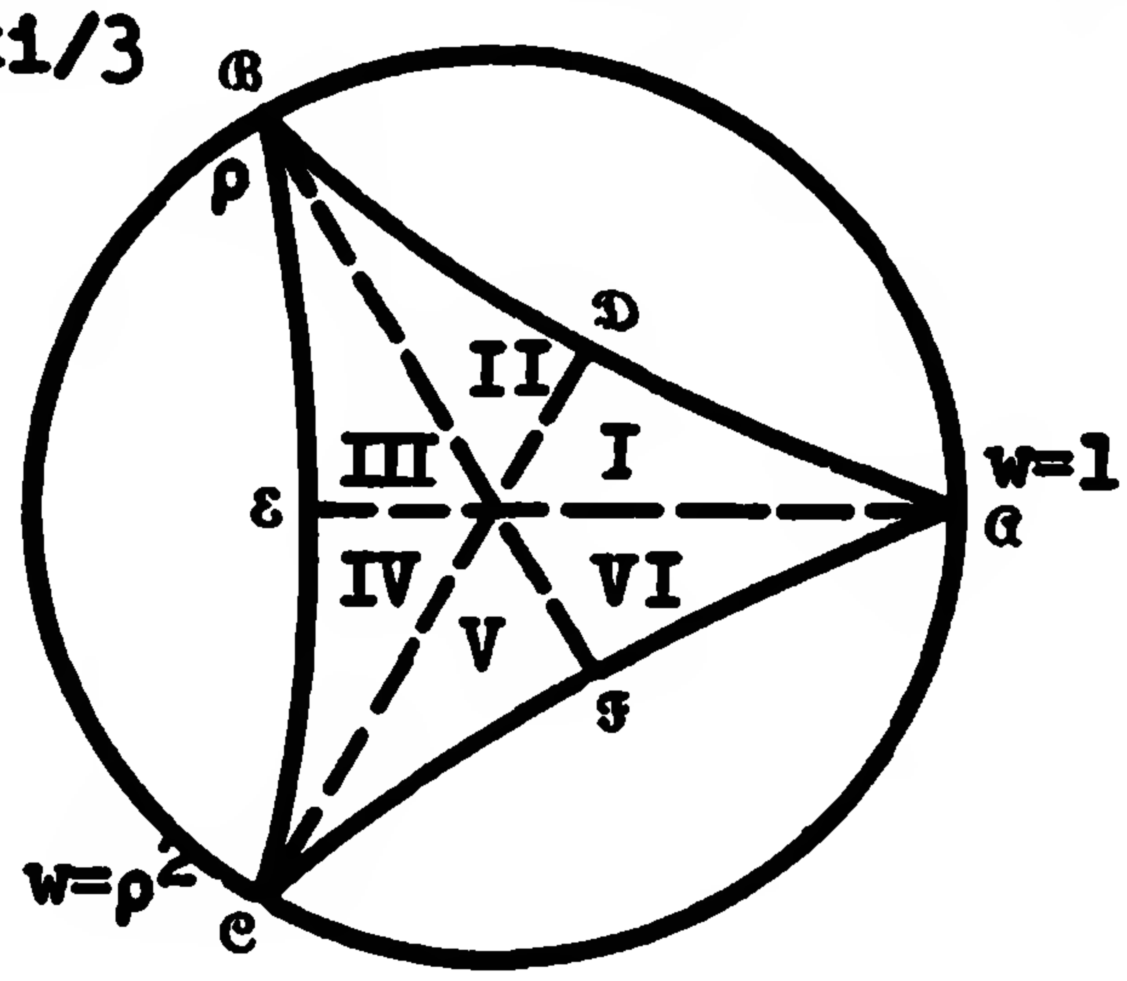
By $\tau' = \frac{\gamma+\delta\tau}{\alpha+\beta\tau}$ the region (I+I') of the τ-plane is mapped on the area of a curvilinear triangle, also with angles $\pi/3$, $\pi/3$, 0. Each of these areas is mapped on the cut w-plane by $w = J(\tau')$. All the triangles together cover the half-plane $\Im(\tau) > 0$.



Of any two curvilinear triangles, with angles $\pi/2$, $\pi/3$, 0 which are separated by an arc of a circle or by part of a line the one is mapped on $v > 0$, the other on $v < 0$. The equation of a circle which passes through $\tau = n$ $(n = 0, \pm 1, \ldots)$ is $|\tau - n - 1/k| = |1/k|$ $(k = \pm 1, \pm 2, \ldots)$; some of them are shown in the figure. Also there are other circles, all of them with centre on $y = 0$.

14.3 **Equilateral equiangular circular triangle**, with angles o, **on circle.**

$0 \leq \alpha < \pi$; $\beta = \frac{\alpha}{2\pi}$; radius of $\mathfrak{A}\mathfrak{D}\mathfrak{B}$: $\frac{\sqrt{3}}{2}\left|\cos\left(\frac{\alpha}{2} + \frac{\pi}{3}\right)\right|^{-1}$

z - plane	w - plane
	$\rho = e^{2\pi i/3}$ 

$$\boxed{w = z\,\frac{\Gamma(\frac{2}{3})\Gamma(\frac{5}{6}+\beta)F(\frac{5}{6}-\beta,\frac{1}{2}-\beta,\frac{4}{3};z^3)}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{6}+\beta)F(\frac{1}{6}-\beta,\frac{1}{2}-\beta,\frac{2}{3};z^3)}}$$

where

$$F(a,b,c;\zeta) = 1 + \frac{ab}{1!c}\zeta + \frac{a(a+1)b(b+1)}{2!c(c+1)}\zeta^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{3!c(c+1)(c+2)}\zeta^3 + \dots$$

$(|\zeta| < 1)$ is the Gaussian hypergeometric series.

Case $\alpha = \frac{\pi}{3}$: see §13.8.

Case $\alpha = 0$: $\boxed{z = \lambda\left(\frac{\rho^2 - w\rho}{w-1}\right)}$ (cf. §14.1) maps the interior of the circular triangle $\mathfrak{A}\mathfrak{B}\mathfrak{C}$, with angles 0, 0, 0, on half-plane $y > 0$.

z - plane	$\tau = \frac{\rho^2 - w\rho}{w-1}$ plane	w - plane
$z = 0;\ 1;\ \infty$	$\tau = \infty;\ 0;\ 1$	$w = 1$ (i.e. $\mathfrak{A}$); ρ ($\mathfrak{B}$); ρ^2 ($\mathfrak{C}$)
$z = 2;\ \frac{1}{2}$	$\tau = \frac{1+i}{2};\ i$	$w = \sqrt{3}-2$ ($\mathfrak{E}$); $(2-\sqrt{3})e^{i\pi/3}$ ($\mathfrak{D}$)
$z = -1;\ e^{i\pi/3}$	$\tau = i+1;\ -\rho^2 = e^{i\pi/3}$	$w = (2-\sqrt{3})e^{-i\pi/3}$ ($\mathfrak{F}$); 0 ($\mathfrak{o}$).

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GEOMETRICAL SUBJECT INDEX

ABBREVIATIONS: C = circle, P = plane, R = rectangle, St = strip. HC = semicircle, HP = half plane, HSt = semi-infinite strip. Q = quarter; s = slit, e.g: sC = circle with one slit inside it, ssP = cut plane, with two slits; QC = quadrant of circle. Double slit: two disjoint semi-infinite slits in line.

Letters in brackets, behind the number of the page, indicate the region on which the interior, unless stated otherwise, of the configuration concerned is mapped.

GEOMETRICAL SUBJECT INDEX

GEOMETRICAL SUBJECT INDEX